Chapter 1

Introduction

The purpose of this thesis is to present some results on finite element Galerkin methods for linear elliptic, parabolic and hyperbolic interface problems.

1.1 Problem Description

Interface problems are often referred as differential equations with discontinuous coefficients. The discontinuity of the coefficients corresponds to the fact that the medium consists of two or more physically different materials. To begin with, we first introduce elliptic, parabolic and hyperbolic interface problems.

**Elliptic interface problems:** Let $\Omega$ be a convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial \Omega$. Further, let $\Omega_1 \subset \Omega$ be an open domain with $C^2$ smooth boundary $\Gamma$ and $\Omega_2 = \Omega \setminus \Omega_1$ (see, Figure 1.1). We now consider the following linear elliptic interface problems of the form

$$L u = f(x) \quad \text{in} \; \Omega \quad (1.1.1)$$

with Dirichlet boundary condition

$$u(x) = 0 \quad \text{on} \; \partial \Omega \quad (1.1.2)$$

and interface conditions

$$[u] = 0, \quad \left[ \beta \frac{\partial u}{\partial n} \right] = g(x) \quad \text{along} \; \Gamma. \quad (1.1.3)$$
The symbol \([v]\) is a jump of a quantity \(v\) across the interface \(\Gamma\), i.e., \([v](x) = v_1(x) - v_2(x), \ x \in \Gamma\), where \(v_i(x) = v(x) \mid_{\Omega_i}, \ i = 1, 2\) and \(\mathbf{n}\) denotes the unit outward normal to the boundary \(\partial \Omega_1\). Here, \(\mathcal{L}\) is a second order elliptic partial differential operator of the form
\[
\mathcal{L}v = -\nabla \cdot (\beta(x) \nabla v).
\]
We assume that the coefficient function \(\beta\) is positive and piecewise constant, i.e.,
\[
\beta(x) = \beta_i \text{ in } \Omega_i, \ i = 1, 2.
\]

**Parabolic interface problems**: We consider the following linear parabolic interface problems of the form
\[
u_t + \mathcal{L}u = f(x,t) \text{ in } \Omega \times (0,T] \tag{1.1.4}
\]
with initial and boundary conditions
\[
u(x,0) = u_0(x) \text{ in } \Omega; \ u(x,t) = 0 \text{ on } \partial \Omega \times (0,T] \tag{1.1.5}
\]
and interface conditions
\[
[u] = 0, \ \left[\beta \frac{\partial u}{\partial n}\right] = g(x,t) \text{ along } \Gamma, \tag{1.1.6}
\]
The domain \(\Omega\), operator \(\mathcal{L}\), symbols \([v]\) and \(\mathbf{n}\) are defined as before, and \(T < \infty\).
**Hyperbolic interface problems:** We shall also consider the following hyperbolic interface problems of the form

\[ u_{tt} + Lu = 0 \text{ in } \Omega \times (0, T) \]  

(1.1.7)

with initial and boundary conditions

\[ u(x, 0) = u_0(x) \& u_t(x, 0) = v_0(x) \text{ in } \Omega; \quad u(x, t) = 0 \text{ on } \partial \Omega \times (0, T) \]  

(1.1.8)

and interface conditions

\[ [u] = 0, \quad \left[ \beta \frac{\partial u}{\partial n} \right] = g(x, t) \text{ along } \Gamma, \]  

(1.1.9)

The domain \( \Omega \), operator \( L \), symbols \([v]\) and \( n \) are defined as before, and \( T < \infty \).

The equations of the form (1.1.1)-(1.1.3) are often encountered in the theory of magnetic field, heat conduction theory, the theory of elasticity and in reaction diffusion problems (see, [23, 29, 49]). Many interface problems in material science and fluid dynamics are modeled after above problem when two or more distinct materials or fluids with different conductivities or densities or diffusions are involved. For the literature relating to applications of elliptic differential equations with discontinuous coefficients, one may refer to Ewing [22], Nielsen [37] or Peaceman [38] for the model of the pressure equation arising in reservoir simulation, Reddy [41] for reactor dynamics, Z. Li et al. [33] for the model of the potential in the computation of micromagnetics for the ferromagnetic materials or electrostatics for macromolecules.

