10.1 In 1982, Khan [92] obtained a number of fixed point theorems for certain commuting mappings in Banach spaces. The results obtained by Khan [92] can be viewed as applications of those due to Goebel - Žloltiewicz [52] and Dotson [39] others, are indeed generalisation of results due to Shimi [160]. Khan [92] proved the following:

**THEOREM A.** Let $K$ be a closed and convex subset of Banach space $X$. Let $F : K \to K$, $G : K \to K$ satisfy the following conditions:

1. $F$ and $G$ commute,
2. $F^2 = I$ and $G^2 = I$, where $I$ denotes the identity mapping,
3. $\|Fx - Fy\| \leq \alpha \|Gx - Gy\|

for every $x, y \in K$ and $0 \leq \alpha < 2$. Then there exists at least one point $x_0 \in K$ such that $F(x_0) = G(x_0)$. Further, if $0 \leq \alpha < 1$, then $x_0$ is unique and $x_0 = F(x_0) = G(x_0)$.

10.2 Our aim in this section is to improve upon the above result and establish some common fixed point theorems.

for a pair of mappings defined on certain subsets of a Banach space satisfying a new contractive type condition. Our main results are established in the following theorems.

**THEOREM 1.** Let M be a non-empty closed convex subset of a Banach space B. Let \( F : M \rightarrow M \), \( G : M \rightarrow M \) satisfy the following conditions:

- (10.2.1) \( F \) and \( G \) commute,
- (10.2.2) \( F^2 = I \) and \( G^2 = I \), where \( I \) denotes the identity mappings,
- (10.2.3) \[ \| Fx - Fy \|^2 \leq q \max \left\{ \| Gx - Fx \|, \| Gy - Fy \|, \| Gx - Fy \| \right\} \]
  \[ \| Gx - Fy \| + \| Gy - Fy \| \]
  for all \( x, y \in M \), where \( q \in (0, 1) \). Let \( x_1 \in M \) be arbitrary, \( t \in (0, 1) \) and \( G_{n+1} = (1-t)G_n + tF_n \) for each integer \( n \geq 1 \). If the sequence \( \{G_n\} \) converges to a point \( u \in M \), then \( u \) is the unique common fixed point of \( F \) and \( G \).

**PROOF:** For each \( n \geq 1 \), we have \( G_{n+1} = (1-t)G_n + tF_n \) which implies

\[ \| G_{n+1} - FGu \|^2 \leq [ (1-t) \| G_n - FGu \| + t \| F_n - FGu \| ] \]
Since \( F \) and \( G \) satisfy (10.2.3), we have by using (10.2.2)

\[
\|F_{x_n} - F_{G_n}\|^2 \leq q \max \left\{ \|G_{x_n} - F_{x_n}\| \|u - F_{G_n}\|, \|G_{x_n} - F_{x_n}\| \|G^2 u - F_{G_n}\|, \|G_{x_n} - F_{G_n}\| \|G^2 u - F_{G_n}\| \right\}
\]

From (10.2.4) and (10.2.5), we obtain

\[
(10.2.6) \quad \|G_{x_{n+1}} - F_{G_n}\|^2 \leq (1-t)^2 \|G_{x_n} - F_{G_n}\|^2 + 2t(1-t) + t^2 \|F_{x_n} - F_{G_n}\|_2 + \|F_{x_n} - F_{G_n}\|
\]

\[
\|G_{x_n} - F_{x_n}\| \left( \|u - G_{x_n}\| + \|G_{x_n} - F_{x_n}\| \right),
\]

\[
\|G_{x_n} - F_{x_n}\| \left( \|u - G_{x_n}\| + \|G_{x_n} - F_{x_n}\| \right),
\]

\[
\|G_{x_n} - F_{G_n}\| \left( \|u - G_{x_n}\| + \|G_{x_n} - F_{x_n}\| \right),
\]

\[
\|G_{x_n} - F_{G_n}\| \left( \|u - G_{x_n}\| + \|G_{x_n} - F_{x_n}\| \right),
\]

\[
\|G_{x_n} - F_{G_n}\| \left( \|u - G_{x_n}\| + \|G_{x_n} - F_{x_n}\| \right),
\]
Since \( \{Gx_n\} \) converges to \( u \) and \( Gx_{n+1} - Gx_n = t(Fx_n - Gx_n) \),
then \( \{Fx_n - Gx_n\} \) tends to zero. By letting \( n \to \infty \) in
(10.2.6), we obtain

\[
\| u - FGu \|^2 \leq (1-t)^2 \| u - FGu \|^2 + 2t(1-t) \| u - FGu \|^2
\]
\[
+ t^2 q \max \{ \alpha, \beta, \gamma, \| u - FGu \|^2 \}
\]
\[
= (1-t)^2 \| u - FGu \|^2 + 2t(1-t) \| u - FGu \|^2
\]
\[
+ t^2 q \| u - FGu \|^2
\]
\[
= \beta \| u - FGu \|^2
\]

