Chapter 4

$L^\infty(L^2)$ and $L^\infty(H^1)$ norms Error Estimates for Parabolic Interface Problems

The purpose of this chapter is to establish some new a priori error estimates in finite element method for parabolic interface problems. Optimal $L^\infty(H^1)$ and $L^\infty(L^2)$ norms error estimates have been derived under practical regularity assumption of the true solution for fitted finite element method with straight interface triangles.

4.1 Introduction

In $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$, we shall again recall the following parabolic interface problem

$$u_t + Lu = f(x,t) \quad \text{in } \Omega \times (0,T)$$

with initial and boundary conditions

$$u(x,0) = u_0 \quad \text{in } \Omega; \quad u(x,t) = 0 \quad \text{on } \partial \Omega \times (0,T)$$

and jump conditions on the interface

$$[u] = 0, \quad \left[ \beta \frac{\partial u}{\partial n} \right] = 0 \quad \text{along } \Gamma,$$

where, $f = f(x,t)$ is real valued functions in $\Omega \times (0,T)$, and $u_t = \frac{\partial u}{\partial t}$. Further, $u_0 = u_0(x)$ is real valued function in $\Omega$. The operator $L$, symbols $[v]$ and $n$ are defined as in Chapter 1, and $T < \infty$. 
Due to low global regularity of the solutions, it is difficult to achieve optimal $L^\infty(L^2)$ and $L^\infty(H^1)$ error estimates for parabolic interface problems. More recently, Deka and Sinha ([19]) have studied the pointwise-in-time convergence in finite element method for parabolic interface problems. They have shown optimal error estimates in $L^\infty(H^1)$ and $L^\infty(L^2)$ norms under the assumption that grid line exactly follow the actual interface. This may causes some technical difficulties in practice for the evaluation of the integrals over those curved elements near the interface. Therefore, in present work an attempt has been made to extend the results obtained in [19] for a more practical finite element discretization discussed in [11]. In this chapter, we are able to show that the standard energy technique of finite element method can be extended to parabolic interface problems under the assumptions that solution as well as its normal derivative along interface are continuous. Optimal order pointwise-in-time error estimates in the $L^2$ and $H^1$ norms are established for the semidiscrete scheme. In addition, a fully discrete method based on backward Euler time-stepping scheme is analyzed and related optimal pointwise-in-time error bounds are derived. To the best of our knowledge, optimal pointwise in time error estimates for a finite element discretization based on [11] have not been established earlier for the parabolic interface problem.

A brief outline of this chapter is as follows. In section 4.2, we introduce some standard notations, recall some basic results from the literature and obtain the a priori estimate for the solution. In section 4.3, we describe a finite element discretization for the problem (4.1.1)-(4.1.3) and prove some approximation properties related to the auxiliary projection used in our analysis. While Section 4.4 is devoted to the error analysis for the semidiscrete finite element approximation, error estimates for the fully discrete backward Euler time stepping scheme are derived in section 4.5.

4.2 Preliminaries

The purpose of this chapter is to introduce some new a priori estimates for the solutions of parabolic interface problems.

In order to introduce the weak formulation of the problem, we now define the local bilinear form $A^l(.,.) : H^1(\Omega_l) \times H^1(\Omega_l) \to \mathbb{R}$ by

$$ A^l(w, v) = \int_{\Omega_l} \beta_i \nabla w \cdot \nabla v dx, \quad l = 1, 2. $$
Then the global bilinear form $A(\cdot,\cdot) : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ is defined by

$$A(w,v) = \int_{\Omega} \beta(x) \nabla w \cdot \nabla v dx = A_1(w,v) + A_2(w,v) \quad \forall \ w, v \in H^1_0(\Omega).$$

Then weak form for the problem (4.1.1)-(4.1.3) is defined as follows: Find $u : (0,T] \to H^1_0(\Omega)$ such that

$$ (u_t, v) + A(u, v) = (f, v) \quad \forall \ v \in H^1_0(\Omega), \ a.e. \ t \in (0,T] \quad (4.2.1)$$

with $u(x,0) = u_0(x)$. 

**Remark 4.2.1** Let $f(x,0) = f_0(x)$. Then it is clear from (1.1.1) that $u_t(0) \in H^2(\Omega)$ provided $u_0 \in H^1_0(\Omega) \cap H^4(\Omega)$ and $f_0 \in H^2(\Omega)$. From therein, we assume that $u_0 \in H^1_0(\Omega) \cap H^4(\Omega)$, $f \in H^1(0,T; L^2(\Omega))$ and $f_0 \in H^2(\Omega)$.

Under the assumption $f \in H^1(0,T; L^2(\Omega))$, we have

$$u_{ttt} - \nabla \cdot (\beta(x) \nabla u_t) = f_t \quad \text{in } \Omega_i, \ i = 1, 2. \quad (4.2.2)$$

Further $u_t$, satisfies the following initial and boundary condition

$$u_t(x,0) = u_t(0) \quad \text{and} \quad u_t(x,t) = 0 \quad \text{on } \partial \Omega \times (0,T] \quad (4.2.3)$$

along with the jump conditions

$$[u_t] = 0 \quad \text{and} \quad \left[ \beta \frac{\partial u_t}{\partial n} \right] = 0 \quad \text{along } \Gamma. \quad (4.2.4)$$

Thus $v = u_t \in \Omega_i, \ i = 1, 2$ satisfies a parabolic interface problem (4.2.2)-(4.2.4). Therefore, for $f_t \in H^1(0,T; L^2(\Omega))$ and $u_t(0) \in H^2(\Omega)$, apply Theorem 3.2.1 to have the following result.

