Chapter 3

$L^2(L^2)$ and $L^2(H^1)$ norms Error Estimates for Parabolic Interface Problems

In this chapter, we extend the finite element analysis of elliptic interface problems discussed in Chapter 2 to parabolic interface problems. Optimal order error estimates in $L^2(L^2)$ and $L^2(H^1)$ norms are derived for the linear parabolic interface problems.

3.1 Introduction

In this chapter, we consider a linear parabolic interface problem of the form

$$u_t + Lu = f(x,t) \quad \text{in } \Omega \times (0,T)$$

(3.1.1)

with initial and boundary conditions

$$u(x,0) = u_0 \quad \text{in } \Omega; \quad u(x,t) = 0 \quad \text{on } \partial \Omega \times (0,T)$$

(3.1.2)

and jump conditions on the interface

$$[u] = 0, \quad \left[ \beta \frac{\partial u}{\partial n} \right] = g(x,t) \quad \text{along } \Gamma,$$

(3.1.3)

where, $f = f(x,t)$ and $g = g(x,t)$ are real valued functions in $\Omega \times (0,T]$, and $u_t = \frac{\partial u}{\partial t}$. Further, $u_0 = u_0(x)$ is real valued function in $\Omega$. The domain $\Omega$, operator $L$, symbols $[v]$ and $n$ are defined as in Chapter 1, and $T < \infty$. 

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To derive $O(h^m) \ (m \geq 0)$ error estimates for non-interface parabolic problems in the literature generally require $u \in L^2(0, T; H^{m+1}(\Omega)) \cap H^1(0, T; H^{-m}(\Omega))$, see, [47].

The purpose of the present chapter is to extend the convergence analysis of fitted finite element method for elliptic interface problems to parabolic interface problems. The convergence of finite element solution to the exact solution has been discussed in terms of $L^2(H^1)$ and $L^2(L^2)$ norms. The main crucial technical tools used in our analysis are Sobolev embedding inequality, approximation result for the linear interpolant and elliptic projection (see, Lemma 3.2.2), parabolic duality arguments and some known results on elliptic interface problems. The previous work on finite element analysis for parabolic interface problems can be found in Chapter 1.

The outline of this chapter is as follows. In section 3.2, the approximation properties related to the auxiliary projections are presented and section 3.3 is devoted to the error analysis for the semidiscrete scheme. Finally, in section 3.4, a fully discrete scheme based on backward Euler method is proposed and optimal $L^2(H^1)$ norm is established.

### 3.2 Preliminaries

In this section, some approximation properties related to the auxiliary projection is obtained. Due to the presence of discontinuous coefficients the solution $u$, in general, does not belong to $H^2(\Omega)$. Regarding the regularity for the solution of the interface problem (3.1.1)-(3.1.3), we have the following result (cf. [11, 30, 44]).

**Theorem 3.2.1** Let $f \in H^1(0, T; L^2(\Omega)), \ g \in H^1(0, T; H^{-\frac{1}{2}}(\Gamma))$ and $u_0 \in H^1_0(\Omega)$. Then the problem (3.1.1)-(3.1.3) has a unique solution $u \in L^2(0, T; X) \cap H^1(0, T; Y)$. Further, $u$ satisfies the following a priori estimate

$$
\|u\|_{L^2(0,T;X)} \leq C \left\{ \|f\|_{L^2(0,T;L^2(\Omega))} + \|u_0\|_{H^1(\Omega)} + \|g(x,0)\|_{H^\frac{1}{2}(\Gamma)} + \|g(x,T)\|_{H^\frac{1}{2}(\Gamma)} + \|g\|_{H^1(0,T;H^\frac{1}{2}(\Gamma))} \right\}.
$$

(3.2.1)

Now, we shall recall the finite element space $V_h \subset H^1_0(\Omega)$ consisting of piecewise linear polynomials vanishing on the boundary $\partial \Omega$ where interface triangles are straight triangles as discussed in Chapter 2. Further, we assume that $V_h$ satisfy the inverse estimate

$$
\|\phi\|_{H^1(\Omega)} \leq C h^{-1} \|\phi\|_{L^2(\Omega)} \quad \forall \ \phi \in V_h.
$$

(3.2.2)
Approximating the interface function \( g(x) \) by its discrete specimen \( g_h = \sum_{j=1}^{m_h} g(P_j) \Phi^h_j \), where \( \{\Phi^h_j\}_{j=1}^{m_h} \) is the set of standard nodal basis functions corresponding to the nodes \( \{P_j\}_{j=1}^{m_h} \) on the interface \( \Gamma \), we have the following approximation result. For a proof, we refer to [11].

