11.1 DEFINITIONS AND NOTATIONS: Let \( \{a_n\} \) denote the sequence of partial sum of a given infinite series \( \sum a_n \) and let \( \{p_n\} \) be a sequence of constants, real or complex such that
\[
P_n = p_0 + p_1 + p_2 + \cdots + p_n, \quad p_{-1} = p_{-1} = 0.
\]
The sequence-to-sequence transformation
\[
(11.1.1) \quad t_n = \frac{1}{p_n} \sum_{\nu=0}^{n} p_{n-\nu} s_\nu, \quad p_n \neq 0
\]
defines the sequence \( \{t_n\} \) of Nörlund means\(^1\) of the sequence \( \{s_n\} \) generated by the sequence of coefficients \( \{p_n\} \). The series \( \sum a_n \) is said to be summable \((N, p_n)\) to the sum \( s \) if \( \lim_{n \to \infty} t_n \) exists and equal to \( s \).

Again as infinite series \( \sum a_n \) with sequence of partial sums \( \{s_n\} \) is said to be summable to \( s \) by Nörlund means\(^2\) \((N, p_n^\alpha)\) \((\alpha > -1)\), if

1. Nörlund [119] substantially the same definition is due to Woronoi [183]. See English translation by J.D. Tamarkin in Woronoi [177].
2. This definition is due to Cass [19].
(11.1.2) \[ \lim_{n \to \infty} t_n^{(\omega)} = s, \] as \( n \to \infty \)

where

(11.1.3) \[ t_n^{(\omega)} = \frac{1}{P_n^{(\omega)}} \sum_{\gamma=0}^{n} \frac{P_{n-\gamma}}{\gamma!} s_{\gamma} \]

(11.1.4) \[ P_n^{(\omega)} = \sum_{\gamma=0}^{n} \frac{P^{(\omega)}_{n-\gamma}}{\gamma!}, \quad p_n^{(\omega)} = \sum_{\gamma=0}^{n} A_{n-\gamma} B_{\gamma} \]

where for \( \omega \) real \( A_{\omega} = 1 \) and \( A_{n} = \frac{(\omega+1)(\omega+2)\ldots(\omega+n)}{n!} \)

\( n = 0, 1, 2, \ldots \) and \( p_0^{(\omega)} > 0 \) for all \( n \geq 0 \).

For \( \omega = 1 \), this method reduces to the \( (N, \text{P}_n) \) method of summation.

The condition for the regularity of the method of summability \( (N, p_n^{(\omega)}) \) defined by (11.1.3) are

(11.1.5) \[ \lim_{n \to \infty} \frac{p_n^{(\omega)}}{P_n^{(\omega)}} = 0 \]

and

(11.1.6) \[ \sum_{k=0}^{n} |P_k^{(\omega)}| = O \left( \left| P_n^{(\omega)} \right| \right), \] as \( n \to \infty \).

If \( p_n^{(\omega)} \) is real and non-negative, (11.1.6) is automatically satisfied and then (11.1.5) is the necessary and sufficient condition for the regularity of the method.

Let \( f(t) \) be a periodic function with period \( 2\pi \).
and integrable in the sense of Lebesgue over \((-\pi, \pi)\).

Let its Fourier series be

\[
(11.1.7) \quad \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)
\]

Then the conjugate series of (11.1.7) is

\[
(11.1.8) \quad \sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} B_n(t)
\]

We write throughout

\[
\psi(t) = \psi(x-t) = \left\{ f(x+t) - f(x-t) \right\}
\]

\[
\psi(t) = \int_0^t |\psi(u)| \, du
\]

11.2 INTRODUCTION: In 1979, J. Jin [78] has obtained the following theorem concerning \((N, P_n)\) summability of the conjugate series of a Fourier series at a point.

**THEOREM A.** Let \(\{P_n\}\) be a positive and monotonic non-decreasing sequence and let \(p_n = P(n), \, \, \, \mathcal{P}(u) = \int_0^u P(y) \, dy\)

and let \(g(t)\) be a positive non-decreasing function of \(t\) and

\[
(11.2.1) \quad \psi(t) = o\{t|\log|t|\} \quad \text{as} \quad t \to 0
\]

Then, in order that \(\sum_{n=1}^{\infty} B_n(x)\) is \((N, P_n)\) summable to \(\tilde{f}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(t) \cos nt \, dt\), whenever it exists, it is necessary and sufficient that
The object of this chapter is to improve Theorem A by replacing condition (11.2.1) and (11.2.2) by more general conditions by introducing a variable parameter $\alpha$ ($\alpha > -1$) such that when this is equal to unity, we get Theorem A.