**Lemma 4.2.1** Let $f \in H^2(0,T; L^2(\Omega))$, $f_0 \in H^2(\Omega)$ and $u_0 \in H^1_0(\Omega) \cap H^4(\Omega)$. Then the problem (1.1.1)-(1.1.3) has a unique solution $u \in H^1(0,T; H^2(\Omega_1) \cap H^2(\Omega_2)) \cap H^1_0(\Omega) \cap H^2(0,T; L^2(\Omega_1) \cap H^2(\Omega_2))$. Further, $u_t$ satisfies the following a priori estimate

$$\|u_t\|_{H^2(\Omega_1)} + \|u_t\|_{H^2(\Omega_2)} \leq C\{\|f_t\|_{L^2(\Omega)} + \|u_t\|_{L^2(\Omega)}\}.$$
Proof. The proof of the existence of unique solution \( u \in H^1(0,T; H^2(\Omega_1) \cap H^2(\Omega_2)) \cap H^1_0(\Omega) \cap H^2(0,T; L^2(\Omega)) \) follows from the assumptions and Theorem 3.2.1.

Next, to obtain the a priori estimate we consider the following elliptic interface problem: For a.e \( t \in (0,T] \), find \( w = w(x,t) \in H^1_0(\Omega) \cap X \) satisfying

\[
-\nabla \cdot (\beta(x) \nabla w(x,t)) = f_t(x,t) - u_{tt}(x,t) \quad \text{in } \Omega,
\]

\[
w = 0 \quad \text{on } \partial \Omega,
\]

\[
[w] = 0, \quad \left[ \beta \frac{\partial w}{\partial n} \right] = 0 \quad \text{along } \Gamma.
\]

From the elliptic regularity estimate for elliptic interface problem (cf. [11]), it follows that

\[
\|w\|_{H^2(\Omega_1)} + \|w\|_{H^2(\Omega_2)} \leq C \{ \| f_t \|_{L^2(\Omega)} + \| u_{tt} \|_{L^2(\Omega)} \}. \quad (4.2.6)
\]

Now, multiplying (4.2.5) by \( \phi \in L^2(\Omega) \cap H^1(\Omega_1) \cap H^1(\Omega_2) \cap \{ \psi \in L^2(\Omega) : \psi = 0 \text{ on } \partial \Omega, \ [w] = 0 \text{ on } \Gamma \} \) and then integrating over \( \Omega_1 \) and \( \Omega_2 \), we get

\[
A^1(w,\phi) + A^2(w,\phi) = (f_t - u_{tt}, \phi). \quad (4.2.7)
\]

Similarly from (4.2.2), we get

\[
A^1(u_t,\phi) + A^2(u_t,\phi) = (f_t - u_{tt}, \phi). \quad (4.2.8)
\]

Thus, for all such \( \phi \), we have

\[
A^1(w - u_t,\phi) + A^2(w - u_t,\phi) = 0.
\]

Again, \( u_t \in L^2(\Omega) \cap H^1(\Omega_1) \cap H^1(\Omega_2) \cap \{ \psi \in L^2(\Omega) : \psi = 0 \text{ on } \partial \Omega, \ [w] = 0 \text{ on } \Gamma \} \). Finally, setting \( \phi = w - u_t \) in the above equation and using the coercivity of each local bilinear map, we have \( w = u_t \) in \( \Omega_i, \ i = 1, 2 \). Then the desire estimate follows from (4.2.6). \( \square \)

4.3 Some Auxiliary Projections

In this chapter, we introduce linear interpolant and some auxiliary projections. Further, the convergence of such operators are obtained under global minimum regularity assumption of the true solutions.
Since the global regularity of the true solution is low, it is not favorable to work on $H^1(\Omega)$ in estimating pointwise-in-time error estimates. Therefore, we introduce $X^*$ be the collection of all $v \in L^2(\Omega)$ with the property that $v \in H^2(\Omega_1) \cap H^2(\Omega_2) \cap \{\psi : \psi = 0$ on $\partial\Omega\}$ and $[v] = 0$ along $\Gamma$. Let $\Pi_h$ be the Lagrange’s interpolation operator defined in Chapter 2. Then, for $K \in T_h$ and $v \in X^*$, we now define

$$v_I = \begin{cases} 
\Pi_h\tilde{v}_1 & \text{if } K \subseteq \Omega^h_1 \\
\Pi_h\tilde{v}_2 & \text{if } K \subseteq \Omega^h_2.
\end{cases} \tag{4.3.1}$$

For a finite dimensional space $V_h \subset H^1_0(\Omega)$ discussed in Chapter 2, it is easy to verify that $v_I \in V_h$.

Following the lines of proof for Lemma 2.2.3, it is possible to obtain the following optimal error bounds for linear interpolant $v_I$ in $X^*$. We include the proof for the completeness of this work.

**Lemma 4.3.1** For any $v \in X^*$, we have

$$\|v - v_I\|_{H^1(\Omega_1)} + \|v - v_I\|_{H^1(\Omega_2)} \leq Ch(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}).$$

**Proof.** For $H^1$ norm estimate, we have

$$\begin{align*}
\|v - v_I\|_{H^1(\Omega_1)} + \|v - v_I\|_{H^1(\Omega_2)} &\leq \sum_{K \in T_h \setminus T^*_i} \|v - v_I\|_{H^1(K)} + \sum_{K \in T^*_i} \{\|v - v_I\|_{H^1(K_1)} + \|v - v_I\|_{H^1(K_2)}\} \\
&\leq Ch\{\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}\} \\
&\quad + \sum_{K \in T^*_i} \{\|v - v_I\|_{H^1(K_1)} + \|v - v_I\|_{H^1(K_2)}\}. \tag{4.3.2}
\end{align*}$$