**Lemma 3.2.1** Let \( g \in H^2(\Gamma) \). If \( \Omega^\pm_1 \) is the union of all interface triangles then we have

\[
\left| \int_{\Gamma} gv_h ds - \int_{\Gamma_h} g_h v_h ds \right| \leq Ch^2 \| g \|_{H^2(\Gamma)} \| v_h \|_{H^1(\Omega^+_1 \Gamma)} \quad \forall v_h \in V_h.
\]

We now define an operator \( P_h : X \cap H^1_0(\Omega) \to V_h \) by

\[
A_h(P_h v, \phi) = A(v, \phi) \quad \forall \phi \in V_h, \ v \in X \cap H^1_0(\Omega). \tag{3.2.3}
\]

Earlier, in [11], the approximation results obtained for \( P_h \) in \( L^2 \) and \( H^1 \)-norms are not optimal. However, the loss in accuracy for the \( H^1 \) norm is recovered in [45] under the assumption that the solution \( u \in X \cap W^{1,\infty}(\Omega_1 \cap \Omega_0) \cap W^{1,\infty}(\Omega_2 \cap \Omega_0) \). The following lemma shows that optimal approximation of \( P_h \) in \( L^2 \) and \( H^1 \)-norms can be derived for \( u \in X \cap H^1_0(\Omega) \) only. This lemma is very crucial for our later analysis.

**Lemma 3.2.2** Having the projection \( P_h \) fixed in (3.2.3), there is a positive constant \( C \) independent of the mesh size parameter \( h \) such that

\[
\| u - P_h u \|_{L^2(\Omega)} + h \| u - P_h u \|_{H^1(\Omega)} \leq Ch^2 \| u \|_X.
\]

*Proof.* We first split \( u - P_h u \) as

\[
u - P_h u = (u - \Pi_h u) + (\Pi_h u - P_h u) .
\]

From Lemma 2.2.3 of Chapter 2 and (3.2.3), we note that

\[
\begin{align*}
&\| \Pi_h u - P_h u \|^2_{H^1(\Omega)} \\
&\leq A_h(\Pi_h u - u, \Pi_h u - P_h u) + A_h(u - P_h u, \Pi_h u - P_h u) \\
&\leq Ch \| u \|_X \| \Pi_h u - P_h u \|_{H^1(\Omega)} + \{ A_h(u, \Pi_h u - P_h u) - A(u, \Pi_h u - P_h u) \} \\
&= Ch \| u \|_X \| \Pi_h u - P_h u \|_{H^1(\Omega)} \\
&+ A_h(u - \Pi_h u, \Pi_h u - P_h u) - A(u - \Pi_h u, \Pi_h u - P_h u) \\
&+ \{ A_h(\Pi_h u, \Pi_h u - P_h u) - A(\Pi_h u, \Pi_h u - P_h u) \} \\
&\leq Ch \| u \|_X \| \Pi_h u - P_h u \|_{H^1(\Omega)} \\
&+ \{ A_h(\Pi_h u, \Pi_h u - P_h u) - A(\Pi_h u, \Pi_h u - P_h u) \} \\
&\equiv: Ch \| u \|_X \| \Pi_h u - P_h u \|_{H^1(\Omega)} + (I). \tag{3.2.4}
\end{align*}
\]
Then using Lemma 2.2.1 of Chapter 2 for the term \((I)\) to have

\[
|I| \leq Ch\|\Pi_h u\|_{H^1(\Omega)}\|\Pi_h u - P_h u\|_{H^1(\Omega)}
\]

\[
\leq Ch(\|\Pi_h u - u\|_{H^1(\Omega)} + \|u\|_{H^1(\Omega)})\|\Pi_h u - P_h u\|_{H^1(\Omega)}
\]

\[
\leq Ch\|u\|_X\|\Pi_h u - P_h u\|_{H^1(\Omega)}.
\]

This in combination with (3.2.4) now leads to

\[
\|\Pi_h u - P_h u\|_{H^1(\Omega)} \leq Ch\|u\|_X.
\]

By Lemma 2.2.3 and using triangle inequality, we obtain

\[
\|u - P_h u\|_{H^1(\Omega)} \leq Ch\|u\|_X. \tag{3.2.5}
\]

For \(L^2\)-norm error estimate, we consider the following interface problem: Find \(w \in H^1_0(\Omega)\) such that

\[
A(w, v) = (u - P_h u, v) \ \forall v \in H^1_0(\Omega), \tag{3.2.6}
\]

and let \(w_h \in V_h\) be its finite element approximation such that

\[
A_h(w_h, v_h) = (u - P_h u, v_h) \ \forall v_h \in V_h. \tag{3.2.7}
\]

Note that \(w \in H^1_0(\Omega)\) is the solution of (3.2.6) with jump conditions

\[
[w] = 0 \quad \text{and} \quad [\beta \frac{\partial w}{\partial n}] = 0 \quad \text{along} \ \Gamma.
\]

Then apply Theorem 2.3.2 for the above interface problem to have

\[
\|w - w_h\|_{H^1(\Omega)} \leq Ch\|w\|_X \leq Ch\|u - P_h u\|_{L^2(\Omega)}. \tag{3.2.8}
\]

