CHAPTER XIII

A NOTE ON FOURIER COEFFICIENTS

13.1 DEFINITIONS AND NOTATIONS.

Let \( \lambda_n \) be non-negative and let

\[
\lambda_n = \lambda_0 + \lambda_1 + \lambda_2 + \ldots + \lambda_n, \quad \lambda_0 = \lambda_\infty
\]

Then \((R', \lambda_{n-1}, l)\) mean of a sequence \( \{t_n\} \) is given by

\[
\frac{1}{\lambda_n} \sum_{\nu=0}^{\infty} \lambda_\nu \, t_\nu
\]

By super-imposing \((R', \lambda_{n-1}, l)\) mean on \((C, l)\) mean the \((R', \lambda_{n-1}, l) (C, l)\) mean of any sequence \( \{y_n\} \) is defined by

\[
T_n = \frac{1}{\lambda_n} \sum_{\nu=0}^{\infty} \lambda_\nu \frac{1}{\nu+1} \sum_{m=0}^{\nu} s_m
\]

If \( T_n \rightarrow s \) as \( n \rightarrow \infty \), we say that the sequence \( \{s_n\} \) is summable \((R', \lambda_{n-1}, l) (C, l)\) to the same.

Let \( f(t) \) be a periodic function with period \( 2\pi \) and integrable in the sense of Lebesgue over \((-\pi, \pi)\).

Let its Fourier series be

\[
(13.1.1) \quad \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n \, t + b_n \sin n \, t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \tilde{a}_n(t)
\]

Then the conjugate series of \((13.1.1)\) is

(13.1.2) \[ \sum_{n=1}^{\infty} \left( b_n \cos nt - a_n \sin nt \right) = \sum_{n=1}^{\infty} \beta_n(t). \]

Let \( x \) and \( l \) be fixed real numbers. Then, we write

(13.1.3) \[ \Psi(t) = f(x+t) - f(x-t) - f(x) \]

(13.1.4) \[ E(n, t) = \sum_{k=0}^{n} \frac{\mu_k}{k+1} \left( \sum_{\nu=0}^{n} k \sin k t \right) \]

(13.1.5) \[ \Psi(t) = \int_{0}^{t} |\Psi(u)| \, du \]

(13.1.6) \[ \lambda_{n-1} = \exp \left\{ \frac{1}{n} \lambda(n) \right\} \]

13.2 INTRODUCTION:

In 1954, Mohanty and Nanda [114] studied Cesàro summability of order one of the sequence \( \{ n \beta_n(x) \} \) and proved the following:

THEOREM A. If

\[ \Psi(t) = o \left( \frac{1}{\log(1/t)} \right), \, t \to 0 \]

and \( a_n \) and \( b_n \) are of \( O(n^{-S}) \), \( 0 < S < 1 \). Then \( \{ n \beta_n(x) \} \) is summable \((C, 1)\) to the value \( 1/\pi \) where \( \pi \) is a finite number.

Varshney [180] obtained the following result concerning the summability of \( \{ n \beta_n(x) \} \).
THEOREM B. If

\[ \psi(t) = o\left(\frac{t}{\log(1/t)}\right), \quad t \to 0; \]

then \( \{n \theta_n(x)\} \) is summable \((N, \frac{1}{n+1}) (C, 1)\) to the value \(1/\pi\).

It is known by Dikshit [37] that the inclusion

\[(13.2.1) \quad (R', \exp(n/\log n), 1) \subset (N, \frac{1}{n+1}) \]

is strict and hence it follows that

\[(13.2.2) \quad (R', \exp(n/\log n), 1) (C, 1) \subset (N, \frac{1}{n+1}) (C, 1) \]

is also strict.