Here, $K_1 = K \cap \Omega_1$ and $K_2 = K \cap \Omega_2$. Again, for any $K \in T_h$ either $K \subseteq \Omega^h_1$ or $K \subseteq \Omega^h_2$. Let $K \subseteq \Omega^h_1$, then $v_I = \Pi_h\tilde{v}_1$ and hence, we have

$$\|v - v_I\|_{H^1(K_1)} = \|\tilde{v}_1 - \Pi_h\tilde{v}_1\|_{H^1(K_1)} \leq \|\tilde{v}_1 - \Pi_h\tilde{v}_1\|_{H^1(K)} \leq Ch\|\tilde{v}_1\|_{H^2(K)} \leq Ch\|v_1\|_{H^2(\Omega_1)}. \tag{4.3.3}$$
Again, since \( v \in H^2(\Omega_2) \) and \( K_2 \subseteq \Omega_2 \) with \( \text{meas}(K_2) \leq C h^3 \), we have
\[
\|v - v_I\|_{H^1(K_2)} \leq C h^{3(p-2)/2} \|v - v_I\|_{W^{1,p}(K_2)} \quad \forall p > 2
\]
\[
= C h \|v - v_I\|_{W^{1,6}(K_2)} = C h \|v_2 - \Pi_h \tilde{v}_1\|_{W^{1,6}(K_2)}
\]
\[
\leq C h \|\tilde{v}_2 - \tilde{v}_1\|_{W^{1,6}(K_2)} + C h \|\tilde{v}_1 - \Pi_h \tilde{v}_1\|_{W^{1,6}(K_2)}
\]
\[
\leq C h \|\tilde{v}_2 - \tilde{v}_1\|_{W^{1,6}(K)} + C h \|\tilde{v}_1\|_{H^2(K)}
\]
\[
\leq C h \|\tilde{v}_1\|_{H^2(\Omega)} + C h \|\tilde{v}_2\|_{H^2(\Omega)}
\]
\[
\leq C h (\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}).
\] (4.3.4)

Then Lemma 4.3.1 follows immediately from the estimates (4.3.2)-(4.3.4). □

In the error analysis of parabolic problems the term \( \rho = u - P_h u \) and \( \rho_t = u_t - P_h u_t \) plays very crucial role, where \( P_h \) is the standard elliptic projection (c.f. [47]). But in our present case solution \( u \in H^1(\Omega) \) and \( u_t \in L^2(\Omega) \), and therefore the standard elliptic projection \( P_h \) at \( u_t \) is not defined in usual manner. Therefore a modification in the definition of elliptic projection has been proposed and analyzed in this work. For any \( v \in X^* \) with \( [\beta \partial v / \partial \mathbf{n}] = 0 \) along \( \Gamma \), we define
\[
f^* = \begin{cases} 
- \nabla \cdot (\beta_1 \nabla v) & \text{in } \Omega_1 \\
- \nabla \cdot (\beta_2 \nabla v) & \text{in } \Omega_2.
\end{cases}
\]

Clearly \( f^* \in L^2(\Omega) \). We denote \( X^{**} \) to be the collection of all such \( v \in X^* \). Then define \( R_h : X^* \rightarrow V_h \) by
\[
A_h(R_h v, v_h) = (f^*, v_h) \quad \forall v_h \in V_h.
\] (4.3.5)

The existence and uniqueness of such \( R_h v \) can be verified by setting \( R_h v = \sum c_i \Phi_i \) in (4.3.5) and then applying the coercivity of \( A_h(\cdot, \cdot) \). Here, \( \Phi_i \) represents basis function corresponding to the \( i \)th grid. Again,
\[
(f^*, v_h) = - \int_{\Omega_1} \nabla \cdot (\beta_1 \nabla v) v_h dx - \int_{\Omega_2} \nabla \cdot (\beta_2 \nabla v) v_h dx
\]
\[
= - \int_{\Gamma} \beta_1 \frac{\partial v}{\partial \eta} v_h ds + \int_{\Omega_1} \beta_1 \nabla v \cdot \nabla v_h dx + \int_{\Gamma} \beta_2 \frac{\partial v}{\partial \eta} v_h ds + \int_{\Omega_2} \beta_2 \nabla v \cdot \nabla v_h dx
\]
\[
= \int_{\Omega_1} \beta_1 \nabla v \cdot \nabla v_h dx + \int_{\Omega_2} \beta_2 \nabla v \cdot \nabla v_h dx + \int_{\Gamma} \left[ \beta \frac{\partial v}{\partial \eta} \right] v_h ds
\]
\[
= A^1(v, v_h) + A^2(v, v_h).
\] (4.3.6)
In the last equality, we have used the fact that $[\beta \frac{\partial u}{\partial n}] = 0$ along $\Gamma$. Combining (4.3.5) and (4.3.6), we have

$$A_h(R_h v, v_h) = A^1(v, v_h) + A^2(v, v_h) \quad \forall v_h \in V_h. \quad (4.3.7)$$

Regarding the approximation properties of $R_h$ operator defined by (4.3.7), we have the following results

**Lemma 4.3.2** Let $R_h$ be defined by (4.3.7), then for any $v \in X^{**}$ there is a positive constant $C$ independent of the mesh parameter $h$ such that

$$\|R_h v - v\|_{H^1(\Omega_1)} + \|R_h v - v\|_{H^1(\Omega_2)} \leq Ch(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}).$$

**Proof.** Coercivity of each local bilinear map and the definition of $R_h$ projection leads to

$$\|v - R_h v\|^2_{H^1(\Omega_1)} + \|v - R_h v\|^2_{H^1(\Omega_2)} \leq C\{A^1(v - R_h v, v - v_h) + A^2(v - R_h v, v - v_h)\}
+ CA^1(v, v_h - R_h v) - CA^1(R_h v, v_h - R_h v)
+ CA^2(v, v_h - R_h v) - CA^2(R_h v, v_h - R_h v)
= C\{A^1(v - R_h v, v - v_h) + A^2(v - R_h v, v - v_h)\}
+ C\{A^1_h(R_h v, v_h - R_h v) - A^1(R_h v, v_h - R_h v)\}
+ C\{A^2_h(R_h v, v_h - R_h v) - A^2(R_h v, v_h - R_h v)\}
= C\{A^1(v - R_h v, v - v_h) + A^2(v - R_h v, v - v_h)\}
+ C\{A_h(R_h v, v_h - R_h v) - A(R_h v, v_h - R_h v)\}. \nonumber$$