The equations of the form (1.1.4)-(1.1.6) involving discontinuous coefficients are sometimes called diffraction problems of parabolic types. This type of interface problem is critical in many applications of engineering and sciences, including non-stationary heat conduction problems, electromagnetic problems, shape optimization problems to name just a few. For a detailed discussion on parabolic problems with discontinuous coefficients, see Dautry and Lions [14], Gilberg and Trudinzer [25], Hackbush [27], Ladyzhenskaya et al. [30], Li and Ito [32] and Marti [36].

The model equations of the form (1.1.7)-(1.1.9) involving discontinuous coefficients are used in many applications such as ocean acoustics, elasticity, and seismology to model the propagation of small disturbances in fluids or solids. In electromagnetism, the equation (1.1.7) corresponds to a problem in which the material occupying the interior is a dielectric rather than a metal (cf. [2]). In the study of wave equations for
some physical problems, such as acoustic or elastic waves travelling through heterogeneous media, there can be discontinuities in the coefficients of the equation. As a model, consider the problem of transverse vibrations of an infinite string, with a discontinuity in density $\rho$ at a location $x = \alpha$. Let $\psi$ represent the non dimensionalized displacement normal to the string. Then we have the equation

$$\rho \psi_{tt} - (\tau_0 \psi_x)_x = 0$$

which is equivalent to the problem

$$\psi_{tt} - \beta(x) \psi_{xx} = 0$$

where

$$\beta(x) = \begin{cases} 
\beta_1 = \frac{\tau_0}{\rho_1} & \text{if } x < \alpha \\
\beta_2 = \frac{\tau_0}{\rho_2} & \text{if } x > \alpha
\end{cases}$$

along with the initial condition

$$\psi(x, 0) = f(x), \quad \psi_t(x, 0) = 0.$$

For this physical model, we have the following jump conditions at the interface $x = \alpha$

$$[\psi] = 0, \quad [\psi_x] = 0.$$

The interface conditions correspond to the facts that displacement and normal component of the tension in the deflected string are continuous. The one dimensional acoustic wave equation is often used as a model in seismology. For example, consider the one dimensional acoustic wave equation

$$\rho u_t + p_x = 0 \quad \& \quad p_t + ku_x = 0,$$

where $\rho$ is the density, $u$ is the velocity, $p$ is the pressure and $k$ is compression(bulk) modulus. At $x = \alpha$, the coefficients are given as

$$(\rho, k) = \begin{cases} 
(\rho^-, k^-) & \text{if } x < \alpha \\
(\rho^+, k^+) & \text{if } x > \alpha
\end{cases}$$

The velocity and pressure must be continuous across the interface, and therefore the jump conditions at the interface are

$$[u] = 0, \quad [p] = 0.$$
The above problem can also be rewritten as hyperbolic problems
\[ \rho u_{tt} - ku_{xx} = 0, \quad p_{tt} - \frac{k}{\rho} p_{xx} = 0. \]
with discontinuous coefficients.

1.2 Notation and Preliminaries

In this section, we shall introduce some standard notation and preliminaries to be used throughout this work.

All functions considered here are real valued. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^d \), \( d \)-dimensional Euclidian space and \( \partial \Omega \) denote the boundary of \( \Omega \). Let \( x = (x_1, x_2, \ldots, x_d) \in \Omega \), and let \( dx = dx_1, \ldots, dx_d \). Further, let \( \alpha = (\alpha_1, \ldots, \alpha_d) \) be a \( d \)-tuple with nonnegative integer components and denote order of \( \alpha \) as \( |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_d \). Then, by \( D^\alpha \phi \), we shall mean the \( \alpha \)th derivative of \( \phi \) defined by
\[ D^\alpha \phi = \frac{\partial^{|\alpha|}\phi}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}. \]