Where \( \beta = [1 - (1-q)t^2] \). Since \( \beta \in (0,1) \), \( FG \) has at
least one fixed point \( u \in M \) is.

(10.2.7) \( FGu = u \).

Now using (10.2.2), we have

(10.2.8) \( Fu = F^2Gu = Gu \).

Again using (10.2.1), (10.2.2), (10.2.3) and (10.2.7),
(10.2.8) we have

\[
\| u - Fu \|^2 = \| F(Fu) - Fu \|^2
\]
\[
\leq q \max \{ \| GFu - F^2u \|, \| Gu - Fu \|, \| GFu - Fu \|, \| Gu - F^2u \|, \| GFu - Fu \|, \| Gu - Fu \| \}
\]
\[
- q \max \{ \alpha, \beta, \gamma, \| u - Fu \|^2, \alpha, \beta \}
\]
\[
- q \| u - Fu \|^2
\]

Since \( q < 1 \), it follows \( Fu = u \) i.e. \( u \) is the fixed point.
fixed point of $F$ and $G$. Now to show the uniqueness of $u$, let us consider $v$ \((v \neq u)\) be another common fixed point of $F$ and $G$.

\[
\begin{align*}
\|u - v\|^2 &= \|F^2u - Fv\|^2 - \|F(Fu) - F(Fv)\|^2 \\
&
\leq q \max \left\{ \|GFu - F^2u\|, \|GFv - F^2v\|, \|GFu - F^2v\|, \|GFv - F^2u\| \right\} \\
&= q \max \left\{ 0, \|u - v\|^2, 0, 0 \right\} \\
&= q \|u - v\|^2.
\end{align*}
\]

Since $q \in (0, 1)$, it follows $u = v$, proving the uniqueness of $u$. This completes the proof of the theorem.

Now, wish to investigate the solvability of certain non-linear functional equations in a Banach space.

**Theorem 2.** Let \(\{f_n\}\) be a sequence of elements in a Banach space $B$. Let $v_n$ the unique solution of the equation \(u - FGu = f_n\), where $F$ and $G$ are mappings of $B$ into itself satisfy conditions (10.2.1), (10.2.2) and (10.2.3) of Theorem 1. If $\|f_n\| \to 0$ as $n \to \infty$, then the sequence $\{v_n\}$ converges to the solution of the equation $u = Fu - Gu$.

**Proof.** By (10.2.3), we have for $m > n$
On letting $n \to \infty$ and using the hypothesis, we obtain

$$\|v_n - v_m\|^2 \leq q \|v_n - v_m\|^2$$

which is impossible since $q \in (0, 1)$. It follows therefore that $\{v_n\}$ is a Cauchy sequence. Hence it converges to $v$, say.

Also it is an easy exercise to show that

$$\|v - FGv\|^2 \leq q \|v - FGv\|^2$$

which implies $v = FGv$. Now $Fv = F^2Gv = Gv$. The rest of the proof of this theorem goes in a similar fashion as that of Theorem 1, so we omit the proof.

Finally, we furnish an example to discuss the validity of the hypothesis and degree of generality of Theorem 1.
EXAMPLE 1. Let \( B = \mathbb{R}^3 \), where \( \mathbb{R}^3 \) is the set of all 3-tuples \( x = (x(1), x(2), x(3)) \) of real numbers and the norm \( \| x \| \) is defined by

\[
\| x \| = \left( \sum_{i=1}^{3} |x(i)|^2 \right)^{1/2}, \quad x \in \mathbb{R}^3
\]

Further, let \( M = \{ x : \| x - 0 \| \leq 1, \quad 0 < x \in \mathbb{R}^3 \} \) and we define mapping \( F \) and \( G \) of \( M \) into itself such that for any arbitrary \( x = (x(1), x(2), x(3)) \in M \)

\[
Fx = (-x(1), -x(2), -x(3)) \quad \text{and} \quad Gx = (x(2), x(1), x(3))
\]