11. 3 MAIN RESULT: We now establish the following

Theorem:

THEOREM 1. Let $\{P_n\}$ be a positive and monotonic decreasing sequence and let $\tilde{P}_n(x) = P_n(x)$, $\tilde{P}(x) = \int P(y) dy$

and let $g(t)$ be a positive non-decreasing function of $t$ and for $-1 < \alpha < 1$

(11.3.1) $\tilde{f}(x) = \int \frac{1}{2\pi} \int P(y) \cot \frac{t}{2} dt$

Then, in order that $\sum B_n(\alpha)$ is $(N, \tilde{P}_n(x))$ summable to

(11.3.2) $\tilde{f}(x) = \int \frac{1}{2\pi} \int P(y) \cot \frac{t}{2} dt$

whenever it exists, it is necessary and sufficient that

(11.3.3) $\int \frac{P(t)}{t^{x+\alpha} g(t)} dt = O (P(x))$ (as $u \to \infty$)

In order to prove the theorem, we require the following lemmas:
LEMMA 1. If \( \{ b_n \} \) is positive and non-increasing sequence,
then for \( a \leq \alpha \leq b \leq \infty \), \( a \leq t \leq \infty \) and for any \( n \)
\[
\left| \sum_{k=a}^{b} b_k \exp(i(n-k)t) \right| \leq K \left( \frac{1}{t} \right)^n
\]
where \( K \) is an absolute constant, the proof of the lemma follows on the lines of McFadden [111].

LEMMA 2 (Jain [78]). Let \((11.3.1)\) hold and let the integral in \((11.3.2)\) exist. Then, for \( \alpha < s < \pi \) and \( m \geq n \)
\[
\bar{S}_n(x) = \frac{1}{2\pi} \int_{0}^{\pi} \Psi(t) \cot \frac{t}{2} \, dt = \frac{1}{2\pi} \int_{0}^{\pi} \frac{\psi(t)}{\sin \frac{t}{2}} \cos \left( \frac{n}{2} \right) t \, dt + o(1)
\]
where \( \bar{S}_n(x) \) denotes the nth partial sum of the series
\[
\sum_{n=1}^{\infty} \beta_n(x)
\]

11.4 PROOF OF THEOREM 1:

Let \( t_n \) denote the Nörlund transform of

of the series \( \sum_{n=1}^{\infty} \beta_n(x) \)

We have,
\[
t_n = \frac{1}{2\pi} \int_{0}^{\pi} \psi(t) \cot \frac{t}{2} \, dt
\]
\[
= - \frac{1}{2\pi} \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \frac{\psi(t)}{\sin \frac{t}{2}} \sum_{\nu=0}^{n} \frac{\eta_{\nu}}{n} \cos \left( \frac{\nu}{2} \right) t \, dt + o(1)
\]
\[
= - \frac{1}{2\pi} \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \frac{\psi(t)}{\sin \frac{t}{2}} \sum_{\nu=0}^{n} \frac{\eta_{\nu}}{n} \cos \left( \frac{\nu}{2} \right) t \, dt + o(1)
\]
\[ = O \left[ \int_{n^{1}}^{b} \frac{|\psi(t)|}{\ln n t / 2} \frac{p^{(\omega)}(1/t)}{t} \, dt \right] \]  

(by Lemma 1)

\[ = O \left[ \int_{n^{1}}^{b} \frac{|\psi(t)|}{\ln n t / 2} \frac{p^{(\omega)}(1/t)}{t} \, dt \right] \]

\[ = O \left\{ \frac{1}{p^{(\omega)}} \int_{n^{1}}^{b} \psi(t) \frac{p^{(\omega)}(1/t)}{t} \, dt \right\} - \int_{n^{1}}^{b} \psi(t) \frac{d}{dt} \left( \frac{p^{(\omega)}(1/t)}{t} \right) \, dt \right\} \]

\[ = O \left( \int_{n^{1}}^{b} \frac{1}{t} \frac{p^{(\omega)}(1/t)}{2} \, dt \right) + O \left( \int_{n^{1}}^{b} \frac{1}{n^{1-\omega}} \frac{1}{g(n)} \, dt \right) + O \left[ \frac{1}{p^{(\omega)}} \int_{n^{1}}^{b} \psi(t) \frac{d}{dt} \left( \frac{p^{(\omega)}(1/t)}{t} \right) \, dt \right] \]

\[ = O \left\{ \frac{1}{p^{(\omega)}} \int_{n^{1}}^{b} \psi(t) \frac{d}{dt} \left( \frac{p^{(\omega)}(1/t)}{t} \right) \, dt \right\} + o(1) \]