We shall make frequent reference to the following well-known function spaces. For \( 1 \leq p < \infty \), \( L^p(\Omega) \) denotes the linear space of equivalence classes of measurable functions \( \phi \) in \( \Omega \) such that \( \int_\Omega |\phi(x)|^p dx \) exists and is finite. The norm on \( L^p(\Omega) \) is given by
\[ \|u\|_{L^p(\Omega)} = \left( \int_\Omega |\phi(x)|^p dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty. \]
For \( p = \infty \), \( L^\infty(\Omega) \) denotes the space of functions \( \phi \) on \( \Omega \) such that
\[ \|\phi\|_{L^\infty(\Omega)} = \text{ess sup}_{x \in \Omega} |\phi(x)| < \infty. \]
When \( p = 2 \), \( L^2(\Omega) \) is a Hilbert space with respect to the inner product
\[ (\phi, \psi) = \int_\Omega \phi(x)\psi(x) dx. \]
By support of a function \( \phi \), \( \text{supp} \phi \), we mean the closure of all points \( x \) with \( \phi(x) \neq 0 \), i.e.,
\[ \text{supp} \phi = \{x : \phi(x) \neq 0\}. \]
For any nonnegative integer $m$, $C^m(\Omega)$ denotes the space of functions with continuous derivatives upto and including order $m$ in $\Omega$. $C^m_0(\Omega)$ is the space of all $C^m(\Omega)$ functions with compact support in $\Omega$. Also, $C^\infty_0(\Omega)$ is the space of all infinitely differential functions with compact support in $\Omega$.

We now introduce the notion of Sobolev spaces. Let $m \geq 0$ and real $p$ with $1 \leq p < \infty$. The Sobolev space of order $(m, p)$ on $\Omega$, denoted by $W^{m, p}(\Omega)$, is defined as a linear space of functions (or equivalence class of functions) in $L^p(\Omega)$ whose distributional derivatives upto order $m$ are also in $L^p(\Omega)$, i.e.,

$$W^{m, p}(\Omega) = \{ \phi : D^\alpha \phi \in L^p(\Omega) \text{ for } 0 \leq |\alpha| \leq m \}.$$ 

The space $W^{m, p}(\Omega)$ is endowed with the norm

$$\| \phi \|_{m, p} = \left( \int_{\Omega} \sum_{0 \leq |\alpha| \leq m} |D^\alpha \phi(x)|^p \, dx \right)^{\frac{1}{p}} = \left( \sum_{0 \leq |\alpha| \leq m} \|D^\alpha \phi\|^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$ 

When $p = \infty$, the norm on the space $W^{m, \infty}(\Omega)$ is defined by

$$\| \phi \|_{m, \infty} = \max_{0 \leq |\alpha| \leq m} \|D^\alpha \phi(x)\|_{L^\infty(\Omega)}.$$ 

For $p=2$, these spaces will be denoted by $H^m(\Omega)$. The space $H^m(\Omega)$ is a Hilbert space with natural inner product defined by

$$(\phi, \psi) = \sum_{0 \leq |\alpha| \leq m} \int_{\Omega} D^\alpha \phi D^\alpha \psi \, dx, \quad \phi, \psi \in H^m(\Omega).$$ 

The sobolev space $H^m(\Omega)$ (respectively, $H^m_0(\Omega)$) is also defined as the closure of $C^m(\Omega)$ (respectively, $C^\infty_0(\Omega)$) with respect to the norm $\| \phi \|_m = \| \phi \|_{m, 2}$. This result is true under some smoothness assumption on the boundary $\partial \Omega$. Clearly, $L^2(\Omega) = H^0(\Omega)$ and $H^m(\Omega) = W^{m, 2}(\Omega)$. We also need the fractional space $H^{\frac{1}{2}}(\Omega)$ equipped with the norm

$$\| \psi \|_{H^{\frac{1}{2}}(\Omega)} = \inf_{w \in H^1(\Omega)} \{ \|w\|_{H^1(\Omega)} : \gamma_0 w = \psi \},$$

where $\gamma_0$ is a trace operator. For a more complete discussion on Sobolev spaces, see Adams [1].
We shall also use the following spaces in our error analysis. For a given Banach space \( \mathcal{B} \), we define, for \( m = 0,1 \) and \( 1 \leq p < \infty \)

\[
W^{m,p}(0,T;\mathcal{B}) = \left\{ u(t) \in \mathcal{B} \text{ for a.e. } t \in (0,T) \text{ and } \sum_{j=0}^{m} \int_{0}^{T} \left\| \frac{\partial^{j} u(t)}{\partial t^{j}} \right\|_{\mathcal{B}}^{p} dt < \infty \right\}
\]

equipped with the norm

\[
\|u\|_{W^{m,p}(0,T;\mathcal{B})} = \left( \sum_{j=0}^{m} \int_{0}^{T} \left\| \frac{\partial^{j} u(t)}{\partial t^{j}} \right\|_{\mathcal{B}}^{p} dt \right)^{\frac{1}{p}}.
\]

We write \( H^{m}(0,T;\mathcal{B}) = W^{m,2}(0,T;\mathcal{B}) \) and \( L^{2}(0,T;\mathcal{B}) = H^{0}(0,T;\mathcal{B}) \). When no risk of confusion exists we shall write \( L^{2}(\mathcal{B}) \) for \( L^{2}(0,T;\mathcal{B}) \).