In the last inequality, we have used regularity estimate for elliptic interface problem (3.2.6). Now, setting \(v = u - P_h u\) in (3.2.6) and, using (3.2.5) and (3.2.8), we have

\[
\|u - P_h u\|_{L^2(\Omega)}^2 = A(w - w_h, u - P_h u) + A(w_h, u) - A(w_h, P_h u)
\]

\[
= A(w - w_h, u - P_h u) + \{A(u, w_h) - A(P_h u, w_h)\}
\]

\[
\equiv A(w - w_h, u - P_h u) + (II)
\]

\[
\leq C\|w - w_h\|_{H^1(\Omega)}\|u - P_h u\|_{H^1(\Omega)} + (II)
\]

\[
\leq Ch^2\|u\|_X\|u - P_h u\|_{L^2(\Omega)} + (II). \tag{3.2.9}
\]
For the term \((II)\), we use (3.2.3) and Lemma 2.2.1 of Chapter 2 to have
\[
|\(II\)| = |\(A_h(P_h u, w_h) - A(P_h u, w_h)\)| \leq Ch\|P_h u\|_{H^1_0(\Omega)}\|w_h\|_{H^1_0(\Omega)}  \\
\leq Ch(\|P_h u - u\|_{H^1_0(\Omega)} + \|u\|_{H^1_0(\Omega)})(\|w_h - w\|_{H^1_0(\Omega)} + \|w\|_{H^1_0(\Omega)})  \\
\leq Ch^2\|u\|_X Ch^2\|w\|_X \leq Ch^2\|P_h u - u\|_{L^2(\Omega)}. \quad (3.2.10)
\]

In the last inequality, we have used Lemma 2.2.2 of Chapter 2. Then combining the estimates (3.2.9)-(3.2.10), we can conclude that
\[
\|u - P_h u\|_{L^2(\Omega)} \leq Ch^2\|u\|_X. \quad (3.2.11)
\]

This completes the proof of Lemma 3.2.2. □

We need the standard \(L^2\) projection \(L_h : L^2(\Omega) \rightarrow V_h\) defined by
\[
(L_h v, \phi) = (v, \phi) \quad \forall v \in L^2(\Omega), \; \phi \in V_h, \quad (3.2.12)
\]
satisfying the stability estimate
\[
\|L_h v\|_{H^1(\Omega)} \leq C\|v\|_{H^1(\Omega)} \quad \forall v \in H^1_0(\Omega). \quad (3.2.13)
\]

It is well known that \(L_h v \in V_h\) is the best approximation of \(v \in L^2(\Omega)\) with respect to the \(L^2\) norm. Thus Lemma 3.2.2 immediately implies

**Lemma 3.2.3** Let \(L_h\) be defined by (3.2.12). Then, for \(m = 0, 1\), we have
\[
\|L_h v - v\|_{H^m(\Omega)} \leq Ch^{2-m}\|v\|_X \quad \forall v \in H^1_0(\Omega) \cap X.
\]

**Proof.** The \(L^2\)-norm estimate follows immediately from the fact that \(L_h w \in V_h\) is the best approximation in the \(L^2\) norm to \(w \in L^2(\Omega)\) and Lemma 3.2.2. For \(H^1\)-norm estimate, we use the inverse inequality (3.2.2) and Lemma 3.2.2 to have
\[
\|v - L_h v\|_{H^1(\Omega)} \leq \|v - P_h v\|_{H^1(\Omega)} + \|P_h v - L_h v\|_{H^1(\Omega)}  \\
\leq Ch\|v\|_X + Ch^{-1}\|P_h v - L_h v\|_{L^2(\Omega)}  \\
\leq Ch\|v\|_X + Ch^{-1}\{\|P_h v - v\|_{L^2(\Omega)} + \|v - L_h v\|_{L^2(\Omega)}\}  \\
\leq Ch\|v\|_X.
\]

This completes the rest of the proof. □
3.3 Continuous time Galerkin Method

This section deals with the error analysis for the spatially discrete scheme for parabolic interface problems (3.1.1)-(3.1.3) and derive optimal error estimates in $L^2(0, T; H^1)$ and $L^2(0, T; L^2)$ norms.

The weak formulation of the problem (3.1.1)-(3.1.3) is stated as follows: Find $u \in H^1_0(\Omega)$ such that

$$
(u_t, v) + A(u, v) = (f, v) + \langle g, v \rangle \quad \forall v \in H^1_0(\Omega), \ t \in (0, T)
$$

with $u(0) = u_0$. Here, $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle_{\Gamma}$ are used to denote the inner products of $L^2(\Omega)$ and $L^2(\Gamma)$ spaces, respectively.