Recently, Jain [78] replaces the summobility method \((N, \frac{1}{n+1}) (C, 1)\) of Theorem B by weaker summobility method \((R', \exp(n/\log n), 1) (C, 1)\) by imposing stronger condition than that of Theorem B on the generating function of the sequence \(\{n \theta_n(x)\}\) and proved the following:

THEOREM C. Let \(g\) be a function satisfying the conditions:

\[(13.2.3) \quad g(u) \uparrow \infty \text{ with } u, \]

\[(13.2.4) \quad w^{-1}(g'(u))^{-1} \in L(\delta, \infty), \quad (\delta > 0) \]

\[(13.2.5) \quad G(u) = \int_{\omega}^{\infty} y^{-1} (g(y))^{-1} dy = o(1), \text{ as } n \to \infty \]

Then if
(13.2.6) \( \Psi(t) = O\left(\frac{t}{g(t)^{1/\alpha}}\right) \quad (t \to 0) \)

The sequence \( \{n B_n(x)\} \) is \( (\mathbb{R}^r, \exp(n/\log n), 1) (\mathbb{C}, 1) \)
summable to \( 1/\pi \).

The object of this chapter is to replace the type 'exp(n/log n)' by 'exp(n/\( \tilde{\lambda}(n) \))' where \( \tilde{\lambda}(n) \) is an increasing function of \( n \) and \( \tilde{\lambda}(n) \to \infty \) as \( n \to \infty \) and imposing stronger condition than that of Theorem 8 on the generating function of the sequence \( \{B_n(x)\} \) such that the inclusion

(13.2.7) \( (\mathbb{R}^r, \exp(n/\tilde{\lambda}(n)), 1) (\mathbb{C}, 1) \subset (\mathbb{N}, \frac{1}{n+1}) (\mathbb{C}, 1) \)

is strict.

13.3 MAIN THEOREM. We establish the following theorem.

**Theorem 1.** Let \( \beta \geq 1 \) and \( g \) be a function satisfying the conditions:

(13.3.1) \( g(w) \uparrow \infty \) with \( w \),

(13.3.2) \( \int_0^\infty \frac{\beta (\log w)^{-1} \exp(-w)}{w} \, dw = \alpha(1), \quad w \to \infty \)

(13.3.3) \( \int_0^\infty \frac{\beta (\log y)^{-1} \exp(-y)}{y} \, dy = \alpha(1), \quad y \to \infty \)

(13.3.4) \( \frac{N_v}{v+1} \quad \frac{1}{\tilde{\lambda}(n)} \quad \text{as} \quad n \uparrow \infty \) such that
Then, if

\[(13.3.6) \quad \psi(t) = \mathcal{O}\left(\frac{\beta^3}{g(\beta t)}\right), \quad (t \to 0)\]

The sequence \(\{n \Theta_n(x)\}\) is \((\mathbb{N}', \exp(n/\gamma(\infty)), 1) \quad (C, 1)\)

summable to \(1/n\).

13. 4 ORDER ESTIMATES: We shall use the following order estimates uniformly in \(0 < t \leq \pi\) in the proof of the theorem.

\[(13.4.1) \quad E(n, t) = \mathcal{O}(n \lambda_n)\]
\[(13.4.2) \quad E(n, t) = \mathcal{O}(\lambda_n^{-1} t)\]
\[(13.4.3) \quad E(n, t) = \mathcal{O}(\lambda_n^{-2} t^2)\]

Proof of (13.4.1) and (13.4.2) are trivial.

PROOF OF (13.4.3). We have

\[
|E(n, t)| = \left| \sum_{\nu=0}^{n} \frac{\mu_{\nu}}{\nu+1} \sum_{k=0}^{\nu} k \sin k t \right| = \left| \sum_{\nu=0}^{n} \frac{\mu_{\nu}}{\nu+1} \sum_{k=0}^{\nu} \frac{d}{dt} (\cos kt) \right| = \left| \sum_{\nu=0}^{n} \frac{\mu_{\nu}}{\nu+1} \frac{1}{2} \frac{d}{dt} \left( \frac{\sum_{k=0}^{\nu} \left( \sin \left( k + \frac{1}{2} \right) t - \sin \left( k - \frac{1}{2} \right) t \right)}{\sin \frac{t}{2}} \right) \right|
\]
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PROOF OF THEOREM 1: We have

\begin{align*}
\mathbb{E}_n(x) &= \frac{1}{\pi} \int_0^{\pi} \left[ f(x + t) - f(x - t) \right] n \sin nt \, dt \\
&= \frac{1}{\pi} \int_0^{\pi} \psi(t) n \sin nt \, dt + \frac{1}{\pi} \{ 1 - (-1)^n \}
\end{align*}