Then it follows from Lemma 2.2.1 of Chapter 2 and Young’s inequality that

$$\|v - R_h v\|^2_{H^1(\Omega_1)} + \|v - R_h v\|^2_{H^1(\Omega_2)} \leq C\|v - R_h v\|_{H^1(\Omega_1)}\|v - v_h\|_{H^1(\Omega_1)} + C\|v - R_h v\|_{H^1(\Omega_2)}\|v - v_h\|_{H^1(\Omega_2)}
+ Ch\|R_h v\|_{H^1(\Omega)}\|v_h - R_h v\|_{H^1(\Omega)}
\leq \epsilon\|v - R_h v\|^2_{H^1(\Omega_1)} + \frac{C}{\epsilon}\|v - v_h\|^2_{H^1(\Omega_1)} + \epsilon\|v - R_h v\|^2_{H^1(\Omega_2)}
+ \frac{C}{\epsilon}\|v - v_h\|^2_{H^1(\Omega_2)} + \frac{Ch^2}{\epsilon}\|R_h v\|^2_{H^1(\Omega)} + \epsilon\|v_h - R_h v\|^2_{H^1(\Omega)}. \nonumber$$
Again applying the fact \( \| R_h v \|_{H^1(\Omega)} \leq C(\| v \|_{H^1(\Omega_1)} + \| v \|_{H^1(\Omega_2)}) \) and for suitable \( \epsilon > 0 \), we have

\[
\| v - R_h v \|_{H^1(\Omega_1)} + \| v - R_h v \|_{H^1(\Omega_2)} \leq C \| v \|_{H^1(\Omega_1)} + C \| v - v_h \|_{H^1(\Omega_2)} + C h^2 \{ \| v \|_{H^1(\Omega_1)}^2 + \| v \|_{H^1(\Omega_2)}^2 \}.
\]

Now, setting \( v_h = v_I \) and then using Lemma 4.3.1, we have

\[
\| v - R_h v \|_{H^1(\Omega_1)} + \| v - R_h v \|_{H^1(\Omega_2)} \leq C h \{ \| v \|_{H^2(\Omega_1)} + \| v \|_{H^2(\Omega_2)} \}.
\]

This completes the proof of Lemma 4.3.2. \( \square \)

**Corollary 4.3.1** Let \( u \) be the exact solution of the interface problem (4.1.1)-(4.1.3), then

\[
\| u - R_h u \|_{H^1(\Omega_1)} + \| u - R_h u \|_{H^1(\Omega_2)} \leq C h \{ \| u \|_{H^2(\Omega_1)} + \| u \|_{H^2(\Omega_2)} \}.
\]

**Proof.** Since the solution \( u \in X \cap H^1_0(\Omega) \) with \( [u] = 0 \) and \( [\beta \partial u / \partial \eta] = 0 \), thus \( u \in X^\ast \) and hence the result follows from the previous result.

**Corollary 4.3.2** Let \( u \) be the exact solution of the interface problem (4.1.1)-(4.1.3), then

\[
\| u_t - R_h u_t \|_{H^1(\Omega_1)} + \| u_t - R_h u_t \|_{H^1(\Omega_2)} \leq C h \{ \| u_t \|_{H^2(\Omega_1)} + \| u_t \|_{H^2(\Omega_2)} \}.
\]

**Proof.** Again \( u_1 = u_2 \) and \( \beta_1 \partial u_1 / \partial \eta = \beta_2 \partial u_2 / \partial \eta \) along \( \Gamma \), therefore taking time derivative, we have

\[
\frac{\partial u_1}{\partial t} = \frac{\partial u_2}{\partial t} \text{ and } \beta_1 \frac{\partial u_1}{\partial \eta} = \beta_2 \frac{\partial u_2}{\partial \eta} \Rightarrow [u_t] = 0 \text{ and } \left[ \beta \frac{\partial u_t}{\partial \eta} \right] = 0 \text{ along } \Gamma.
\]

Therefore, \( u_t \in X^\ast \) and hence an application of Lemma 4.3.2 leads to the desired result.

**Lemma 4.3.3** Let \( R_h \) be defined fixed in (4.3.7), then for any \( v \in X^\ast \) there is a positive constant \( C \) independent of the mesh size parameter \( h \) such that

\[
\| R_h v - v \|_{L^2(\Omega)} \leq C h^2 (\| v \|_{H^2(\Omega_1)} + \| v \|_{H^2(\Omega_2)}).
\]

45
Proof. For $L^2$ norm error estimate, we will use the duality argument. For this purpose, we consider the following interface problem

$$-\nabla \cdot (\beta \nabla \phi) = v - R_h v$$

with the boundary condition $\phi = 0$ on $\partial \Omega$ and interface conditions $[\phi] = 0$, $[\beta \frac{\partial \phi}{\partial \eta}] = 0$ along $\Gamma$.