The continuous in time Galerkin finite element approximation to (3.3.1) which may be stated as follows: Find $u_h : [0, T] \rightarrow V_h$ such that

$$
(u_{ht}, v_h) + A_h(u_h, v_h) = (f, v_h) + \langle g_h, v_h \rangle_{\Gamma_h} \quad \forall v_h \in V_h, \ t \in (0, T).
$$

(3.3.2)

We shall need the following Lemma for the semidiscrete solution $u_h$ satisfying (3.3.2) for our future use. For a proof, we refer to [15].

**Lemma 3.3.1** Let $f \in L^2(\Omega)$ and $g \in H^2(\Gamma)$. Then we have

$$
\int_0^t \|u_h\|_{H^1(\Omega)}^2 ds \leq C \left( \int_0^t \{\|f\|_{L^2(\Omega)}^2 + \|g\|_{H^2(\Gamma)}^2\} ds + \|u_0\|_{L^2(\Omega)}^2 \right).
$$

Now, we are in a position to discuss the main results of this section which is stated in the following theorems.

**Theorem 3.3.1** Let $u$ and $u_h$ be the solutions of (3.1.1)-(3.1.3) and (3.3.2), respectively. Then, for $u_0 \in H^1_0(\Omega), f \in L^2(\Omega)$ and $g \in H^2(\Gamma)$, there is a positive constant $C$ independent of $h$ such that

$$
\|u - u_h\|_{L^2(0, T; H^1(\Omega))} \leq C(u_0, u, f, g) h.
$$

**Theorem 3.3.2** Let $u$ and $u_h$ be the solutions of (3.1.1)-(3.1.3) and (3.3.2), respectively. Then, for $u_0 \in H^1_0(\Omega), f \in L^2(\Omega)$ and $g \in H^2(\Gamma)$, there is a positive constant $C$ independent of $h$ such that

$$
\|u - u_h\|_{L^2(0, T; L^2(\Omega))} \leq C(u_0, u, f, g) h^2.
$$
Proof of Theorem 3.3.1. Subtracting (3.3.2) from (3.3.1), for all \( v_h \in V_h \), we have

\[
(u_t - u_h, v_h) + A(u - u_h, v_h) = \langle g, v_h \rangle_{\Gamma} - \langle g_h, v_h \rangle_{\Gamma_h} + A_h(u_h, v_h) - A(u_h, v_h). \tag{3.3.3}
\]

Define the error \( e(t) \) as \( e(t) = u(t) - u_h(t) \). Setting \( v_h = L_h u \) in (3.3.3) and using (3.2.12), we obtain

\[
\frac{1}{2} \frac{d}{dt} \| e(t) \|_{L^2(\Omega)}^2 + A(e, e) = (I)_1 + (I)_2 + (I)_3 + \frac{1}{2} \frac{d}{dt} \| u - L_h u \|_{L^2(\Omega)}^2, \tag{3.3.4}
\]

where the terms \((I)_i, i = 1, 2, 3\) are given by

\[
(I)_1 = \langle g, L_h u - u_h \rangle_{\Gamma} - \langle g_h, L_h u - u_h \rangle_{\Gamma_h},
\]

\[
(I)_2 = A_h(u_h, L_h u - u_h) - A(u_h, L_h u - u_h),
\]

\[
(I)_3 = A(u_h - u, L_h u - u).
\]

Now, we estimate the terms \((I)_1, (I)_2\) and \((I)_3\) one by one. By Lemma 3.2.1, Lemma 3.2.3 and the triangle inequality, we obtain

\[
| (I)_1 | \leq C h^\frac{3}{2} \| g \|_{H^2(\Gamma)} \| L_h u - u_h \|_{H^1(\Omega)}
\]

\[
\leq C h^\frac{3}{2} \| g \|_{H^2(\Gamma)} \| u \|_X + C h^\frac{3}{2} \| g \|_{H^2(\Gamma)} \| e(t) \|_{H^1(\Omega)}
\]

\[
\leq C h^\frac{3}{2} \| g \|_{H^2(\Gamma)} \| u \|_X + C h^3 \| g \|_{H^2(\Gamma)}^2 + \frac{1}{4} \| e(t) \|_{H^1(\Omega)}^2
\]

\[
\leq C h^2 (\| u \|_X^2 + \| g \|_{H^2(\Gamma)}^2) + \frac{1}{4} \| e(t) \|_{H^1(\Omega)}^2. \tag{3.3.5}
\]

In the last inequality, we used Young’s Inequality. Similarly, for \((I)_2\), using Lemma 2.2.1 and Lemma 3.2.3 to have

\[
| (I)_2 | \leq C h \| u_h \|_{H^1(\Omega)} \| L_h u - u - u_h \|_{H^1(\Omega)}
\]

\[
\leq C h \| u_h \|_{H^1(\Omega)} (\| L_h u - u \|_{H^1(\Omega)} + \| u - u_h \|_{H^1(\Omega)})
\]

\[
\leq C h^2 \| u_h \|_{H^1(\Omega)}^2 + C \| L_h u - u \|_{H^1(\Omega)}^2 + \frac{1}{4} \| u - u_h \|_{H^1(\Omega)}^2
\]

\[
\leq C h^2 \| u_h \|_{H^1(\Omega)}^2 + C h^2 \| u \|_X^2 + \frac{1}{4} \| e(t) \|_{H^1(\Omega)}^2. \tag{3.3.6}
\]