Further, we denote \( L^{\infty}(0,T;\mathcal{B}) \) to be the collection of all functions \( v \in \mathcal{B} \) such that

\[
\text{ess sup}_{t \in (0,T]} \|v(x,t)\|_{\mathcal{B}} < \infty.
\]

Below, we shall discuss some preliminary materials which will be of frequent use in error analysis in the subsequent chapters. The bilinear form \( A(\cdot,\cdot) \) associated with the operator \( \mathcal{L} \), given by

\[
A(u,v) = \int_{\Omega} \beta(x) \nabla u \cdot \nabla v dx,
\]

satisfies the following boundedness and coercive properties: For \( \phi,\psi \in H^{1}(\Omega) \), there exists positive constants \( C \) and \( c \) such that

\[
A(\phi,\psi) \leq C \|\phi\|_{H^{1}(\Omega)} \|\psi\|_{H^{1}(\Omega)}
\]

and

\[
A(\phi,\phi) \geq c \|\phi\|_{H^{1}(\Omega)}^{2}.
\]

From time to time we shall also use the following inequalities (see, Hardy et al. [28]):

(i) Young’s inequality: For \( a, b \geq 0 \) and \( \epsilon > 0 \), the following inequality

\[
ab \leq \frac{a^{2}}{2\epsilon} + \frac{\epsilon b^{2}}{2}
\]

holds.
(ii) Cauchy-Schwarz inequality: For \(a, b \geq 0\), \(1 < p < \infty\) and \(\frac{1}{p} + \frac{1}{q} = 1\),

\[
ab \leq \frac{a^p}{p} + \frac{b^q}{q}.
\]

In integral form, if \(\phi\) and \(\psi\) are both real valued and \(\phi \in L^p\) and \(\psi \in L^q\), then

\[
\int_{\Omega} \phi \psi \leq \|\phi\|_p \|\psi\|_q.
\]

For \(p = q = 2\), the above inequality is known as Schwarz’s inequality. The discrete version of Schwarz’s inequality may be stated as:

(iii) Let \(\phi_j, \psi_j, j = 1, 2, \ldots, n\) be positive real numbers. Then

\[
\sum_{j=1}^{n} \phi_j \psi_j \leq \left( \sum_{j=1}^{n} \phi_j^2 \right)^{\frac{1}{2}} \left( \sum_{j=1}^{n} \psi_j^2 \right)^{\frac{1}{2}}
\]

Below, we state without proof, the following two versions of Gronwall’s lemma. For a proof, see [40].

**Lemma 1.2.1 (Continuous Gronwall’s Lemma)** Let \(G(t)\) be a continuous function and \(H(t)\) a nonnegative continuous function on its interval \(t_0 \leq t \leq t_0 + a\). If a continuous function \(F(t)\) has the property

\[
F(t) \leq G(t) + \int_{t_0}^{t} F(s)H(s)ds \quad \text{for } t \in [t_0, t_0 + a],
\]

then

\[
F(t) \leq G(t) + \int_{t_0}^{t} G(s)H(s) \exp \left[ \int_{s}^{t} H(\tau)d\tau \right] ds \quad \text{for } t \in [t_0, t_0 + a].
\]

In particular, when \(G(t) = C\) a nonnegative constant, we have

\[
F(t) \leq C \exp \left[ \int_{t_0}^{t} H(s)ds \right] \quad \text{for } t \in [t_0, t_0 + a].
\]

**Lemma 1.2.2 (Discrete Gronwall’s Lemma)** If \(\langle y_n \rangle\), \(\langle f_n \rangle\) and \(\langle g_n \rangle\) are non-negative sequences and

\[
y_n \leq f_n + \sum_{0 \leq k < n} g_k y_k, \quad n \geq 0,
\]

then

\[
y_n \leq f_n + \sum_{0 \leq k < n} g_k f_k \exp \left( \sum_{k < j < n} g_j \right), \quad n \geq 0.
\]
In addition, we shall also work on the following spaces:

\[ X = H^1(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2) \quad \& \quad Y = L^2(\Omega) \cap H^1(\Omega_1) \cap H^1(\Omega_2) \]

equipped with the norms

\[ \|v\|_X = \|v\|_{H^1(\Omega)} + \sum_{i=1}^{2} \|v\|_{H^2(\Omega_i)} \quad \& \quad \|v\|_Y = \|v\|_{L^2(\Omega)} + \sum_{i=1}^{2} \|v\|_{H^1(\Omega_i)}, \]

respectively.

