Chapter 3

Mathematical Formulation

3.1 Preliminary Remarks

Complex variable approach [6] developed by Muskhelishvili finds wide varieties of applications in stress concentration, contact stresses, fracture mechanics, fluid flow problems, etc. The present problem of isotropic/anisotropic plate with a hole is solved based on two dimensional theory of elasticity using complex variable method [6]. Stresses are presented in terms of Muskhelishvili’s stress functions and failure strengths are calculated using Tsai-Hill and quadratic interaction criterion. The thin anisotropic composite plate is considered which is subjected to in-plane/ bending loading applied at the edge of the plate. The plane stress condition is assumed in the formulations. Each layer is considered to be orthotropic with uniform thickness and all layers are perfectly bonded together. The genetic algorithm is adopted to obtain the optimum stacking sequence in composite plate weakened by a hole. The strength is considered as an objective function and ply orientations are the design variables.
3.2 Solution of Plate Subjected to In-plane Loading

The homogeneous anisotropic infinite plate containing a hole and subjected to in-plane loading at infinity is as shown in Fig. 3.1.

![Figure 3.1: Plate with hole subjected to in-plane loading](image)

The thickness of the plate is very small and plane stress condition is applied (stresses in z-direction are zero). The top and bottom surface of the plate is free of external forces. For small deformations, the model of elastic anisotropic body which obeys the generalized Hooke’s law is taken into consideration for the present study. When the lamina is loaded along arbitrary axis x and y, the generalized Hooke’s law can be written as follows:

\[
\begin{align*}
\varepsilon_x &= A_{11}\sigma_x + A_{12}\sigma_y + A_{16}\tau_{xy}, \\
\varepsilon_y &= A_{12}\sigma_x + A_{22}\sigma_y + A_{26}\tau_{xy}, \\
\gamma_{xy} &= A_{16}\sigma_x + A_{26}\sigma_y + A_{66}\tau_{xy},
\end{align*}
\]

(3.1)

where, \( A_{ij} \) (\( i, j = 1, 2, 6 \)) are material constants that can be calculated as:

\[
[A] = [a]^{-1}, \quad a = \begin{pmatrix}
a_{11} & a_{12} & a_{16} \\
a_{21} & a_{22} & a_{26} \\
a_{61} & a_{62} & a_{66}
\end{pmatrix},
\]
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where, $\beta$ = fiber orientation. Now, stresses can be represented in terms of Airy’s stress functions:

$$
\sigma_x = \frac{\partial^2 U}{\partial y^2} ; \sigma_y = \frac{\partial^2 U}{\partial x^2} ; \tau_{xy} = \frac{\partial^2 U}{\partial x \partial y},
$$

(3.2)

and using Eqs. (3.1) and (3.2) into following compatibility equation

$$
\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y},
$$

(3.3)

the 4th order biharmonic characteristic equation can be obtained as

$$
A_{11} \mu^4 - 2A_{16} \mu^3 + (2A_{12} + A_{66}) \mu^2 - 2A_{26} \mu + A_{22} = 0.
$$

(3.4)

The roots ($\mu_j$ ($j=1$ to 4)) of Eq. (3.4) are to be complex or purely imaginary roots [13], known as constants of anisotropy. The roots can not be real roots for the cases of body with $A_{11}$, $2A_{12} + A_{66}$ and $A_{22}$ are finite and not equal to zero. The cases not considered in the present study are 1. $A_{11} = A_{22} = 0$ (first two roots are equal to zero) 2. $A_{22} = A_{26} = 2A_{12} + A_{66} = A_{16} = 0$ (all roots are zero) 3. $A_{11} = A_{16} = 0$ (two or all roots are infinity) 4. $A_{11} = A_{16} = 2A_{12} + A_{66} = A_{26} = 0$ (two or all roots are infinity) [13].