Now in order to prove that \( \sum_{n=1}^{\infty} \Theta_n(x) \) is \((N, p^{\infty})\) summable to \( \hat{f}(x) \) whenever (11.3.1) holds, it is necessary and sufficient that

\[ (11.4.1) \int_{n^{1}}^{b} \frac{t^{2-\omega}}{g(1/t)} \frac{d}{dt} \left\{ \frac{p^{(\omega)}(1/t)}{t} \right\} \, dt = O \left( \frac{p^{(\omega)}}{p^{(\omega)}} \right), \quad \text{as } n \to \infty. \]

Now we will prove that (11.3.3) is sufficient for (11.4.1), we have

\[ \int_{n^{1}}^{b} \frac{t^{2-\omega}}{g(1/t)} \frac{d}{dt} \left\{ \frac{p^{(\omega)}(1/t)}{t} \right\} \, dt \]

\[ = \left\{ \int_{n^{1}}^{b} \frac{t^{2-\omega}}{g(1/t)} \frac{d}{dt} \left\{ \frac{p^{(\omega)}(1/t)}{t} \right\} \, dt \right\} \]
\[
\left. \int_{\tau}^{s} \int_{t}^{s} \frac{\alpha}{g(1/t)} \left\{ \frac{\gamma(1/t)}{t} \frac{d}{dt} \left( \frac{P(1/t)}{t} \right) \right\} \frac{d}{dt} g(1/t) dt \right. \\
+ \int_{\tau}^{s} \frac{\alpha}{g(1/t)} \left\{ \frac{P(1/t)}{t} \frac{d}{dt} g(1/t) \right\} dt \\
= \int_{\tau}^{s} \frac{e^{-\alpha}}{g(1/t)} \left\{ \frac{P(1/t)}{t} \frac{d}{dt} \left( \frac{P(1/t)}{t} \right) \right\} dt \\
+ \int_{\tau}^{s} \frac{t}{(g(1/t))^2} \frac{d}{dt} g(1/t) dt \\
(11.4.2) = \left( \frac{S^{1-\alpha} P(1/t)}{g(1/t)} \right) - \frac{P(n)}{n^{1-\alpha} g(n)} - \int_{\tau}^{s} \frac{P(1/t)}{t^{\alpha} g(1/t)} dt \\
+ \int_{\tau}^{s} \frac{t}{(g(1/t))^2} \frac{d}{dt} (g(1/t)) dt \\
\]

Hence, we have
\[
\left\lfloor \int_{\tau}^{s} \frac{e^{-\alpha}}{g(1/t)} \left\{ \frac{P(1/t)}{t} \frac{d}{dt} \left( \frac{P(1/t)}{t} \right) \right\} \frac{d}{dt} g(1/t) dt \right. \\
\frac{d}{dt} \left( \frac{P(1/t)}{t} \right) \right\} dt \\
\leq \frac{S^{1-\alpha} P(1/t)}{g(1/t)} + \frac{P(n)}{n^{1-\alpha} g(n)} + \int_{\tau}^{s} \frac{P(1/t)}{t^{\alpha} g(1/t)} dt \\
+ \int_{\tau}^{s} \frac{t}{(g(1/t))^2} \frac{d}{dt} (g(1/t)) dt \\
= \mathcal{O}(P(n)) + \int_{\tau}^{s} \frac{P(1/t)}{t^{\alpha} g(1/t)} dt \\
\]
Finally we will show that (11.3.3) is necessary. Now from (11.4.2) we have

\[ \int_{1.4}^{S} \frac{t^{-\alpha} P(1/t)}{g(1/t)} \, dt = \frac{S^{1-\alpha} P(1/S)}{g(1/S)} - \frac{1}{n^{1-\alpha} g(n)} - \int_{1.4}^{S} \frac{t^{2-\alpha} d\left( \frac{P(1/t)}{t}\right)}{d(1/t)} \, dt \]

\[ + \int_{1.4}^{S} \frac{t^{1-\alpha} P(3/t)}{g(3/t)^2} \frac{d}{dt} \left( g(1/t) \right) \, dt, \]

\[ = O(1) + O\left( \frac{P(1)}{n^2} \right) + O\left( \frac{P(n)}{n} \right) \]

\[ = O\left( \frac{P(n)}{n} \right) \]

By (11.4.1)

This completes the proof of the theorem.

REMARK: When \( \alpha = 1 \), Theorem A of Jain [78] becomes a corollary to this theorem.