Now multiply the above equation by $w$ with $w \in L^2(\Omega) \cap H^1(\Omega_1) \cap H^1(\Omega_2) \cap \{\psi : \psi = 0 \text{ on } \partial \Omega\}$ and $[w] = 0$ along $\Gamma$, and then integrate over $\Omega$ to have

$$(v - R_h v, w) = \int_\Omega -\nabla \cdot (\beta \nabla \phi) w dx$$

$$= -\int_{\Omega_1} \nabla \cdot (\beta_1 \nabla \phi) w dx - \int_{\Omega_2} \nabla \cdot (\beta_2 \nabla \phi) w dx$$

$$= \int_{\Omega_1} \beta_1 \nabla \phi \cdot \nabla w dx - \int_{\Gamma} \beta_1 \frac{\partial \phi}{\partial \eta} w ds + \int_{\Omega_2} \beta_2 \nabla \phi \cdot \nabla w dx$$

$$+ \int_{\Gamma} \beta_2 \frac{\partial \phi}{\partial \eta} w ds$$

$$= A^1(\phi, w) + A^2(\phi, w) + \int_{\Gamma} [\beta w \frac{\partial \phi}{\partial \eta}] ds.$$ 

Again $w_1 = w_2$ and $\beta_1 \frac{\partial \phi_1}{\partial \eta} = \beta_2 \frac{\partial \phi_2}{\partial \eta}$ along $\Gamma$ implies $[\beta w \frac{\partial \phi}{\partial \eta}] = 0$ along $\Gamma$. Thus, the above equation reduces to

$$A^1(\phi, w) + A^2(\phi, w) = (v - R_h v, w). \quad (4.3.8)$$

Let $\phi_h \in V_h$ be the finite element approximation to $\phi$ defined as: Find $\phi_h \in V_h$ such that

$$A_h(\phi_h, w_h) = (v - R_h v, w_h) \quad \forall w_h \in V_h. \quad (4.3.9)$$

Arguing as deriving Lemma 4.3.2, it can be concluded that

$$\| \phi - \phi_h \|_{H^1(\Omega_1)} + \| \phi - \phi_h \|_{H^1(\Omega_2)}$$

$$\leq C(\| \phi - w_h \|_{H^1(\Omega_1)} + \| \phi - w_h \|_{H^1(\Omega_2)})$$

$$+ Ch(\| \phi \|_{H^2(\Omega_1)} + \| \phi \|_{H^2(\Omega_2)}) \quad \forall w_h \in V_h.$$

Let $\phi_I$ be defined as in (4.3.1) and then set $w_h = \phi_I$ to have

$$\| \phi - \phi_h \|_{H^1(\Omega_1)} + \| \phi - \phi_h \|_{H^1(\Omega_2)} \leq Ch(\| \phi \|_{H^2(\Omega_1)} + \| \phi \|_{H^2(\Omega_2)})$$

$$\leq Ch\| v - R_h v \|_{L^2(\Omega)}.$$
In the last inequality, we used the elliptic regularity estimate \( \| \phi \|_X \leq C \| v - R_h v \|_{L^2(\Omega)} \) (cf. [11]). Thus, we have

\[
\| \phi - \phi_h \|_{H^1(\Omega)} \leq C h \| v - R_h v \|_{L^2(\Omega)}. \tag{4.3.10}
\]

Since \([v - R_h v] = 0\) along \(\Gamma\) and \(v - R_h v \in L^2(\Omega) \cap H^1(\Omega_1) \cap H^1(\Omega_2) \cap \{ \psi : \psi = 0 \text{ on } \partial \Omega \}\), therefore we can set \(w = v - R_h v\) in (4.3.8) to have

\[
\| v - R_h v \|_{L^2(\Omega)}^2 = A^1(\phi, v - R_h v) + A^2(\phi, v - R_h v)
\]

\[
= A^1(\phi - \phi_h, v - R_h v) + A^2(\phi - \phi_h, v - R_h v)
\]

\[
+ \{ A^1(\phi_h, v - R_h v) + A^2(\phi_h, v - R_h v) \}
\]

\[
\leq C \| \phi - \phi_h \|_{H^1(\Omega_1)} \| v - R_h v \|_{H^1(\Omega_1)}
\]

\[
+ C \| \phi - \phi_h \|_{H^1(\Omega_2)} \| v - R_h v \|_{H^1(\Omega_2)}
\]

\[
+ \{ A^1(\phi_h, v) + A^2(\phi_h, v) \} - \{ A^1(\phi_h, R_h v) + A^2(\phi_h, R_h v) \}
\]

\[
\leq C h \| v - R_h v \|_{L^2(\Omega)} \cdot C h (\| v \|_{H^2(\Omega_1)} + \| v \|_{H^2(\Omega_2)})
\]

\[
+ A_h(R_h v, \phi_h) - A(R_h v, \phi_h)
\]

\[
= C h^2 \| v - R_h v \|_{L^2(\Omega)} (\| v \|_{H^2(\Omega_1)} + \| v \|_{H^2(\Omega_2)})
\]

\[
+ \{ A_h(R_h v, \phi_h) - A(R_h v, \phi_h) \}
\]

\[
= C h^2 \| v - R_h v \|_{L^2(\Omega)} (\| v \|_{H^2(\Omega_1)} + \| v \|_{H^2(\Omega_2)}) + (J). \tag{4.3.11}
\]

Now, we apply Lemma 2.2.1 to have

\[
| (J) | \leq C h \sum_{K \in T^*_h} \| R_h v \|_{H^1(K)} \| \phi_h \|_{H^1(K)}
\]

\[
\leq C h \sum_{K \in T^*_h} \| R_h v \|_{H^1(K_1)} \| \phi_h \|_{H^1(K_1)}
\]

\[
+ C h \sum_{K \in T^*_h} \| R_h v \|_{H^1(K_2)} \| \phi_h \|_{H^1(K_2)}
\]

\[
= (J)_1 + (J)_2. \tag{4.3.12}
\]
Again, using Corollary 4.3.1 and estimate (4.3.10), we have
\[
\|R_h v\|_{H^1(K_2)} \|\phi_h\|_{H^1(K_2)} \\
\leq \{\|R_h v - v\|_{H^1(K_2)} + \|v\|_{H^1(K_2)}\}\{\|\phi_h - \phi\|_{H^1(K_2)} + \|\phi\|_{H^1(K_2)}\} \\
\leq \{\|R_h v - v\|_{H^1(\Omega_2)} + \|\tilde{v}_2\|_{H^1(K_2)}\}\{\|\phi_h - \phi\|_{H^1(\Omega_2)} + \|\phi\|_{H^1(K_2)}\} \\
\leq C\{h\|v\|_{H^2(\Omega_1)} + h\|v\|_{H^2(\Omega_2)} + \|\tilde{v}_2\|_{H^1(K)}\} \\
\times \{h\|v - R_h v\|_{L^2(\Omega)} + \|\phi\|_{H^1(K)}\}. \quad (4.3.13)
\]