We now turn to the literature concerning the regularity of elliptic, parabolic and hyperbolic problems with discontinuous coefficients. Due to the presence of discontinuous coefficients the solution \( u \), in general, does not belong to \( H^2(\Omega) \) even if the coefficients are sufficiently smooth in each individual subdomain \( \Omega_i \), \( i = 1, 2 \). Concerning the elliptic interface problems, we have the following regularity result. For a proof, see Chen and Zou [11], and Ladyzhenskaya et al. [30].

**Theorem 1.2.1** Let \( f \in L^2(\Omega) \) and \( g \in H^{1/2}(\Gamma) \). Then the problem (1.1.1)-(1.1.3) has a unique solution \( u \in X \cap H^1_0(\Omega) \) and \( u \) satisfies a priori estimate

\[ \|u\|_X \leq C \left( \|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\Gamma)} \right). \]

Regarding the parabolic interface problems (1.1.4)-(1.1.6), we have the following regularity result (cf. [11, 30]).

**Theorem 1.2.2** Let \( f \in H^1(0,T;L^2(\Omega)) \), \( g \in H^1(0,T;H^{1/2}(\Gamma)) \) and \( u_0 \in H^1(\Omega) \). Then the problem (1.1.4)-(1.1.6) has a unique solution \( u \in L^2(0,T;X) \cap H^1(0,T,Y) \cap H^1_0(\Omega) \).

We now recall the following regularity result for the solution \( u \) of the interface problem (1.1.7)-(1.1.9) (cf. [13, 30]).

**Theorem 1.2.3** Let \( u_0, v_0 \in H^1_0(\Omega) \). Then the problem (1.1.7)-(1.1.9) has a unique solution \( u \in L^2(0,T;X \cap H^1_0(\Omega)) \cap H^1(0,T;H^2(\Omega_1) \cap H^2(\Omega_2)) \cap H^2(0,T;Y) \).

### 1.3 A Brief Survey on Numerical Methods

In this section, we shall discuss a brief survey of the relevant literature concerning of elliptic, parabolic and hyperbolic interface problems.
Solving differential equations with discontinuous coefficients by means of classical finite element methods usually leads to the loss in accuracy (cf. [5, 11]). One major difficulty is that the solution has low global regularity and the elements do not fit with the interface of general shape. For non-interface problems, one can assume full regularities of the solutions (at least $H^2(\Omega)$) on whole physical domain. But for the interface problems, the global regularity of the solution is low. So the classical analysis is difficult to apply for the convergence analysis of the interface problems. Thus the numerical solution to the interface problem is challenging as well as interesting also.