Then, the last term \((I)_3\) can be bounded by using Lemma 3.2.3

\[
| (I)_3 | \leq C h \| u \|_X \| e(t) \|_{H^1(\Omega)}
\]

\[
\leq C h^2 \| u \|_X^2 + \frac{1}{4} \| e(t) \|_{H^1(\Omega)}^2. \tag{3.3.7}
\]
Integrating the identity (3.3.4) from 0 to \(t\) and using the estimates (3.3.5)-(3.3.7), we obtain
\[
\int_0^t \|e\|_{H^1(\Omega)}^2 \, ds \leq C h^2 \int_0^t \|u\|_X^2 \, ds + C h^2 \int_0^t \|u_h\|_{H^1(\Omega)}^2 \, ds + \frac{3}{4} \int_0^t \|e\|_{H^1(\Omega)}^2 \, ds + \|u - L_h u\|_{L^2(\Omega)}^2,
\]
This, together with Lemma 3.2.3 and Lemma 3.3.1 completes the rest of the proof of Theorem 3.3.1. □

Proof of Theorem 3.3.2. For the \(L^2\) norm error estimate we shall use the parabolic duality trick. For any time \(t > 0\) and \(e = u - u_h\), let \(w(s) \in H^1_0(\Omega)\) and \(w_h(s) \in V_h\), respectively, be the solutions of the backward problems
\[
(\phi, w_s) - A(\phi, w) = (\phi, e) \quad \forall \phi \in H^1_0(\Omega), \quad s < t, \quad (3.3.8)
\]
\[
w(t) = 0;
\]
\[
(\phi_h, w_{hs}) - A_h(\phi_h, w_h) = (\phi_h, e) \quad \forall \phi_h \in V_h, \quad s < t, \quad (3.3.9)
\]
\[
w_h(t) = 0
\]
with \([w] = 0\) and \(g(x, t) = 0\) across the interface \(\Gamma\). From (3.3.8) and (3.3.9), we obtain
\[
(\phi_h, w_s - w_{hs}) - A(\phi, w - w_h) = A(\phi_h, w_h) - A_h(\phi_h, w_h) \quad (3.3.10)
\]
for all \(\phi_h \in V_h\). Following the standard argument of [34], it is easy to show that
\[
\int_0^t \|w_s - w_{hs}\|_{L^2(\Omega)}^2 \, ds \leq C \int_0^t \|e\|_{L^2(\Omega)}^2 \, ds. \quad (3.3.11)
\]
Further, we assume that the following identity
\[
\int_0^t \left( h^{-2} \|w - w_h\|_{H^1(\Omega)}^2 \right) \, ds \leq C \int_0^t \|e\|_{L^2(\Omega)}^2 \, ds \quad (3.3.12)
\]
holds true. The estimate (3.3.12) is obtained by reversing time in the proof of Theorem 3.3.1 and further using Theorem 3.2.1 for the problem (3.3.8)-(3.3.9). Set \(\phi = e\) in
(3.3.8). Then, using the identity (3.3.3), we obtain

\[
\|e\|_{L^2(\Omega)}^2 = (e, w_s) - A(e, w)
\]

\[
= (e, w_{hs}) + (e, w_s - w_{hs}) - A(e, w - w_h) - A(e, w_h)
\]

\[
= \frac{d}{ds}(e, w_h) + (e, w_s - w_{hs}) - A(e, w - w_h)
\]

\[
- (e_s, w_h) - A(e, w_h)
\]

\[
= \frac{d}{ds}(e, w_h) + (e, w_s - w_{hs}) - A(e, w - w_h)
\]

\[
+ \{A(u_h, w_h) - A_h(u_h, w_h)\} + \{(g_h, w_h)_{\Gamma_h} - (g, w_h)_{\Gamma}\}.
\]

With an aid of (3.3.10), the above equation may be rewritten as

\[
\|e\|_{L^2(\Omega)}^2 = \frac{d}{ds}(e, w_h) + (u - P_h u, w_s - w_{hs}) - A(u - P_h u, w - w_h)
\]

\[
+ (P_h u - u_h, w_s - w_{hs}) - A(P_h u - u_h, w - w_h)
\]

\[
+ \{A(u_h, w_h) - A_h(u_h, w_h)\} + \{(g_h, w_h)_{\Gamma_h} - (g, w_h)_{\Gamma}\}
\]

\[
= \frac{d}{ds}(e, w_h) + (u - P_h u, w_s - w_{hs}) - A(u - P_h u, w - w_h)
\]

\[
+ \{A(P_h u - u_h, w_h) - A_h(P_h u - u_h, w_h)\}
\]

\[
+ \{A(u_h, w_h) - A_h(u_h, w_h)\} + \{(g_h, w_h)_{\Gamma_h} - (g, w_h)_{\Gamma}\}
\]

\[
= \frac{d}{ds}(e, w_h) + (u - P_h u, w_s - w_{hs})
\]

\[
- A(u - P_h u, w - w_h) + (II)_1 + (II)_2,
\]

(3.3.13)

where \((II)_1 = A(P_h u, w_h) - A_h(P_h u, w_h)\) and \((II)_2 = \{(g_h, w_h)_{\Gamma_h} - (g, w_h)_{\Gamma}\}.