Setting \( p = 4 \) in the Sobolev embedding inequality (2.2.8), we obtain
\[
\|\tilde{v}_2\|_{H^1(K)} = \|\tilde{v}_2\|_{L^2(K)} + \|\nabla \tilde{v}_2\|_{L^2(K)} \\
\leq Ch^{\frac{1}{2}}\|\tilde{v}_2\|_{L^4(K)} + Ch^{\frac{1}{2}}\|\nabla \tilde{v}_2\|_{L^4(K)} \\
\leq Ch^{\frac{1}{2}}\|\tilde{v}_2\|_{H^1(K)} + Ch^{\frac{1}{2}}\|\nabla \tilde{v}_2\|_{H^1(K)} \\
\leq Ch^{\frac{1}{2}}\|\tilde{v}_2\|_{H^2(K)} \leq Ch^{\frac{1}{2}}\|\tilde{v}_2\|_{H^2(\Omega_2)} \quad (4.3.14)
\]
where we have used the fact that \( \text{meas}(K) \leq Ch^2 \). Similarly, for \( \|\phi\|_{H^1(K)} \), we have
\[
\|\phi\|_{H^1(K)} \leq Ch^{\frac{1}{2}}\|\phi\|_{X} \leq Ch^{\frac{1}{2}}\|v - R_h v\|_{L^2(\Omega)}. \quad (4.3.15)
\]

Combining (4.3.13)-(4.3.15), we have
\[
\|R_h v\|_{H^1(K_2)} \|\phi_h\|_{H^1(K_2)} \\
\leq Ch\{\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}\}\|v - R_h v\|_{L^2(\Omega)}.
\]

Therefore, for \((J)_2\), we have
\[
(J)_2 \leq Ch^2\{\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}\}\|v - R_h v\|_{L^2(\Omega)}. \quad (4.3.16)
\]

Similarly, for \((J)_1\), we have
\[
(J)_1 \leq Ch^2\{\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}\}\|v - R_h v\|_{L^2(\Omega)}. \quad (4.3.17)
\]

Then, using the estimates (4.3.16) and (4.3.17) in (4.3.12), we have
\[
|(J)| \leq Ch^2\|v - R_h v\|_{L^2(\Omega)}(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}). \quad (4.3.18)
\]

Finally, (4.3.11) and (4.3.18) leads to the following optimal \( L^2 \) norm estimate
\[
\|v - R_h v\|_{L^2(\Omega)} \leq Ch^2(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}).
\]

This completes the rest of the proof.
Corollary 4.3.3 Let $u$ be the exact solution of the interface problem (4.1.1)-(4.1.3), then

$$
\|u - R_h u\|_{L^2(\Omega)} \leq Ch^2 \|u\|_X,
$$

$$
\|u_t - R_h u_t\|_{L^2(\Omega)} \leq Ch^2 (\|u_t\|_{H^2(\Omega_1)} + \|u_t\|_{H^2(\Omega_2)}).
$$

4.4 Error Analysis for the Semidiscrete Scheme

In this section, we discuss the semidiscrete finite element method for the problem (4.1.1)-(4.1.3) and derive optimal error estimates in $L^2$ and $H^1$ norms.

The continuous-time Galerkin finite element approximation to (4.2.1) is stated as follows: Find $u_h(t) \in V_h$ such that

$$
(u_{ht}, v_h) + A_h(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h, \quad t \in (0, T].
$$

(4.4.1)

Subtracting (4.4.1) from (4.2.1), we have

$$
(u_t - u_{ht}, v_h) + A(u, v_h) - A_h(u_h, v_h) = 0.
$$

(4.4.2)

Define the error $e(t) = u - u_h = u - R_h u + R_h u - u_h = \rho + \theta$, with $\rho = u - R_h u$ and $\theta = R_h u - u_h$. Again, using (4.3.7) for $v = u \in X^{**}$ and further differentiating with respect to $t$, we have

$$
A_h((R_h u)_t, v_h) = A^1(u_t, v_h) + A^2(u_t, v_h).
$$

Also,

$$
A_h(R_h u_t, v_h) = A^1(u_t, v_h) + A^2(u_t, v_h).
$$

From the above two equations, we have

$$
A_h((R_h u)_t - R_h u_t, v_h) = 0 \quad \forall v_h \in V_h.
$$

Setting $v_h = (R_h u)_t - R_h u_t$ in the above equation, we obtain $(R_h u)_t = R_h u_t$.

Now, by the definition of $R_h$ operator and (4.4.2), we obtain

$$
(\theta_t, v_h) + A_h(\theta, v_h) = ((R_h u)_t - u_{ht}, v_h) + A_h(R_h u - u_h, v_h)
$$

$$
= (R_h u_t, v_h) - (u_{ht}, v_h) + A_h(R_h u, v_h) - A_h(u_h, v_h)
$$

$$
= (u_t - \rho_t, v_h) - (u_{ht}, v_h) + A(u, v_h) - A_h(u_h, v_h)
$$

$$
= (-\rho_t, v_h) + (u_t - u_{ht}, v_h) - (u_t - u_{ht}, v_h)
$$

$$
= (-\rho_t, v_h).
$$
For $v_h = \theta$, we have
\[
(\theta_t, \theta) + C\|\theta\|^2_{H^1(\Omega)} \leq \|\rho_t\|_{L^2(\Omega)} \|\theta\|_{L^2(\Omega)} \\
\leq C\|\rho_t\|^2_{L^2(\Omega)} + \frac{\epsilon}{2} \|\theta\|^2_{H^1(\Omega)}.
\]

Integrating the above equation from 0 to $t$ and using Corollary 4.3.3, we obtain
\[
\|\theta(t)\|^2_{L^2(\Omega)} \leq C \int_0^t \|\rho_t\|^2_{L^2(\Omega)} ds + \|\theta(0)\|^2_{L^2(\Omega)} \\
\leq C \int_0^t \|\rho_t\|^2_{L^2(\Omega)} ds \\
\leq Ch^4 \int_0^t (\|u_t\|_{H^2(\Omega_1)} + \|u_t\|_{H^2(\Omega_2)})^2 ds.
\]

(4.4.3)

Now, combining Corollary 4.3.3 and (4.4.3), we have the following optimal pointwise-in-time $L^2$-norm error estimates.