The standard finite difference and finite element methods may not be successful in giving satisfactory numerical results for such problems. Hence, many new methods have been developed. Some of them are developed with the modifications in the standard methods, so that they can deal with the discontinuities and the singularities. For the literature on the recent developments of the numerical methods for such problems, we refer to [15, 35] which includes extensive list of relevant literature. The numerical solutions of interface problems by means of finite element Galerkin procedures have been investigated by several authors. One of the first finite element methods treating interface problem has been studied by Babuška in [5]. In [5], the author has formulated the problem as an equivalent minimization problem and then finite element methods are used to solve the minimization problem. Under some approximation assumptions on finite element spaces, Babuška has obtained sub-optimal order error estimate in $H^1$ norm. The algorithm in [5] requires the exact evaluation of line integrals on the boundary of the domain and on the interface, and exact integrals on the interface finite elements are also needed. In the absence of variational crimes, finite element approximation of interface problem has been studied by Barrett and Elliott in [6]. They have shown that the finite element solution converges to the true solution at optimal rate in $L^2$ and $H^1$ norms over any interior subdomain. In [6], it is assumed that the solution and the normal derivatives of the solution are continuous along the interface, and fourth order differentiable on each subdomain. For the problems (1.1.1)-(1.1.3), Bramble and King [8] have considered a finite element method in which the domains $\Omega_1$ and $\Omega_2$ are replaced by polygonal domains $\Omega_{1,h}$ and $\Omega_{2,h}$, respectively. Then, the Dirichlet data and the interface function are transferred to the polygonal boundaries. Finally, discontinuous Galerkin finite element method is applied to the perturbed problem defined on the polygonal domains.
Optimal order error estimates are derived for rough as well as smooth boundary data. Under practical regularity assumptions on the true solution, the convergence of conforming finite element method is studied in [11], [37] and [43]. In [11], Chen and Zou have considered a practical piecewise linear finite element approximation for solving second order elliptic interface problem with \( \mathcal{L}u = -\nabla.(\beta \nabla u) \) in a polygonal domain, where the coefficient \( \beta \) is assumed to be positive and piecewise constant in each subdomains. They have proved almost optimal order of convergence in \( L^2 \) and energy norms. More precisely, the error bounds obtained by Chen and Zou [11] are optimal up to the factor \( \log h \). Under the assumptions on the source term \( f|_{\Omega_1} = 0 \) and the interface function \( g = 0 \), Neilsen [37] has proved optimal order of convergence in \( H^1 \) norm in the presence of arbitrarily small ellipticity. The algorithm in [37] requires that the interface triangles follow exactly the actual interface \( \Gamma \). In [43], the finite element solution converges to the exact solution at an optimal rate in \( L^2 \) and \( H^1 \) norms if the grid lines coincide with the actual interface by allowing interface triangles to be curved triangles. Further, if the grid lines form an approximation to the actual interface, optimal order of convergence in \( H^1 \) norm and sub-optimal order in \( L^2 \) norm are derived for elliptic problems. More recently, in [16], the author has discussed quadrature finite element method for elliptic interface problems in a two dimensional convex polygonal domain. Optimal order error estimates in \( L^2 \) and \( H^1 \) norms are derived for a practical finite element discretization with straight interface triangles.

We now turn to the finite element Galerkin approximation to parabolic interface problems (1.1.4)-(1.1.6). Although a good number of articles is devoted to the finite element approximation of elliptic interface problems, the literature seems to lack concerning the convergence of finite element solutions to the true solutions of parabolic interface problems (1.1.4)-(1.1.6). For the backward Euler time discretization, Chen and Zou [11] have studied the convergence of fully discrete solution to the exact solution using fitted finite element methods. They have proved almost optimal error estimates in \( L^2(L^2) \) and \( L^2(H^1) \) norms when global regularity of the solution is low. Then an essential improvement was made in [21]. The authors of [21] have used a finite element discretization where interface triangles are assumed to be curved triangles instead of straight triangles like classical finite element methods. Optimal order error estimates in \( L^2(L^2) \) and \( L^2(H^1) \) norms are shown to hold for both semi discrete and fully dis-
crete scheme in [21]. More recently, for similar triangulation, Deka and Sinha ([19]) have studied the pointwise-in-time convergence in finite element method for parabolic interface problems. They have shown optimal error estimates in $L^\infty(H^1)$ and $L^\infty(L^2)$ norms under the assumption that grid line exactly follow the actual interface. Similar results are also obtained by Attanayake and Senaratne in [4] for immersed finite element method.

Finally, we turn to the numerical methods for hyperbolic interface problems (1.1.7)-(1.1.9). Numerical solutions of hyperbolic equations with discontinuous coefficients draws significant attention in a variety of fields such as the oil exploration industry and mineral finding as well as the study of earthquakes. Numerical simulation of seismic wave propagation problems in heterogeneous media can be traced back to as early as Alterman and Karal([3]) in 1968 and Boore([7]) in 1972. Alterman and Karal developed a finite difference scheme to solve the equations of elasticity in one spatial dimension and they applied their scheme to the problem of a layered half space with a buried point source emitting a compressional pulse. The interface between different layers was placed at $z = h$ on the grid line, where $z$ is the coordinate representing the depth below the surface of the Earth. A general introduction on the numerical treatment for hyperbolic interface problems by means of finite difference method can be found in Le Veque’s Book [31]. Three numerical schemes namely Wendroff, TVD and WENO have been discussed in [31]. These schemes use values of the sound speed on discrete points or averaged values on grid cells. As a consequence, they do not describe accurately the position and the shape of interfaces cutting grid cells. Furthermore, due to low regularity of the true solution the method leads to the loss in accuracy near the interface. It is then a new approach called explicit jump immersed interface method was introduced in [48]. These numerical methods ensure a given accuracy at grid points near interface, but they are difficult to implement with higher order schemes. To overcome this difficulty an explicit simplified interface method was introduced by Piraux et al. in [39] for one dimensional acoustic velocity and acoustic pressure.
1.4 Objectives