Suppose \( \{ x_n \} \) be a sequence of elements of \( M \) such that

\[
Gx_{n+1} = (1-t)Gx_n + tFx_n \quad \text{for} \ n \geq 1 \quad \text{and} \ t \in (0, 1).
\]

setting \( t = \frac{1}{3} \) and consider \( x_1 = (1, 0, 0) \in M \), then it is easy to see that \( x_2 = (\frac{2}{3}, \frac{1}{3}, 0), \ x_3 = (\frac{5}{9}, -\frac{4}{9}, 0) \)

etc. Taking \( x = x_1, \ y = x \) and \( \frac{1}{6} \leq q < 1 \), we see that all the conditions of theorem 1 are satisfied and \( 0 \) is the unique common fixed point of \( F \) and \( G \).

REMARK: Using \( (a_3) \) of Khan's condition in example 1

for \( x = x_1 \) and \( y = x_2 \) we have

\[
\| Fx_1 - Fx_2 \| \leq \kappa \| Gx_1 - Gx_2 \|
\]

i.e., \( \frac{\sqrt{2}}{3} \leq \kappa \frac{\sqrt{2}}{3} \)
a contradiction, since \( \alpha \in (0, 1) \) for uniqueness of common fixed point of maps \( F \) and \( G \). Hence condition \((a_3)\) fails to prove the existence of the unique common fixed point of maps \( F \) and \( G \). But this pair of mappings \( F \) and \( G \) is contained in our class of mappings and hence it guarantees the uniqueness of common fixed point of maps \( F \) and \( G \).

10.3 In 1974, K. Iseki [69] proved that if \( F \) is a self-mapping of a Banach space \( X \) into itself and \( F \) satisfy the conditions:

\[
\text{(a) } F^2 \neq I, \text{ where } I \text{ is the identity mapping},
\]

\[
\text{(b) } \|Fx - Fy\| \leq \alpha \|x - y\| + \beta [\|x -Fx\| + \|y - Fy\|]
\]

for all \( x, y \in X \) where \( \alpha < 1, \beta \) and \( \alpha < 1 + 4\beta < 2 \). Then \( F \) has at least one fixed point.

Sharma and Bajaj [156] generalizes the theorem of Khan [92] in the following:

**Theorem 8.** Let \( K \) be a closed and convex subset of a Banach space \( X \). Let \( F : X \to X, G : K \to K \) satisfy the following conditions:

\( (b_1) \) \( F \) and \( G \) commutes;

\( (b_2) \) \( F^2 = I \) and \( G^2 = I \), where \( I \) denotes the identity mapping;

\( (b_3) \) \( \|Fx - Fy\| \leq \alpha \|Gx - Gy\| + \beta [\|Gx - Fx\| + \|Gy - Fy\|] \)
for every \( x, y \in K \) and \( \alpha \leq \alpha, \beta \) and \( \alpha + 4 \beta < 2 \). Then there exists at least one fixed point \( x_0 \in K \) such that \( F(x_0) = G(x_0) \). Further if \( \alpha \leq \alpha < 1 \), then \( x_0 \) is the unique fixed point of \( F \) and \( G \).

**REMARK.** If we put \( \beta = \alpha \) in Theorem 8 we get theorem A of Khan \([92]\).

Now we present the theorem which generalizes the theorem of Iséki \([69]\), Khan \([92]\) and Sharma and Bajaj \([158]\).

**THEOREM 3.** Let \( K \) be a closed and convex subset of a Banach space \( X \). Let \( F, G \) and \( H \) are three mappings of \( K \) into itself such that

\[
(10.3.1) \quad FG = GF, \quad GF = HG \quad \text{and} \quad FH = HF;
\]

\[
(10.3.2) \quad F^2 = I, \quad G^2 = I \quad \text{and} \quad H^2 = I, \quad \text{where} \quad I \quad \text{denotes the identity mapping};
\]

\[
(10.3.3) \quad \| Fx - Fy \| \leq \alpha \| GHx - GHy \| + \beta \left[ \| GHx - Fx \| + \| GHy - Fy \| \right]
\]

for every \( x, y \in K \) and \( \alpha \leq \alpha, \beta \) such that \( \alpha + 4 \beta < 2 \).