**Theorem 4.4.1** Let $u$ and $u_h$ be the solution of the problem (4.1.1)-(4.1.3) and (4.4.1), respectively. Assume that $u_h(0) = R_h u_0$. Then there exists a constant $C$ independent of $h$ such that
\[
\|e(t)\|_{L^2(\Omega)} \leq Ch^2 \left\{ \|u\|_X + \left( \int_0^t (\|u_t\|_{H^2(\Omega_1)} + \|u_t\|_{H^2(\Omega_2)})^2 ds \right)^{\frac{1}{2}} \right\}.
\]

For $H^1$-norm estimate, we first use Corollary 4.3.1 to have
\[
\sum_{i=1}^2 \|\rho(t)\|_{H^1(\Omega_i)} \leq Ch \sum_{i=1}^2 \|u\|_{H^2(\Omega_i)}.
\]

(4.4.4)

Applying inverse estimate 2.2 of Chapter 3, we obtain
\[
\sum_{i=1}^2 \|\theta(t)\|_{H^1(\Omega_i)} \leq Ch^{-1} \sum_{i=1}^2 \|\theta(t)\|_{L^2(\Omega_i)} \\
\leq Ch^{-1} \int_0^t (\|u_t\|_{H^2(\Omega_1)} + \|u_t\|_{H^2(\Omega_2)}) \right\}^{\frac{1}{2}} \\
= Ch \left\{ \int_0^t (\|u_t\|_{H^2(\Omega_1)} + \|u_t\|_{H^2(\Omega_2)}) \right\}^{\frac{1}{2}}.
\]

(4.4.5)

Combining (4.4.4) and (4.4.5), we have the following optimal pointwise-in-time $H^1$-norm error estimates.
Theorem 4.4.2 Let $u$ and $u_h$ be the solution of the problem (4.1.1)-(4.1.3) and (4.4.1), respectively. Assume that $u_h(0) = R_h u_0$. Then there exists a constant $C$ independent of $h$ such that
\[
\|e(t)\|_{H^1(\Omega)} \leq C h \left\{ \sum_{i=1}^{2} \|u\|_{X} + \left( \int_{0}^{t} \left( \|u_t\|_{H^2(\Omega_1)} + \|u_t\|_{H^2(\Omega_2)} \right)^2 \right)^{\frac{1}{2}} \right\}.
\]

4.5 Error Analysis for the Fully Discrete Scheme

A fully discrete scheme based on backward Euler method is proposed and analyzed in this section. Optimal $L^2$ norm error estimate is obtained for fully discrete scheme.

We first partition the interval $[0, T]$ into $M$ equally spaced subintervals by the following points
\[0 = t_0 < t_1 < \ldots < t_M = T\]
with $t_n = nk$, $k = \frac{T}{M}$, be the time step. Let $I_n = (t_{n-1}, t_n]$ be the $n$-th subinterval. Now we introduce the backward difference quotient
\[\Delta_k \phi^n = \frac{\phi^n - \phi^{n-1}}{k},\]
for a given sequence $\{\phi^n\}_{n=0}^{M} \subset L^2(\Omega)$.

The fully discrete finite element approximation to the problem (4.2.1) is defined as follows: For $n = 1, \ldots, M$, find $U^n \in V_h$ such that
\[
(\Delta_k U^n, v_h) + A_h(U^n, v_h) = (f^n, v_h) \quad \forall v_h \in V_h
\]
(4.5.1)

with $U^0 = R_h u_0$. For each $n = 1, \ldots, M$, the existence of a unique solution to (4.5.1) can be found in [11]. We then define the fully discrete solution to be a piecewise constant function $U_h(x,t)$ in time and is given by
\[U_h(x,t) = U^n(x) \quad \forall t \in I_n, \ 1 \leq n \leq M.\]

We now prove the main result of this section in the following theorem.

Theorem 4.5.1 Let $u$ and $U$ be the solutions of the problem (4.1.1)-(4.1.3) and (4.5.1), respectively. Assume that $U^0 = R_h u_0$. Then there exists a constant $C$ independent of $h$
and $k$ such that
\[
\|U(t_n) - u(t_n)\|_{L^2(\Omega)} \leq C(h^2 + k) \sum_{i=1}^{2} \left\{ \|u^0\|_{H^2(\Omega_i)} + \|u_t\|_{L^2(0,T;H^2(\Omega_i))} + \|u_{tt}\|_{L^2(0,T;L^2(\Omega_i))} \right\}
\]

**Proof.** We write the error $U^n - u^n$ at time $t_n$ as
\[
U^n - u^n = (U^n - R_hu^n) + (R_hu^n - u^n) \equiv \theta^n + \rho^n
\]
where $\theta^n = U^n - R_hu^n$ and $\rho^n = R_hu^n - u^n$.