The bi-harmonic stress function $U(x, y)$ can be represented in terms of $\varpi = x + iy$ [6]:

\[
a_{11} = \frac{1}{1 - \nu_{12} \nu_{21}} (E_1 c^4 + E_2 s^4 + 2\nu_{21} E_1 c^2 s^2 + 4(1 - \nu_{12} \nu_{21}) G_{12} c^2 s^2),
\]

\[
a_{22} = \frac{1}{1 - \nu_{12} \nu_{21}} (E_1 s^4 + E_2 c^4 + 2\nu_{21} E_1 c^2 s^2 + 4(1 - \nu_{12} \nu_{21}) G_{12} c^2 s^2),
\]

\[
a_{12} = \frac{1}{1 - \nu_{12} \nu_{21}} (E_1 c^2 s^2 + E_2 c^2 s^2 + \nu_{21} E_1 (c^4 + s^4) - 4(1 - \nu_{12} \nu_{21}) G_{12} c^2 s^2),
\]

\[
a_{66} = \frac{1}{1 - \nu_{12} \nu_{21}} (E_1 c^2 s^2 + E_2 c^2 s^2 - \nu_{21} E_1 c^2 s^2 + 4(1 - \nu_{12} \nu_{21}) G_{12} (c^2 - s^2)^2),
\]

\[
a_{16} = \frac{1}{1 - \nu_{12} \nu_{21}} (E_1 c^3 s - E_2 c s^3 + 2\nu_{21} E_1 (c s^3 - c^3 s) + 2(1 - \nu_{12} \nu_{21}) G_{12} (c s^3 - c^3 s)),
\]

\[
a_{26} = \frac{1}{1 - \nu_{12} \nu_{21}} (E_1 c s^3 - E_2 c^3 s + 2\nu_{21} E_1 (c^3 s - c s^3) + 2(1 - \nu_{12} \nu_{21}) G_{12} (c^3 s - c s^3)),
\]

$c = \cos \beta,$

$s = \sin \beta,
\[ U(x, y) = 2 \text{Re}[F_1(\varpi_1) + F_2(\varpi_2)], \tag{3.5} \]

where, \( \varpi_j = x + \mu_j y, j = 1, 2. \)

We denote
\[
\frac{dF_1}{d\varpi_1} = \phi(\varpi_1), \quad \frac{dF_2}{d\varpi_2} = \psi(\varpi_2). \tag{3.6}
\]

Now, the area external to a given hole (\( \varpi \)-plane) is mapped conformally to the area outside the unit circle (\( \zeta \)-plane) using the following mapping function:
\[
\varpi(\zeta) = R(\zeta + \sum_{k=1}^{N} m_k \zeta^{-k}), \tag{3.7}
\]

where, \( \zeta = \rho e^{i\theta} \) where \( \rho, \theta \) are the coordinates in \( \zeta \)-plane and \( \rho = 1 \) for the unit circle, for polygonal hole \( m_k = \prod_{p=1}^{n} \frac{(p-1)n-2}{n^k(nk-1)k!} \), \( n= \) number of sides of polygonal hole and \( N= \) number of terms in mapping function. The radius at the corner of this hole can be calculated using:
\[
r = R \frac{(1 - (Nn - 1)D)^2}{(1 + (Nn - 1)^2D_N)}, \tag{3.8}
\]

where, \( D_N = \sum_{k=1}^{N} m_k \zeta^{-k} \). Here, all polygonal holes are inscribed in the circle with radius \( 2R(1 + D_N) \). Now, introducing the complex parameters into Eq. (3.7), the following equation is derived:
\[
\varpi_j(\zeta) = \frac{R}{2} [A_j(\zeta^{-1} + \sum_{k=1}^{N} m_k \zeta^{nk-1}) + B_j(\zeta + \sum_{k=1}^{N} m_k \zeta^{1-nk})], \tag{3.9}
\]

where, \( A_j = (1 + i\mu_j), B_j = (1 - i\mu_j) \) \( (j=1, 2) \).