This section elucidates our contributions and motivation for the present study. The physical world is replete with examples of free surfaces, material interface and moving boundaries that interact with a surrounding fluid. There are interfaces that separate air and water (in the case of bubbles or free surface flows) and boundaries between two materials of different physical properties (in porous media flow or mixing layers). While the mathematical modelling of the interaction is a difficult problem in itself, another formidable task is developing a numerical method that solves these problems effectively and efficiently.

The analysis of finite element methods for interface problem has become an active research area over the years. The main objective of this work is to establish some new optimal a priori error estimates in fitted finite element method for interface problem with straight interface triangles. The achieved estimates are analogous to the case with a regular solution, however, due to low regularity, the proof requires a careful technical work coupled with a approximation result for the linear interpolant. Other technical tools used in this work are Sobolev embedding inequality, approximations properties for modified elliptic projection, modified duality arguments and some known results on elliptic interface problems.

In the present work, optimal order error estimates in $L^2$ and $H^1$ norms are derived for the linear elliptic interface problems (c.f. [17]) and which improve the earlier results in the articles [11] and [43]. Then the results are extended for parabolic interface problems and optimal order error estimates in $L^2(L^2)$ and $L^2(H^1)$ norms are achieved (c.f. [18]).

Due to low global regularity of the solutions, the error analysis of the standard finite element method for parabolic problems is difficult to adopt for parabolic interface problems. In this work, we are able to fill a theoretical gap between standard energy technique of finite element method for non interface problems and parabolic interface problems. Optimal $L^\infty(H^1)$ and $L^\infty(L^2)$ norms error estimates have been derived under practical regularity assumption of the true solution (c.f. [20]).

Although various FEM for elliptic and parabolic interface problems have been proposed and studied in the literature, but FEM treatment of similar hyperbolic problems is mostly missing. In this work, we are able to prove optimal order pointwise-in-time error estimates in $L^2$ and $H^1$ norms for the hyperbolic interface problem with semidis-
crete scheme. Fully discrete scheme based on a symmetric difference approximation is also analyzed and optimal $L^\infty(H^1)$ norm error is obtained.

1.5 Organization of the Thesis

The organization of the thesis is as follows: Chapter 2 deals with the error analysis for elliptic interface problems in two dimensional convex polygonal domains. Optimal order error estimates in $L^2$ and $H^1$ norms are derived for a practical finite element discretization.

Chapter 3 is devoted to the convergence of finite element method for parabolic interface problems with straight interface triangles. The proposed method yields optimal order error estimates in $L^2(L^2)$ and $L^2(H^1)$ norms for semi-discrete scheme. Convergence of fully discrete solution is also discussed and optimal error estimate in $L^2(H^1)$ norm is achieved.

In Chapter 4, we analyze the continuous time Galerkin method for spatially discrete scheme for parabolic interface problems. Optimal $L^\infty(H^1)$ and $L^\infty(L^2)$ norms error estimates have been derived under practical regularity assumption of the true solution. Further, the fully discrete scheme based on backward Euler method is also proposed and analyzed. Optimal $L^2$ norm error estimate is obtained for fully discrete scheme.

Chapter 5 is concerned with a priori error estimates for hyperbolic interface problems. Optimal error estimates in $L^\infty(L^2)$ and $L^\infty(H^1)$ norms are established for continuous time discretization. Further, the fully discrete scheme based on a symmetric difference approximation is considered and optimal order convergence in $H^1$ norm is established.

Finally, numerical results are presented for two dimensional test problems in Chapter 6 for the completeness of this work.

For clarity of presentation we have repeatedly given equations (1.1.1) – (1.1.3) or (1.1.4) – (1.1.6) or (1.1.7) – (1.1.9) at the beginning of subsequent chapters.