For $\theta^n$, we have the following error equation
\[
(\Delta_k\theta^n, v_h) + A_h(\theta^n, v_h)
= (-\Delta_kR_hu^n + \Delta_kU^n, v_h) + A_h(-R_hu^n + U^n, v_h)
= (\Delta_kU^n, v_h) + A_h(U^n, v_h) - (\Delta_kR_hu^n, v_h) - A_h(R_hu^n, v_h)
= (f^n, v_h) - (\Delta_kR_hu^n, v_h) - A(u^n, v_h)
= (f^n, v_h) - (\Delta_kR_hu^n, v_h) + (u^n_t, v_h) - (f^n, v_h)
\equiv: -(w^n, v_h)
\]
(4.5.2)

where $w^n = \Delta_kR_hu^n - u^n_t$. For simplicity of the exposition, we write $w^n = w^n_1 + w^n_2$, where $w^n_1 = R_h\Delta_ku^n - \Delta_ku^n$ and $w^n_2 = \Delta_ku^n - u^n_t$.

Now, setting $v_h = \theta^n$ in (4.5.2), we have
\[
(\Delta_k\theta^n, \theta^n) + A_h(\theta^n, \theta^n) = -(w^n, \theta^n)
\]
(4.5.3)

Since $A_h(\theta^n, \theta^n) \geq 0$, we have
\[
\|\theta^n\|_{L^2(\Omega)} \leq k\|w^n\|_{L^2(\Omega)} + \|\theta^{n-1}\|_{L^2(\Omega)}
\leq \|\theta^0\|_{L^2(\Omega)} + k\sum_{j=1}^{n} \|w^n_1\|_{L^2(\Omega)} + h\sum_{j=1}^{n} \|w^n_2\|_{L^2(\Omega)}.
\]
(4.5.4)

In $\Omega_1$, the term $w^n_1$ can be expressed as
\[
w^n_1 = R_h\Delta_ku^n_1 - \Delta_ku^n_1 = (R_h - I)(\Delta_ku^n_1)
\]
\[
= (R_h - I)\frac{1}{k} \int_{t_{j-1}}^{t_j} u^n_{1,t} dt = \frac{1}{k} \int_{t_{j-1}}^{t_j} (R_hu^n_{1,t} - u^n_{1,t}) dt,
\]
52
where $u_i$, $i = 1, 2$ is the restriction of $u$ in $\Omega_i$ and $u_{i,t} = \frac{\partial u_i}{\partial t}$.

An application of Corollary 4.3.3 leads to

$$k\|w_1^j\|_{L^2(\Omega_1)} \leq Ch^2 \int_{t_{j-1}}^{t_j} \left\{ \sum_{i=1}^{2} \|u_t\|_{H^2(\Omega_i)} \right\} dt$$

Similarly, we obtain

$$k\|w_2^j\|_{L^2(\Omega_2)} \leq Ch^2 \int_{t_{j-1}}^{t_j} \left\{ \sum_{i=1}^{2} \|u_t\|_{H^2(\Omega_i)} \right\} dt.$$

Using above two estimates, we have

$$k \sum_{j=1}^{n} \|w_1^j\|_{L^2(\Omega)} \leq Ch^2 \int_{0}^{t_n} \left\{ \sum_{i=1}^{2} \|u_t\|_{H^2(\Omega_i)} \right\} dt. \quad (4.5.5)$$

Similarly, for the term $w_2^n$, we have

$$kw_2^j = u^j - u^{j-1} - ku_t^j = - \int_{t_{j-1}}^{t_j} (s - t_{j-1}) u_{tt} ds$$

and hence

$$k\|w_2^j\|_{L^2(\Omega_i)} \leq k \int_{t_{j-1}}^{t_j} \|u_{tt}\|_{L^2(\Omega_i)} ds.$$

Summing over $j$ from $j = 1$ to $j = n$, we obtain

$$k \sum_{j=1}^{n} \|w_2^j\|_{L^2(\Omega)} \leq Ck \int_{0}^{t_n} \left\{ \sum_{i=1}^{2} \|u_{tt}\|_{L^2(\Omega_i)} \right\} dt. \quad (4.5.6)$$

Combining (4.5.4), (4.5.5) and (4.5.6), and further using the fact that $\theta^0 = 0$, we obtain

$$\|\theta^n\|_{L^2(\Omega)} \leq C(h^2 + k) \sum_{i=1}^{2} \int_{0}^{t_n} \left\{ \|u_t\|_{H^2(\Omega_i)} + \|u_{tt}\|_{L^2(\Omega_i)} \right\} dt$$

$$\leq C(h^2 + k) \sum_{i=1}^{2} \left\{ \|u_t\|_{L^2(0,T;H^2(\Omega_i))} + \|u_{tt}\|_{L^2(0,T;L^2(\Omega_i))} \right\} \quad (4.5.7)$$

An application of Corollary 4.3.3 for $\rho^n$ yields

$$\|\rho^n\|_{L^2(\Omega)} \leq Ch^2 \sum_{i=1}^{2} \|u^n\|_{H^2(\Omega_i)}.$$
Again, it is easy to verify that
\[
\|u^n\|_{H^2(\Omega_i)} \leq \|u^0\|_{H^2(\Omega_i)} + \int_0^{t_n} \|u_t\|_{H^2(\Omega_i)} dt
\]
Thus, we have
\[
\|\rho^n\|_{L^2(\Omega)} \leq Ch^2 \sum_{i=1}^2 \left\{ \|u^0\|_{H^2(\Omega_i)} + \|u_t\|_{L^2(0,T;H^2(\Omega_i))} \right\}
\]
(4.5.8)
Combining (4.5.7) and (4.5.8) the desired estimate is easily obtained. □

**Remark 4.5.1** Although the error analysis of Sections 4.4-4.5 depends on standard \(\rho\) and \(\theta\) argument given in Thomee’s monograph ([47]) for non interface problem, the novelty of this chapter are contained in Section 4.3, where we have introduced modified elliptic projection and approximation properties of such projection under minimum regularity assumption of the solution. Due to low global regularity of the solution the classical analysis is difficult to apply for the convergence analysis of the interface problems. Section 4.3 bridges the gap between standard finite element technique for non interface problems and interface problems.