Based on Eqs. (3.2), (3.5) and (3.9), the stress components, bounded throughout the region, are to be given by
\[
\sigma_x = 2 \text{Re}[\mu_1^2 \phi'(\varpi_1) + \mu_2^2 \psi'(\varpi_2)],
\sigma_y = 2 \text{Re}[\phi'(\varpi_1) + \psi'(\varpi_2)],
\tau_{xy} = -2 \text{Re}[\mu_1 \phi'(\varpi_1) + \mu_2 \psi'(\varpi_2)]. \tag{3.10}
\]
Now, the stress functions can be represented based on method of superposition (Eq. (3.11)).

\[
\phi(\varpi_1) = \phi_1(\varpi_1) + \phi_0(\varpi_1); \psi(\varpi_2) = \psi_1(\varpi_2) + \psi_0(\varpi_2),
\]

(3.11)

For the first problem, due to applied stresses at infinity to the hole free plate, the stress functions \(\phi_1(\varpi_1)\) and \(\psi_1(\varpi_2)\) can be written as bellow:

\[
\phi_1(\varpi_1) = \int \Gamma d\varpi_1; \psi_1(\varpi_2) = \int \Gamma^* d\varpi_2,
\]

(3.12)

where, \(\Gamma = B + iC\) and \(\Gamma^* = B^* + iC^*\). The rotational motion is restricted \((C = 0)\). The constants \(B, B^*\) and \(C^*\) can be obtained by substituting the values of Eq. (3.12) into Eq. (3.10) considering stress at infinity \((\sigma_x^\infty = \lambda\sigma; \sigma_y^\infty = \sigma; \tau_{xy}^\infty = 0; |z| \rightarrow \infty\) [18], refer Fig. 3.1), so that solving simultaneous algebraic equations:

\[
\begin{pmatrix}
\sigma_x^\infty \\
\sigma_y^\infty \\
\tau_{xy}^\infty
\end{pmatrix} = 2
\begin{pmatrix}
(p_1^2 - q_1^2) & (p_2^2 - q_2^2) & -2p_2q_2 \\
1 & 1 & 0 \\
-p_1 & -p_2 & q_2
\end{pmatrix}
\begin{pmatrix}
B \\
B^* \\
C^*
\end{pmatrix},
\]

(3.13)

where, \(p_1 = \Re[\mu_1], p_2 = \Re[\mu_2], q_1 = \Im[\mu_1], q_2 = \Im[\mu_2]\) and, \(\sigma_x^\infty, \sigma_y^\infty\) and \(\tau_{xy}^\infty\) (stresses at infinity with respect to x-y axes) can be calculated by employing following equations:
\[ \sigma_x^\infty + \sigma_y^\infty = (1 + \lambda)\sigma, \]
\[ (\sigma_x^\infty - \sigma_y^\infty + 2i\tau_{xy})e^{2i\alpha} = (1 - \lambda)\sigma, \] (3.14)

where, \( \lambda \) and \( \alpha \) are biaxial loading factor and biaxial loading angle respectively. By using different values of \( \lambda \) and \( \alpha \), various loading conditions can be obtained. Now, due to applied stresses at infinity to the hole free plate, the boundary conditions given on contour \( L \) of region \( S \) around the hole, can be written in the following form:

\[ f_1 = \int_0^S F_y ds + C_2 = \phi_1(\overline{w}) + \phi_1(\overline{w}) + \psi_1(\overline{w}), \]
\[ f_2 = \int_0^S F_x ds + C_1 = \mu_1\phi_1(\overline{w}) + \mu_1\phi_1(\overline{w}) + \mu_2\psi_1(\overline{w}) + \mu_2\psi_1(\overline{w}). \] (3.15)

To find the equations for determination of unknown functions \( \phi_0(\zeta) \) and \( \psi_0(\zeta) \), the negative of the boundary conditions \( (f_1^0 = -f_1, f_2^0 = -f_2) \) are considered. The stress functions \( \phi_0(\zeta) \) and \( \psi_0(\zeta) \) can be obtained by inserting the values of boundary conditions into following Schwartz’s equations:

\[ \phi_0(\zeta) = \frac{i}{4\pi(\mu_1 - \mu_2)} \int_\gamma (\mu_2f_1^0 - f_2^0) \frac{\sigma + \zeta \, d\sigma}{\sigma - \zeta} + \lambda_1, \]
\[ \psi_0(\zeta) = \frac{-i}{4\pi(\mu_1 - \mu_2)} \int_\gamma (\mu_1f_1^0 - f_2^0) \frac{\sigma + \zeta \, d\sigma}{\sigma - \zeta} + \lambda_2. \] (3.16)