Then there exists at least one fixed point \( x_0 \in K \) such that \( Fx_0 = GHx_0 \) and \( FGx_0 = Hx_0 \). Further, if \( \alpha \leq \alpha < 1 \), then \( x_0 \) is the unique common fixed point of \( F, G \) and \( H \).
PROOF: From (10.3.1) and (10.3.2) it follows that $(FGH)^2 = I$
and by (10.3.2) and (10.3.3), we have

$$\|FGH Gx - FGH Gy\| \leq \alpha \|((GH)^2Gx - (GH)^2Gy) + \beta [\|Gx - Gy\| + \|Gy - (FGH)Gy\|]\$$

Now if we put $Gx = z$ and $Gy = w$, we get

$$\|FGHz - FGHw\| \leq \alpha \|z - w\| + \beta [\|z - FGHz\| + \|w - FGHw\|]\$$

where $(FGH)^2 = I$ and $\alpha + \beta < 2$. So by theorem of Isâki [69] FGH has at least one fixed point, say $x_0$ in $K$ i.e.

(10.3.4) $FGHx_0 = x_0$

and so

$$GH(FGH)x_0 = GHx_0$$

or

(10.3.5) $Fx_0 = GHx_0$

Also,

$$H(FGH)x_0 = Hx_0$$

or

(10.3.6) $FGx_0 = Hx_0$

Now using (10.3.1), (10.3.2), (10.3.3) and (10.3.4),

(10.3.5), (10.3.6), we have
\[
\| H x_0 - x_0 \| = \| F G x_0 - F^2 x_0 \| = \| F(G x_0) - F(F x_0) \|
\]
\[
\leq \alpha \| G H(G x_0) - G H(F x_0) \| + \beta \left[ \| G H(G x_0) - F(G x_0) \| + \| G H(F x_0) - F(F x_0) \| \right]
\]
\[
= \alpha \| H x_0 - x_0 \| + \beta \left[ \| H x_0 - H x_0 \| + \| x_0 - x_0 \| \right]
\]
\[
= \alpha \| H x_0 - x_0 \|. \]

Since \( \alpha < 1 \), it follows \( H x_0 = x_0 \) i.e. \( x_0 \) is the fixed point of \( H \). Thus we have from (10.3.5) that \( G x_0 = F x_0 \).

Again
\[
\| F x_0 - x_0 \| = \| F x_0 - F^2 x_0 \| = \| F x_0 - F(F x_0) \|
\]
\[
\leq \alpha \| G H(F x_0) - F(F x_0) \| + \beta \left[ \| G H x_0 - F x_0 \| + \| G H(F x_0) - F(F x_0) \| \right]
\]
\[
= \alpha \| F x_0 - x_0 \| + \beta \left[ \| F x_0 - F x_0 \| + \| x_0 - x_0 \| \right]
\]
\[
= \alpha \| F x_0 - x_0 \|.
\]

a contradiction, since \( \alpha < 1 \). Hence it follows that \( F x_0 = x_0 \) but \( F x_0 = G x_0 \), so we have \( F x_0 = G x_0 = H x_0 = x_0 \) i.e. \( x_0 \) is the common fixed point of \( F, G \) and \( H \).

In order to prove the uniqueness of \( x_0 \), let us consider \( y_0 \) be another common fixed point of \( F, G \) and \( H \).
Now on using (10.3.1), (10.3.2), (10.3.3) und (10.3.4) and (10.3.5)

\[ \| x_0 - y_0 \| = \| F^2 x_0 - F^2 y_0 \| = \| F(Fx_0) - F(Fy_0) \| \]

\[ \leq \alpha \| GH(Fx_0) - GH(Fy_0) \| + \beta \left( \| GH(Fx_0) - F(x_0) \| + \| GH(Fy_0) - F(y_0) \| \right) \]

\[ = \alpha \| x_0 - y_0 \| + \beta \| x_0 - x_0 \| + \| y_0 - y_0 \| \]

Since \( \alpha < 1 \), it follows \( x_0 = y_0 \) proving the uniqueness of \( x_0 \). This completes the proof of the theorem.

**REMARK.** (i) If we put \( H = I \) we get theorem B of Sharma and Bajaj [158].

(ii) If we put \( H = I, \beta = 0 \) we get theorem A of Khan [92].

(iii) If we put \( H = G = I \) in our theorem we get the theorem of Iseki [69].