We integrate (3.3.13) from 0 to \(t\) to obtain

\[
\int_0^t \|e\|_{L^2(\Omega)}^2 ds = \int_0^t \{(u - P_h u, w_s - w_{hs}) - A(u - P_h u, w - w_h)\} ds
\]

\[
+ \int_0^t (II)_1 ds + \int_0^t (II)_2 ds
\]

\[
\leq \int_0^t \|u - P_h u\|_{L^2(\Omega)} \|w_s - w_{hs}\|_{L^2(\Omega)} ds
\]

\[
+ C \int_0^t \|u - P_h u\|_{H^1(\Omega)} \|w - w_h\|_{H^1(\Omega)} ds
\]

\[
+ \int_0^t (II)_1 ds + \int_0^t (II)_2 ds.
\]

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We, now use the Young’s inequality to obtain
\[
\int_0^t \|e\|_{L^2(\Omega)}^2 \, ds \leq \epsilon \int_0^t \left\{ \|w_s - w_h\|_{L^2(\Omega)}^2 + h^{-2}\|w - w_h\|_{H^1(\Omega)}^2 \right\} \, ds \\
+ \frac{C}{\epsilon} \int_0^t \left\{ \|u - P_h u\|_{L^2(\Omega)}^2 + h^2 \|u - P_h u\|_{H^1(\Omega)}^2 \right\} \, ds \\
+ \int_0^t (II)_1 \, ds + \int_0^t (II)_2 \, ds.
\]

Apply (3.3.11) and (3.3.12) to have
\[
\int_0^t \|e\|_{L^2(\Omega)}^2 \, ds \leq C \epsilon \int_0^t \|e(t)\|_{L^2(\Omega)}^2 \, ds + \frac{C}{\epsilon} \int_0^t \left\{ \|u - P_h u\|_{L^2(\Omega)}^2 + h^2 \|u - P_h u\|_{H^1(\Omega)}^2 \right\} \, ds \\
+ \int_0^t (II)_1 \, ds + \int_0^t (II)_2 \, ds. \tag{3.3.14}
\]

The term \((II)_1\) can be bounded by using Lemma 2.2.1 and Lemma 2.2.2 of Chapter 2
\[
|\!(II)_1\!| \leq Ch\|P_h u\|_{H^1(\Omega_f)} \|w_h\|_{H^1(\Omega_f)} \\
\leq Ch\|P_h u - u\|_{H^1(\Omega_f)} \|w_h\|_{H^1(\Omega_f)} + Ch\|u\|_{H^1(\Omega_f)} \|w_h\|_{H^1(\Omega_f)} \\
\leq Ch\|u - P_h u\|_{H^1(\Omega_f)} \|w_h\|_{H^1(\Omega_f)} + Ch^\frac{3}{2}\|u\|_{X} \|w - w_h\|_{H^1(\Omega_f)} \\
+ Ch^\frac{3}{2}\|u\|_{X} \|w - w_h\|_{H^1(\Omega_f)} + Ch^2\|u\|_{X} \|w\|_{X}.
\]

Integrating this identity from 0 to \(t\) and using Young’s inequality, we obtain
\[
\int_0^t |\!(II)_1\!| \, ds \leq Ch \int_0^t \|u - P_h u\|_{H^1(\Omega_f)} \|w - w_h\|_{H^1(\Omega_f)} \, ds \\
+ Ch^\frac{3}{2} \int_0^t \|u - P_h u\|_{H^1(\Omega_f)} \|w\|_{X}\, ds \\
+ Ch^\frac{3}{2} \int_0^t \|u\|_{X} \|w - w_h\|_{H^1(\Omega_f)} \, ds + Ch^2 \int_0^t \|u\|_{X} \|w\|_{X}\, ds \\
\leq \frac{C}{\epsilon} h^4 \int_0^t \|u - P_h u\|_{H^1(\Omega)}^2 \, ds + \frac{\epsilon}{2} h^{-2} \int_0^t \|w - w_h\|_{H^1(\Omega)}^2 \, ds \\
+ \frac{C}{\epsilon} h^3 \int_0^t \|u - P_h u\|_{H^1(\Omega)}^2 \, ds + \frac{\epsilon}{2} \int_0^t \|w\|_{X}^2 \, ds \\
+ \frac{C}{\epsilon} h^5 \int_0^t \|u\|_{X}^2 \, ds + \frac{\epsilon}{2} h^{-2} \int_0^t \|w - w_h\|_{H^1(\Omega)}^2 \, ds \\
+ \frac{C}{\epsilon} h^4 \int_0^t \|u\|_{X}^2 \, ds + \frac{\epsilon}{2} \int_0^t \|w\|_{X}^2 \, ds.
\]