Finally, by evaluating above integral (Eq. (3.16)), the stress functions \( \phi_0(\zeta) \) and \( \psi_0(\zeta) \) can be obtained as follows:

\[ \phi_0(\zeta) = \frac{1}{(\mu_1 - \mu_2)} \left[ \{\mu_2(F_1 + F_2) - (F_3 + F_4)\} \zeta^{-1} + \{\mu_2(F_2 + F_1) - (F_4 + F_3)\} \zeta \right], \]
\[ \psi_0(\zeta) = \frac{1}{(\mu_1 - \mu_2)} \left[ \{\mu_1(F_1 + F_2) - (F_3 + F_4)\} \zeta^{-1} + \{\mu_1(F_2 + F_1) - (F_4 + F_3)\} \zeta \right], \]
\[ F_1 = \left( \frac{R}{2} \right) \left[ \Gamma A_1 + \Gamma^* A_2 \right], \]
\[ F_2 = \left( \frac{R}{2} \right) \left[ \Gamma B_1 + \Gamma^* B_2 \right], \]
\[ F_3 = \left( \frac{R}{2} \right) \left[ \mu_1 \Gamma A_1 + \mu_2 \Gamma^* A_2 \right], \]
\[ F_4 = \left( \frac{R}{2} \right) \left[ \mu_1 \Gamma B_1 + \mu_2 \Gamma^* B_2 \right], \]
\[ \zeta_2 = \sum_{k=1}^{N} m_k \zeta^{1-n_k}. \] (3.17)
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The stress function can be obtained by inserting values of Eq. (3.12) and Eq. (3.17) into Eq. (3.11) and then the final stress components are calculated using Eq. (3.10).

3.3 Solution of Plate Subjected to Bending

The thin laminated infinite plate of thickness \( t \), made up of \( N^\ast \) no. of layers, subjected to bending loading, is shown in Fig. 3.3. Each layer is considered to be orthotropic with uniform thickness and perfectly bonded together.

By considering \((x, y)\) as off-axis and \((1, 2)\) as a principal material axis, the transformation equations for \( k^{th} \) layer is employed as:

\[
[Q] = [T]^{-1}[E][T], \tag{3.18}
\]

where,

\[
Q = \begin{pmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{yx} & Q_{yy} & Q_{ys} \\ Q_{sx} & Q_{sy} & Q_{ss} \end{pmatrix}^T; \quad T = \begin{pmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2 - s^2 \end{pmatrix}; \quad E = \begin{pmatrix} E_1 & \nu_{12}E_1 & 0 \\ \nu_{12}E_2 & E_2 & 0 \\ 0 & 0 & G_{12}/2 \end{pmatrix}_k
\]
For the plate subjected to bending, stresses \((σ_x, σ_y, τ_{xy})\) in \(x\) and \(y\) direction can be expressed in-terms of deflection (deflection of the mid-plane in the direction of \(z\)-axis) as follows:

\[
\begin{pmatrix}
σ_x \\
σ_y \\
τ_{xy}
\end{pmatrix}
= -z

\begin{pmatrix}
Q_{xx} & Q_{xy} & Q_{xs} \\
Q_{yx} & Q_{yy} & Q_{ys} \\
Q_{sx} & Q_{ys} & Q_{ss}
\end{pmatrix}

\begin{pmatrix}
w_{,xx} \\
w_{,yy} \\
2w_{,xy}
\end{pmatrix},
\] (3.19)

where, \(z=\) distance from top plane of a lamina to the mid plane of the laminate and the \(w_{,ij}\), \((i, j = x, y)\) are the curvatures \((κ_1, κ_2, κ_3)\) of laminate. The moments in \(x\) and \(y\) directions, \(M_x\), \(M_y\) and \(M_{xy}\) (per unit length of the mid plane) can be obtained by taking:

\[
\begin{pmatrix}
M_x \\
M_y \\
M_{xy}
\end{pmatrix} = -\int_{\frac{t_2}{2}}^{\frac{t_1}{2}} \begin{pmatrix}
σ_x \\
σ_y \\
τ_{xy}
\end{pmatrix} zdz = -\int_{\frac{t_2}{2}}^{\frac{t_1}{2}} \begin{pmatrix}
Q_{xx} & Q_{xy} & Q_{xs} \\
Q_{yx} & Q_{yy} & Q_{ys} \\
Q_{sx} & Q_{ys} & Q_{ss}
\end{pmatrix}

