Cascade System with Random Attenuation Factor

2.1 Introduction

Cascade system were first developed and studied by Pandit and Sriwastav (1975). Cascade reliability is a special kind of standby system for stress-strength models. In a standby system, i.e. a system with standby redundancy, there are a number of components only one of which works at a time and the others remain as standbys. When an impact of stress exceeds the strength of the active component, for the first time, it fails and another from standbys, if there is any, is activated and faces the impact of stresses. The system fails when all the components have failed. In cascade system, the strength of the components are independent and stress on a component is $k$ times the stress on its preceding component, called attenuation factor.

Attenuation factor is generally assumed to be a constant for all the components or to be a parameter having different fixed values for different components (Pandit and Sriwastav, 1975). But an attenuation factor may be a random variable also (Pandit and Sriwastav, 1978). Here we have considered an $n$-cascade system for which the attenuation factor ‘$K$’ is a random variable. Then, instead of talking about the reliability of the system we talk about the expected reliability of the system. The expectation is taken w.r.t. the distribution of $K$. Much of the materials of this chapter are based on Doloi and Borah (2010).

The organization of this chapter is as follows: In Section-2.2, the problem is formulated in mathematical terms. In Section-2.3, a simple case is considered when $K$ is a
uniform r.v. in (0,1). In Sub-Sections 2.3.1, 2.3.2 and 2.3.3, all the distributions are assumed to be exponential, Weibull and exponential-gamma respectively to evaluate the expressions of the unconditional reliability of the system. Some graphs are plotted for each case in Section-2.4. To observe the change in the values of reliabilities with parameters involved, some numerical values of reliabilities are tabulated against the parameters involved in the Table 2.1, Table 2.2 and Table 2.3 (cf. Appendix) and results and discussions are given in Section 2.5.

2.2 Mathematical Formulation

Let us consider an \( n \)-cascade system and suppose that \( n \) components are numbered from 1 to \( n \) in their order of activation. Let \( X_i \) be the strength of the \( i \)\(^{th} \) component, in the order of activation, and when activated faces the stress \( Y_i \), \( i = 1, 2, ..., n \). For a cascade system with attenuation factor ‘ \( K \)’ (considered to be an integer random variable)

\[
Y_i = K^{i-1}Y_1, i = 1, 2, ..., n
\]

The reliability of the system is given by

\[
R_n(K) = R(1, K) + R(2, K) + ... + R(n, K)
\]

where \( R(i, K) = P[X_1 < Y_1, X_2 < K Y_1, ..., X_{i-1} < K^{i-1}Y_1, X_i > K^i Y_1], i = 2, 3, ..., n \)

Now, if the attenuation factor, say ‘ \( K \)’, is a r.v. then \( R_n(K) \) is the conditional reliability of the system on the condition that \( K = k \). Let \( h(k) \) be the density function of \( K \). Then, unconditional reliability of the system is given by the expected value of \( R_n(K) \), where expectation is taken w.r.t \( h(k) \),

\[
\text{i.e. Unconditional reliability of the } n \text{-cascade system } = E[R_n(K)] = \sum_{i=1}^{n} E[R(i, K)]
\]
where \[ E[R(i, K)] = \int_{-\infty}^{\infty} R(i, k)h(k)dk \] (2.2.5)

Thus, when the attenuation factor is a r.v. we talk of expected reliability of the system instead of the reliability of the system.

We may note that \( R(1, K) \) is independent of \( 'K' \) and hence

\[ E[R(1, K)] = R(1, K) = R(1), \text{ say} \] (2.2.6)

2.3 The Distribution of \( K \) is Uniform

We consider here the case when \( 'K' \) follows a uniform distribution in \((0,1)\), i.e.

\[ h(k) = 1, \quad 0 \leq k \leq 1 \] (2.3.1)

Then from (2.2.5)

\[ E[R(i, K)] = \int_{0}^{1} R(i, k)dk, \quad i = 1, 2, \ldots n \] (2.3.2)

The strength \( (X) \) and stress \( (Y) \) may follow any distribution. We consider here only three cases viz., (1) when \( X_i \) and \( Y_i \) both are exponential variates, (2) when \( X_i \) and \( Y_i \) both are Weibull variates and (3) when \( X_i \)'s are exponential variates and \( Y_i \)'s are gamma variates.