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Further, using the regularity result (cf. Theorem 3.2.1), (3.3.8) and (3.3.12), we obtain

\[ \int_0^t |(II)_1| ds \leq C h^3 \int_0^t \| u - P_h u \|^2_{H^1(\Omega)} ds + C \epsilon \int_0^t \| e \|^2_{L^2(\Omega)} ds \]

\[ + \frac{C}{\epsilon} h^4 \int_0^t \| u \|^2_X ds. \]  \( (3.3.15) \)

Finally, Lemma 3.2.1 and similar argument leads to

\[ \int_0^t |(II)_2| ds \leq C h^4 \int_0^t \| g \|^2_{H^2(\Gamma)} ds + C \epsilon \int_0^t \| e \|^2_{L^2(\Omega)} ds. \]  \( (3.3.16) \)

Thus, combining the estimates (3.3.15)-(3.3.16), together with (3.3.14) and Lemma 3.2.2 completes the rest of the proof of Theorem 3.3.2. □

Remark 3.3.1 The convergence results for the linear parabolic interface problems are also extended for the semilinear problems (cf. [18]) into the Brezzi-Rappaz-Raviart ([10]) framework.

3.4 Error Analysis for Fully Discrete Scheme

A fully discrete scheme based on backward Euler method is proposed and analyzed in this section. Optimal \( L^2(0,T; H^1(\Omega)) \) norm error estimate is obtained for fully discrete scheme. For the simplicity, we have assumed \( g(x,t) = 0 \).

We first partition the interval \([0, T]\) into \(M\) equally spaced subintervals by the following points

\[ 0 = t_0 < t_1 < \ldots < t_M = T \]

with \( t_n = nk, \ k = \frac{T}{M}, \) be the time step. Let \( I_n = (t_{n-1}, t_n] \) be the \( n \)-th subinterval. Now we introduce the backward difference quotient

\[ \Delta_k \phi^n = \frac{\phi^n - \phi^{n-1}}{k}, \]

for a given sequence \( \{\phi^n\}_{n=0}^M \subset L^2(\Omega) \).

The fully discrete finite element approximation to the problem (3.3.2) is defined as follows: For \( n = 1, \ldots, M \), find \( U^n \in V_h \) such that

\[ (\Delta_k U^n, v_h) + A_h(U^n, v_h) = (f^n, v_h) \ \forall v_h \in V_h \]  \( (3.4.1) \)
with $U^0 = L_h u_0$. For each $n = 1, \ldots, M$, the existence of a unique solution to (3.4.1) can be found in [11]. We then define the fully discrete solution to be a piecewise constant function $U_h(x, t)$ in time and is given by

$$U_h(x, t) = U^0(x) \quad \forall t \in I_n, \ 1 \leq n \leq M.$$ 

We now prove the main result of this section in the following theorem.

**Theorem 3.4.1** Let $u$ and $U$ be the solutions of the problem (4.1.1)-(4.1.3) and (4.5.1), respectively. Assume that $U^0 = L_h u_0$ and $u_0$ is sufficiently smooth. Then there exists a constant $C$ independent of $h$ and $k$ such that

$$\|U(t_n) - u(t_n)\|_{L^2(\Omega)} \leq C(h^2 + k) \sum_{i=1}^2 \left\{ \|u^0\|_{H^2(\Omega_i)} + \|u_t\|_{L^2(0,T;H^2(\Omega_i))} + \|u_{tt}\|_{L^2(0,T;L^2(\Omega_i))} \right\}.$$ 

Proof. For simplicity of the exposition, we write $u^n = u(x, nk)$, $e^n = u^n - U^n$ and $w^n = u^n - P_h u^n$. Using (3.3.1) and (3.4.1), it follows that

$$(\Delta_k e^n, e^n) + A(e^n, e^n) = (\Delta_k e^n, w^n) + A(e^n, w^n) + (\Delta_k u^n - u^n_t, P_h u^n - U^n)$$

$$+ \left\{ A_h(U^n, P_h u^n - U^n) - A(U^n, P_h u^n - U^n) \right\}$$

$$= C \sum_{j=1}^4 I_j, \quad (3.4.2)$$

where

$$I_1 = (\Delta_k e^n, w^n), \quad I_2 = A(e^n, w^n), \quad I_3 = (\Delta_k u^n - u^n_t, P_h u^n - U^n)$$