\begin{pmatrix}
w_{,xx} \\
w_{,yy} \\
2w_{,xy}
\end{pmatrix} z^2 dz,
\] (3.20)

where, \(t=\)thickness of laminate and \(F_{ij}\) \((i, j = x, y, s)\) are known as flexural stiffness. Using Eq. (3.20) and equilibrium equation, the \(4th\) order characteristic equation can be obtained. Roots of this equation are complex roots \(µ_j, j = 1, 2\) [13]. By taking the general equation of deflection in terms of arbitrary and analytic functions, \(F_1(ϖ_1)\) and \(F_2(ϖ_2)\) as

\[
w(x, y) = 2\Re[F_1(ϖ_1) + F_2(ϖ_2)],
\] (3.21)

where, \(ϖ_j = x + µ_jy, (j = 1, 2)\). Here, the values of \(ϖ_j\) are calculated by using Eq. (3.9). Now, moments can be calculated by substituting the values of \(w(x, y)\) from Eq. (3.21) into Eq. (3.20):
\[
\begin{pmatrix}
M_x \\
M_y \\
M_{xy}
\end{pmatrix} = - \begin{pmatrix}
F_{xx} & F_{xy} & F_{xs} \\
F_{yx} & F_{yy} & F_{ys} \\
F_{sx} & F_{ys} & F_{ss}
\end{pmatrix} \begin{pmatrix}
2\Re[\phi'(\varpi_1) + \psi'(\varpi_2)] \\
2\Re[\mu_1^2 \phi'(\varpi_1) + \mu_2^2 \psi'(\varpi_2)] \\
4\Re[\mu_1 \phi'(\varpi_1) + \mu_2 \psi'(\varpi_2)]
\end{pmatrix},
\]

(3.22)

where, \( \frac{dF_1}{d\varpi_1} = \phi(\varpi_1) \) and \( \frac{dF_2}{d\varpi_2} = \psi(\varpi_2) \).

The stress functions \( \phi_1(\varpi_1) \) and \( \psi_1(\varpi_2) \) are for hole free plate subjected to loading at infinity.

Constants \( B, B^* \) and \( C^* \) in this equation can be derived using following equations:

\[
\begin{pmatrix}
B \\
B^* \\
C^*
\end{pmatrix} = \begin{pmatrix}
D_{11} & D_{12} & D_{13} \\
D_{21} & D_{22} & D_{23} \\
D_{31} & D_{32} & D_{33}
\end{pmatrix}^{-1} \begin{pmatrix}
M_{x}^\infty \\
M_{y}^\infty \\
M_{xy}^\infty
\end{pmatrix},
\]

\[
D_{11} = 2(F_{xx} + (L_1^2 - L_2^2)F_{xy} + 2L_1F_{xs}),
\]

\[
D_{12} = 2(F_{xx} + (L_1^2 - L_3^2)F_{xy} + 2L_1F_{xs}),
\]

\[
D_{13} = -2(2L_3L_3F_{xy} + 2L_4F_{xs}),
\]

\[
D_{21} = 2(F_{xy} + (L_1^2 - L_2^2)F_{yy} + 2L_1F_{ys}),
\]

\[
D_{22} = 2(F_{xy} + (L_1^2 - L_3^2)F_{yy} + 2L_1F_{ys}),
\]

\[
D_{23} = -2(2L_3L_3F_{yy} + 2L_4F_{ys}),
\]

\[
D_{31} = 2(F_{xs} + (L_1^2 - L_2^2)F_{xy} + 2L_1F_{ss}),
\]

\[
D_{32} = 2(F_{xs} + (L_1^2 - L_3^2)F_{xy} + 2L_1F_{ss}),
\]

\[
D_{33} = -2(2L_3L_3F_{xy} + 2L_4F_{ss}),
\]

\[
L_1 = \Re[\mu_1], L_2 = \Im[\mu_1],
\]

\[
L_3 = \Re[\mu_2], L_4 = \Im[\mu_2],
\]