Hence, if \( T_n(x) \) denotes \((R^*, \lambda_{n-1}, 1) \) (C, 1) mean of \( \{n \mathbb{E}_n(x)\} \),

we have
Thus, in order to prove the theorem, it is sufficient to show that, as \( n \to \infty \)

\[
I = \frac{1}{n} \int_0^\pi \psi(t) E(n, t) \, dt = o(1).
\]

Now, we write

\[
I = \frac{1}{\lambda_n} \left[ \left( \int_0^{n^{-1}} + \int_{n^{-1}}^{\alpha} + \int_{\alpha}^1 \right) \psi(t) E(n, t) \, dt \right]
\]

\[
= I_1 + I_2 + I_3 + I_4, \text{ say.}
\]

Now using (13.4.2), we have

\[
|I_1| = \frac{1}{\lambda_n} \int_0^{n^{-1}} |\psi(t)| |E(n, t)| \, dt
\]

\[
= O \left[ n \int_0^{n^{-1}} |\psi(t)| \, dt \right]
\]

\[
= O \left[ n \frac{n^{-\beta}}{g(n)} \right]
\]

(by (13.3.6))

\[
= O \left[ \frac{n^{\beta-1}}{g(n)} \right]
\]

\[
= o(1)
\]

by (13.3.1), as \( n \to \infty \).
\[ \begin{align*}
|I_2| &= O \left[ \frac{1}{\lambda^{\beta-1}(\lambda_{\infty})^{\beta-1}} \int_0^1 |\psi(t)| t^{-1} dt \right] \\
&\quad - O \left[ \frac{1}{\lambda^{\beta-1}(\lambda_{\infty})^{\beta-1}} \int_0^1 \left( \frac{1}{\lambda_{\infty}} \right)^{\beta-1} |\psi(t)| t^{-1} dt \right] \\
&\quad + O(1) \int_0^n y^{\beta-1} \left( g(y) \right)^{-1} dy
\end{align*} \]

(by (13.3.2))

\[ \begin{align*}
\lim_{n \to \infty}
\end{align*} \]

Using (13.4.3), we have

\[ \begin{align*}
|I_3| &= O \left[ \frac{1}{\lambda^{\beta-1}(\lambda_{\infty})^{\beta-1}} \int_0^0 |\psi(t)| \lambda_n^{-1} t^{-2} dt \right] \\
&\quad - O \left[ \frac{1}{\lambda^{\beta-1}(\lambda_{\infty})^{\beta-1}} \int_0^0 |\psi(t)| t^{-2} dt \right]
\end{align*} \]

(by (13.3.5))
- \( \mathcal{O}\left[ \frac{1}{\bar{X}(n)} \right] \) + \( \mathcal{O}\left[ \frac{1}{\bar{X}(n)^{\beta+1}} \int_{\bar{X}(n)^{-1}}^{y} \Phi(t) \, dt \right] \)

And finally, using (13.4.3) again, we have

\[ o(1) + o(1) + o(1) \]

\[ o(1), \quad \text{as} \quad n \to \infty \]
\[ |I_4| = O \left[ \frac{1}{\chi(n)} \int_0^1 t^{-2} |\psi(t)| \, dt \right] = O \left[ \frac{1}{\chi(n)} \right] = o(1), \text{ as } n \to \infty. \]

Thus collecting the results, we obtain

\[ I = o(1), \text{ as } n \to \infty. \]

This terminates the proof of the theorem.

13.6 The following is a direct corollary to our theorem.

**COROLLARY 1.** On taking \( \chi(n) = \log n \) and \( \beta = 1 \) in Theorem 1, we get the theorem C.

The following result is a corollary of our theorem.

**COROLLARY 2.** Let

\[ \psi(t) = o \left( \frac{t}{(\log \log t)^{1+\epsilon}} \right) \quad (\epsilon > 0, \ t \to 0) \]

Then \( \{ n B_n(x) \} \) is summable \( (\mathbb{R}, \exp(n/\log n), 1) \) \( (C, 1) \) to \( 1/\pi \).