$$I_4 = \left\{ A_h(U^n, P_h u^n - U^n) - A(U^n, P_h u^n - U^n) \right\}.$$ 

Summing (3.4.2) over $n$ from $n = 0$ to $n = M$, we have

$$\frac{1}{2} \|\Delta_k e^n\|_{L^2(\Omega)}^2 + k \sum_{n=0}^M A(e^n, e^n) + \frac{1}{2} \sum_{n=0}^M \|\Delta_k e^n\|_{L^2(\Omega)}^2 \leq \frac{1}{2} \|e^0\|_{L^2(\Omega)}^2$$

$$+ k \sum_{n=0}^M (I_1 + I_2 + I_3 + I_4). \quad (3.4.3)$$
Using Lemma 3.2.2 and Young's inequality, we obtain

\[ k \sum_{n=0}^{M} I_1 \leq C h^2 k \sum_{n=0}^{M} \|u^n\|_X^2 + \frac{k}{4} \sum_{n=0}^{M} \|\Delta k e^n\|_{L^2(\Omega)}^2. \]  

(3.4.4)

Similarly,

\[ k \sum_{n=0}^{M} I_2 \leq C(\epsilon) h^2 k \sum_{n=0}^{M} \|u^n\|_X^2 + \epsilon k \sum_{n=0}^{M} \|e^n\|_{H^1(\Omega)}^2. \]  

(3.4.5)

To estimate \( k \sum_{n=0}^{M} I_3 \), we first note that

\[ \Delta_k u^n - \frac{\partial u^n}{\partial t} = -\frac{1}{k} \int_{t_{n-1}}^{t_n} (s - t_{n-1}) u_{ss}(s) ds. \]

and hence using Lemma 3.2.2, we obtain

\[ k \sum_{n=0}^{M} I_3 \leq C k^2 \|u_{tt}\|_{L^2(0,T;L^2(\Omega))} + C h^2 k \sum_{n=0}^{M} \|u^n\|_X^2 + k \sum_{n=0}^{M} \|e^n\|_{L^2(\Omega)}^2. \]  

(3.4.6)

Using Lemma 2.2.1, we obtain

\[ k \sum_{n=0}^{M} I_4 \leq C h k \sum_{n=0}^{M} \left\{ \|U^n\|_{H^1(\Omega)} \|P_h u^n - U^n\|_{H^1(\Omega)} \right\} \]

\[ \leq C h k \sum_{n=0}^{M} \left\{ \|U^n\|_{H^1(\Omega)}^2 + \frac{\epsilon}{4} \|P_h u^n - U^n\|_{H^1(\Omega)}^2 \right\} \]

\[ \leq C h k \sum_{n=0}^{M} \|e^n\|_{H^1(\Omega)}^2 + C h k \sum_{n=0}^{M} \|u^n\|_X^2. \]  

(3.4.7)

In the last inequality, we have used Lemma 3.2.2. Combining (3.4.3)-(3.4.7) and using the standard kickback argument, we arrive at

\[ \|e^M\|_{L^2(\Omega)}^2 + k \sum_{n=0}^{M} \|e^n\|_{H^1(\Omega)}^2 \leq C k^2 \|u_{tt}\|_{L^2(0,T;L^2(\Omega))}^2 + Ch \left( k \sum_{n=0}^{M} \|u^n\|_X^2 \right) \]

\[ + C k \sum_{n=0}^{M} \|e^n\|_{L^2(\Omega)}^2. \]

For sufficiently small \( k \), we obtain

\[ \|e^M\|_{L^2(\Omega)}^2 + k \sum_{n=0}^{M} \|e^n\|_{H^1(\Omega)}^2 \leq C k^2 \|u_{tt}\|_{L^2(0,T;L^2(\Omega))}^2 + Ch \left( k \sum_{n=0}^{M} \|u^n\|_X^2 \right) \]

\[ + C k \sum_{n=0}^{M-1} \|e^n\|_{L^2(\Omega)}^2. \]
An application of discrete version of Gronwall’s lemma leads to
\[
\|e^M\|_{L^2(\Omega)}^2 + k \sum_{n=0}^{M} \|e^n\|_{H^1(\Omega)}^2 \leq Ck^2 \|u_t\|_{L^2(0,T;L^2(\Omega))}^2 + Ch \left( k \sum_{n=0}^{M} \|u^n\|_{X}^2 \right), \tag{3.4.8}
\]
Finally, by a simple calculation we have
\[
\|u - U_h\|_{L^2(0,T;H^1(\Omega))} \leq Ck \|u_t\|_{L^2(0,T;Y)} + C \left( k \sum_{n=0}^{M} \|u^{n+1} - U^{n+1}\|_{H^1(\Omega)}^2 \right)^{\frac{1}{2}}, \tag{3.4.9}
\]
Since \(k = O(h)\), (3.4.9) combine with (3.4.8) leads to the desired result. \(\Box\)