(3.23)

where, \( M_{x}^\infty, M_{y}^\infty \) and \( M_{xy}^\infty \) are the moments applied at infinity. \( M_{x}^\infty, M_{y}^\infty \) and \( M_{xy}^\infty \) can be expressed as follows:
\[ M_x^\infty + M_y^\infty = (1 + \lambda)M, \]
\[ (M_x^\infty - M_y^\infty + 2iM_{xy}^\infty)e^{2i\alpha} = (1 - \lambda)M. \]  

(3.24)

Now, the boundary conditions around the fictitious hole due to applied moments at infinity, can be obtained as:

\[ f_1 = \text{Re} \left[ \frac{F_{xx} + F_{xy} \mu_1^2 + 2F_{x\sigma} \mu_1}{\mu_1} \Gamma \varpi_1 + \frac{F_{xx} + F_{xy} \mu_2^2 + 2F_{x\sigma} \mu_2}{\mu_2} \Gamma^* \varpi_2 \right], \]
\[ f_2 = \text{Re} \left[ \frac{F_{xy} + F_{yy} \mu_1^2 + 2F_{y\sigma} \mu_1}{\mu_1} \Gamma \varpi_1 + \frac{F_{xy} + F_{yy} \mu_2^2 + 2F_{y\sigma} \mu_2}{\mu_2} \Gamma^* \varpi_2 \right]. \]  

(3.25)

By introducing the values of \( \varpi_j, j = 1, 2 \) into above equation and considering negative of the boundary conditions \( (f_1^0 = -f_1, f_2^0 = -f_2) \) on the hole boundary, the stress functions \( (\phi_0(\zeta), \psi_0(\zeta)) \) can be derived by using following equations:

\[ \phi_0(\zeta) = \frac{i\mu_1}{2\pi(c_1\mu_2 - c_2\mu_1)} \int_\gamma ((F_{xx} + F_{xy} \mu_2^2 + 2F_{x\sigma} \mu_2) f_1^0 - (F_{xy} + F_{yy} \mu_2^2 + 2F_{y\sigma} \mu_2) \mu_2 f_2^0) \frac{\sigma + \zeta d\sigma}{\sigma - \zeta} + \lambda_1, \]
\[ \psi_0(\zeta) = \frac{-i\mu_2}{2\pi(c_1\mu_2 - c_2\mu_1)} \int_\gamma ((F_{xx} + F_{xy} \mu_1^2 + 2F_{x\sigma} \mu_1) f_1^0 - (F_{xy} + F_{yy} \mu_1^2 + 2F_{y\sigma} \mu_1) \mu_1 f_2^0) \frac{\sigma + \zeta d\sigma}{\sigma - \zeta} + \lambda_2, \]  

(3.26)

where,

\[ c_1 = (F_{xx} + F_{xy} \mu_1^2 + 2F_{x\sigma} \mu_1)(F_{xy} + F_{yy} \mu_2^2 + 2F_{y\sigma} \mu_2), \]
\[ c_2 = (F_{xy} + F_{yy} \mu_1^2 + 2F_{y\sigma} \mu_1)(F_{xx} + F_{xy} \mu_2^2 + 2F_{x\sigma} \mu_2). \]

Now, by solving Eq. (3.26), the stress function can be derived as:

\[ \phi_0(\zeta) = \frac{\mu_1}{(c_1\mu_2 - c_2\mu_1)} \left[ \frac{a_1}{\zeta} + b_1\zeta_2 \right], \]
\[ \psi_0(\zeta) = \frac{-\mu_2}{(c_1\mu_2 - c_2\mu_1)} \left[ \frac{a_2}{\zeta} + b_2\zeta_2 \right], \]  