2.3.1 Exponential Stress-Strength

Suppose \( X_i \) and \( Y_i \) are i.i.d. as exponential variates with mean \((1/\lambda)\) and \((1/\mu)\), respectively. Then for \( \rho = \frac{\lambda}{\mu} \)

\[ R(1) = \frac{1}{1 + \rho}, \quad \text{(Independent of \( K \))}, \] (2.3.3)
\[ R(2, K) = \left( \frac{1}{1 + \rho K} \right) - \left( \frac{1}{1 + \rho + \rho K} \right) \]  
\[ R(3, K) = \left( \frac{1}{1 + \rho K^2} \right) - \left( \frac{1}{1 + \rho + \rho K^2} \right) - \left( \frac{1}{1 + \rho \rho K + \rho K^2} \right) + \left( \frac{1}{1 + \rho \rho K + \rho K^2} \right) \]  
\[ (2.3.4) \]

So, from (2.2.5), (2.3.4), and (2.3.5) after some simple calculations we get

\[ E[R(2, K)] = \frac{1}{\rho} \log \left( \frac{(1 + \rho)^2}{(1 + 2 \rho)} \right) \]  
\[ (2.3.6) \]

and

\[ E[R(3, K)] = \rho^{-1/2} \left[ -\frac{1}{\sqrt{1 + \rho}} \tan^{-1} \left( \frac{\rho}{1 + \rho} \right)^{1/2} - \frac{2}{\sqrt{4 - \rho}} \tan^{-1} \left( \frac{\rho}{4 - \rho} \right)^{1/2} + \frac{2}{\sqrt{4 + 3 \rho}} \tan^{-1} \left( \frac{\rho}{4 + 3 \rho} \right)^{1/2} \right] \]  
\[ (2.3.7) \]

For \( i \geq 4 \), closed form expression could not be obtained. Of course, one may use numerical integrations.

Then, the unconditional reliability (or expected reliability), \( R_2 \) for a 2- cascade system, from (2.2.4), is given by

\[ R_2 = R(1) + E[R(2, K)] \]  
\[ (2.3.8) \]

where, \( R(1) \) and \( E[R(2, K)] \) are given by (2.3.3) and (2.3.6), respectively.

Similarly, the unconditional reliability \( R_3 \) for a 3- cascade system, is given by
\[ R_3 = R_2 + E[R(3, K)] \tag{2.3.9} \]

A few values of \( R_1, R_2 \) and \( R_3 \) are tabulated in Table 2.1 (cf. Appendix) for different values of \( \rho \).

### 2.3.2 Weibull Stress-Strength

Suppose \( X_i \) and \( Y_i \) are Weibull distribution with p.d.f.'s

\[
\begin{align*}
    f(x) &= cx^{-1} \exp\left\{-\left(\frac{x}{\theta}\right)^c\right\}/\theta^c, \quad x > 0 \\
    g(y) &= ay^{-1} \exp\left\{-\left(\frac{y}{\lambda}\right)^a\right\}/\lambda^a, \quad y > 0
\end{align*}
\]

Then

\[
\begin{align*}
    R(1) &= \int_0^\infty \exp\left\{-t + \left(\frac{\lambda}{\theta}\right)^c t^{c/a}\right\}dt, \tag{2.3.10} \\
    R(2) &= \int_0^\infty \exp\left\{-t + \left(\frac{\lambda k}{\theta}\right)^c t^{c/a}\right\}dt - \int_0^\infty \exp\left\{-t + \left(\frac{\lambda}{\theta}\right)^c + \left(\frac{\lambda k}{\theta}\right)^c\right\}t^{c/a}dt \tag{2.3.11}
\end{align*}
\]

So from (2.2.5) and (2.3.11) we get

\[
E[R(2, k)] = \left[ \int_0^\infty \int_0^\infty \exp\left\{-t + \left(\frac{\lambda k}{\theta}\right)^c t^{c/a}\right\}dt - \int_0^\infty \exp\left\{-t + \left(\frac{\lambda}{\theta}\right)^c + \left(\frac{\lambda k}{\theta}\right)^c\right\}t^{c/a}dt \right]dk \tag{2.3.12}
\]

For \( i \geq 3 \) closed form expression could not be obtained. Using the Gauss Laguerre Integration method and Trapezoidal rule we have evaluated \( R(1) \) and \( E[R(2, K)] \) for different values of \( c, \theta, a \) and \( \lambda \).