(3.27)
where,

\[ a_1 = q_1 (K_3 + K_4) - q_2 \mu_2 (K_1 + K_2); b_1 = q_1 (K_4 + K_3) - q_2 \mu_2 (K_2 + K_1), \]

\[ a_2 = p_1 (K_3 + K_4) - p_2 \mu_2 (K_1 + K_2); b_2 = p_1 (K_4 + K_3) - p_2 \mu_2 (K_2 + K_1), \]

\[ K_1 = \frac{R}{2} \left[ \frac{p_1}{\mu_1} \Gamma A_1 + \frac{q_1}{\mu_2} \Gamma^* A_2 \right]; K_2 = \frac{R}{2} \left[ \frac{p_1}{\mu_1} \Gamma B_1 + \frac{q_1}{\mu_2} \Gamma^* B_2 \right], \]

\[ K_3 = \frac{R}{2} \left[ p_2 \Gamma A_1 + q_2 \Gamma^* A_2 \right]; K_4 = \frac{R}{2} \left[ p_2 \Gamma B_1 + q_2 \Gamma^* B_2 \right], \]

\[ p_1 = F_{xx} + F_{xy} \mu_1^2 + 2F_{xs} \mu_2; q_1 = F_{xx} + F_{xy} \mu_2^2 + 2F_{xs} \mu_2, \]

\[ p_2 = F_{xy} + F_{yy} \mu_1^2 + 2F_{ys} \mu_1; q_2 = F_{xy} + F_{yy} \mu_2^2 + 2F_{ys} \mu_2, \]

\[ \zeta_2 = \sum_{k=1}^{N} m_k \zeta_1^{1-n_k}. \]

Now, using values of Eq. (3.12) and Eq. (3.27), the final moments can be calculated from Eq. (3.22). The stresses/moments in polar coordinates can be obtained by using coordinate transformation:

\[ M_x + M_y = M_r + M_\theta, \]

\[ M_\theta - M_r + 2iM_r \theta = e^{2i\theta} (M_y - M_x + 2iM_{xy}). \] (3.28)

### 3.4 Failure Strength Calculation

#### 3.4.1 Tsai-Hill Criterion

Azzi and Tsai \[132\] adapted Hill’s criterion to orthotropic composite materials. It is assumed that failure will occur when the transverse stress reaches the transverse strength value. Tsai-Hill criterion for a two dimensional stress:

\[
\sigma_f^2 = \frac{1}{N_1 \left( \frac{\sigma_{xx}}{\sigma_f} \right)^2 + N_2 \left( \frac{\sigma_{yy}}{\sigma_f} \right)^2 + N_3 \left( \frac{\tau_{xy}}{\sigma_f} \right)^2 + N_4 \left( \frac{\sigma_{xy}}{\sigma_f} \right)^2 + N_5 \left( \frac{\sigma_{yx}}{\sigma_f} \right)^2 + N_6 \left( \frac{\sigma_{xx} \sigma_{yy}}{\sigma_f^2} \right)^2},
\]

\[
N_1 = \left( \frac{c^4}{X^2} + \frac{s^4}{Y^2} + \left( \frac{1}{S^2} - \frac{1}{X^2} \right) c^2 s^2 \right) \sigma_f^2,
\]

\[
N_2 = \left( \frac{s^4}{X^2} + \frac{c^4}{Y^2} + \left( \frac{1}{S^2} - \frac{1}{X^2} \right) c^2 s^2 \right) \sigma_f^2,
\]

\[
N_3 = \left( \frac{8c^2 s^2}{X^2} + \frac{4c^2 s^2}{Y^2} + \frac{(c^2 - s^2)^2}{S^2} \right) \sigma_f^2.
\]
\[ N_4 = \left( \frac{-(c^2 - s^2)^2}{X^2} + \left( \frac{1}{Y^2} - \frac{1}{S^2} \right) c^2 s^2 \right) \sigma^2, \]
\[ N_5 = \left( \frac{(6cs^3 - 2sc^3)}{X^2} - \frac{c^3 s c}{Y^2} + \left( \frac{2c^3 s - 2cs^3}{S^2} \right) \right) \sigma^2, \]
\[ N_6 = \left( \frac{(6sc^3 - 2cs^3)}{X^2} - \frac{s^3 c c}{Y^2} + \left( \frac{2s^3 c - 2sc^3}{S^2} \right) \right) \sigma^2, \]

\[ c = \cos\beta; \ s = \sin\beta, \quad (3.29) \]

where, \( \beta \) is the fiber orientation and \( X \) is the tensile strength in longitudinal direction, \( Y \) is the tensile strength in transverse direction and \( S \) is the shear strength. The stresses are calculated by using Eq. (3.10).

### 3.4.2 Quadratic Interaction Criterion

The failure strength of laminated composites can be described using quadratic interaction criterion [133]. It includes the interaction between stress and strain. The strength of laminated composites will be based on the strength of the individual plies within the laminate. The strength of the laminate is defined by examining the strength ratio of each ply. The ply which has lowest strength ratio will fail first and is considered as a failure of whole laminate (first ply failure criterion). The strength ratio can be calculated by using following equation:

\[ [H^\beta_{kf} N_k N_f] R^2_{(\beta)} + [H^\beta_j N_i] R_{\beta} - 1 = 0, \quad (3.30) \]

where, \( \beta \) is the fiber orientation, \( R(\beta) \) = strength ratio of the \( \beta \) fiber angle ply, \( N_k (k = 1, 2, 6) \) = the stress resultant in \( x, y \) and \( xy \) direction obtained from complex variable approach (Eq. (3.10)), \( G_{ij} (i, j = 1, 2, 6) \) is the strength parameter in strain space for \( \beta \) fiber angle ply and, \( H^\beta_{kf} (k, f = 1, 2, 6) \) and \( H^\beta_j (j = 1, 2, 6) \), can be obtained as:

\[
\begin{pmatrix}
H_1 \\
H_2 \\
H_6
\end{pmatrix}^\beta =
\begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix}
\begin{pmatrix}
G_1 \\
G_2 \\
G_6
\end{pmatrix}^\beta,
\]
3.4.3 Failure Strength of Plate with Hole under Bending

The procedure to calculate failure strength of symmetric laminated plate weakened by a hole is presented below. First, curvatures ($\kappa_1 = -w_{,xx}$, $\kappa_2 = -w_{,yy}$, $\kappa_3 = -2w_{,xy}$) around the hole are calculated by using Eq. (3.20):

$$[\kappa] = [F]^{-1} [M].$$  \hspace{1cm} (3.31)

The values of stresses are calculated by using the values of curvatures (Eq. (3.19)) for each ply as follows:

$$[\sigma]^{k \rightarrow N^*} = -z [Q]^{k \rightarrow N^*} [\kappa],$$  \hspace{1cm} (3.32)

where, $k =$ lamina number from the middle layer of laminate and $N^*$=total number of layers. Now, from the values of stresses, the failure strength of each lamina is calculated by using Tsai-Hill failure criterion (Eq. (3.29)). The minimum value of the strength around the hole will give the failure strength of a particular ply. At the end, the first ply failure phenomenon is considered that means the ply with the lowest failure strength is considered to be weakest among all plies and the failure strength of this lamina is considered as the overall strength of the laminate:

$$\sigma_f = \min(\sigma_{f,1}, \sigma_{f,2}, \ldots, \sigma_{f,N^*}).$$  \hspace{1cm} (3.33)
3.5 Strength Optimization of Lamina Subjected to In-plane Loading

The aim of the study is to maximize the strength of the lamina of an orthotropic plate with hole. The ply orientation angles are considered as design variables and the Tsai-Hill theory is taken as the fitness function. The problem may be defined in the mathematical form:

\[
\text{Maximize} \quad \text{Min} \quad \left( \frac{1}{N_1 (\sigma_x)^2 + N_2 (\sigma_y)^2 + N_3 (\tau_{xy})^2 + N_4 (\sigma_x \sigma_y)^2 + N_5 (\sigma_y \tau_{xy})^2 + N_6 (\sigma_x \tau_{xy})^2} \right),
\]

subjected to \(-90^\circ \leq \beta \leq 90^\circ\). (3.34)

3.6 Strength Optimization of Laminate Subjected to In-plane Loading

In laminate case, the value of strength is obtained using the quadratic interaction criterion. Here also, the fiber angles are design variables. The strength of the laminate is defined by examining the strength ratio of each ply. The ply which has lowest strength ratio will fail first and is considered as a failure of whole laminate. The mathematical modeling of this problem is as follows [132,133]:

\[
\text{Maximize} \quad \text{Min} \quad \left( [H_{kj}^\beta N_k N_f] R_{(\beta)}^2 + [H_{ij}^\beta N_i] R_{ij} - 1 = 0 \right),
\]

subjected to \(-90^\circ \leq \beta \leq 90^\circ\). (3.35)