Then the unconditional reliability, \( R_2 \) is given by (2.3.8).

A few values of \( R_1 \) and \( R_2 \) are tabulated in Table 2.2 (cf. Appendix) for different values of \( c, \theta, a, \lambda \).


2.3.3 Exponential Strength and Gamma Stress

Suppose, $X_i$ for all $i$, are i.i.d. exponential variates with mean $\frac{1}{\lambda}$ and $Y_i$ for all $i$, are i.i.d. gamma variates with scale parameter unity and degrees of freedom $l$, respectively. Then,

$$R(1) = \frac{1}{(1+\lambda)^l} \quad (2.3.13)$$

$$R(2, K) = \frac{1}{(1+\lambda K)^l} - \frac{1}{(1+\lambda + \lambda K)^l} \quad (2.3.14)$$

So, from (2.2.5) and (2.3.14), after some simple calculations we get,

$$E[R(2, k)] = \frac{1}{\lambda(-l+1)} \left[2(1+\lambda)^{-l+1} - 1 - (1+2\lambda)^{-l+1}\right] \quad (2.3.15)$$

Then the unconditional reliability $R_2$ can be easily obtained by the expression (2.3.8).

For some particular values of $l$ and $\lambda$ we have tabulated the values of $R_1$ and $R_2$ in Table 2.3 (cf. Appendix).

2.4 Graphical Representations

Some graphs are plotted in Fig. 2.1 to 2.3 by taking different parameters along the horizontal axis and the corresponding reliability along the vertical axis for different parametric values. Fig. 2.1 signifies that $R_2$ decreases steadily with increasing $\rho$. These graphs may be used to read the intermediate values directly. For example, for $\rho = 0.25$, we get from the graph, $R_2 = 0.9643$ whereas by actual calculation we get $R_2 = 0.9633$. The difference is only $0.10\%$. Fig. 2.2 represents the curves of $R_1$ which were drawn against $c$ for different parameter values of $\theta, a$ and $\lambda$. From these graphs we get $R_1 = 0.8542$. For
$c=1.5, \theta=10, a=6, \lambda=2$ while the computed value is $R_1=0.8536$. The difference is only 0.06%. Taking $l$ along the horizontal axis and the corresponding $R_2$ along the vertical axis graphs are plotted for different values of $\lambda$ in Fig. 2.3. One can read the values of $R_2$ for intermediate values of $l$, from these graphs. Thus, for $\lambda=0.2$ we get $R_2=0.7731$ for $l=5$ from graphical extrapolation, while the computed value is $R_2=0.7716$. The difference is only 0.15%.

**Fig. 2.1** Exponential Stress-Strength: Graph for $R_2$
Fig. 2.2 Weibull Stress-Strength: Graph for \( R_1 \) for different fixed values of \( \theta, a \) and \( \lambda \) i.e. \( R_1(\theta, a, \lambda) \).

Fig. 2.3 Exponential Strength and Gamma Stress: Graph for \( R_2 \) for different fixed values of \( \lambda \) i.e. \( R_2(\lambda) \).
2.5 Results and Discussions

For some specific values of the parameters involved in the expressions of $R_i$, $i=1,2,3$ we evaluate $R_1$, $R_2$, $R_3$ for exponential distribution for different values of $\rho$ and $R_1$, $R_2$ for Weibull and exponential-gamma distributions from their expressions obtained in Sub-Section 2.3.1-2.3.3. The computation is carried out using the software Matlab 6.

Table 2.1 (cf. Appendix) presents a few values of $R_1$, $R_2$, $R_3$ for different values of the parameter $\rho$ for exponential distribution. From the table it is clear that reliabilities decreases with increasing values of $\rho$.

A few values of $R_1, R_2$ are tabulated for Weibull distribution, in Table 2.2 (cf. Appendix) for different values of the parameter $c, \theta, a, \lambda$. Here the change in the values of reliability is as expected. The increase in the values of shape parameter increases the reliability. But increase in the values of scale parameter decreases the reliability.

Table 2.3 (cf. Appendix) presents some values of $R_1$ and $R_2$ for exponential-gamma distributions. Here the values of $R_1$ and $R_2$ decreases with increasing values of $l$ and $\lambda$ which is expected as well.