Chapter 5

A Cascade Model with Random Parameters

5.1 Introduction

Most of the discussions of interference models assume that the parameters of stress and strength distributions are constants (Beg, 1980, Enis and Geisser, 1971, Harris and Singpurwalla, 1968, Kelley et al., 1976 etc.). But in many cases this assumption may not be true and the parameters may be assumed themselves (parameters) to be random variables. In other words, the distributions with fixed parameters may not represent the stress and or strength distributions adequately; distributions with random parameters may model the situations better. For example, if a particular component, having certain strength distribution is manufactured in different lots, for a particular lot the parameters of the strength distribution may remain fixed but may vary randomly from lot to lot. In such situations the parameters of the strength distribution may themselves be taken as random variables. Similarly, stress applied on a component (or system) is due to different factors such as vibration, pressure, temperature, humidity etc. Generally one of these factors will be dominant and will be the main cause for the stress on the component and stress distribution will be the distribution of this factor. But the other factors may vary at different time or at different places in such a way that, though they may not alter the form of the distribution, they may bring random fluctuations in the values of the parameters of the stress distribution. For example, solutions corrosive action may be highly influenced by variation in its temperature (Kakati, 1983) and hence the distribution of stress (corrosive action) may have different parametric values which vary randomly with temperature or in other words, the stress parameter may be taken as a random variable.
Further, if a prior knowledge exists about the parameters involved, it will be a waste of available data if we do not use a random parameter model i.e. a Bayesian model. In order to use the Bayesian approach the subjective information must be quantified and represented in the form of a prior distribution of the parameter concerned (Kapur and Lamberson, 1977).

Harris and Singpurwalla (1968) have derived an unconditional time to failure distribution by assuming that a parameter of the failure distribution (viz. exponential or Weibull) is a random variable. Using the derived compound distribution and Bayesian techniques they have estimated the system’s reliability. They considered two-point, uniform and gamma distributions as prior distributions. Krishnamoorthy et al. (2007) proposed in estimating the parameter $R$ which is referred to as the reliability parameter. i.e. $R = (X > Y)$, where its strength $X$ and stress $Y$ are independent random variables.

Shooman (1968) has assumed that the parameter of strength distribution is a deterministic function of time. Tarman and Kapur (1975) have assumed that the parameters of the stress-strength distributions are variables but not random variables. They have found optimal values of the parameters involved subject to resource and design constraints.

In this chapter we have calculated the system reliability of a 2-cascade system when the parameters of the stress-strength distributions are random variable. Here stress and strength are considered to be exponential random variable with parameters, say $\lambda$ and $\mu$, respectively. We further assume that either $\lambda$ or $\mu$ is a random variable with a known prior. The following two cases are considered:

- When strength parameter is random but stress parameter is a constant.
- When stress parameter is random but strength parameter is a constant.

The prior distributions considered for stress-strength parameters, in all the above two cases, are either uniform or two-point distributions.
In Section 5.2 the general model is formulated. In Section 5.3 to 5.4 the above two cases are considered. The reliabilities of a cascade system are evaluated in each case. For all the above cases, some numerical values of these reliabilities are tabulated in the Table 5.1, Table 5.2, Table 5.3 and Table 5.4 (cf. Appendix) for different set of values of the parameters. To make the things clear, a few graphs are drawn in Section 5.5 for selected values of the parameters. Results and discussions are devoted to Section 5.6.

5.2 Notations and Formulation of the Model

Here we have assumed that strength ‘X’ and stress ‘Y’ are exponential variates with means $1/\lambda$ and $1/\mu$, respectively. The parameters $\lambda$ and $\mu$ may be random. Let $X_i$’s be i.i.d. with distribution function $F(x)$ and let p.d.f of $Y_i$ be $g(y_i)$

$$R(r) = \int_{-\infty}^{\infty} F(y_1) F(ky_2) F(k^2y_3) \ldots F(k^{r-1}y_1) g(y_1) dy_1$$  \hspace{1cm} (5.2.1)

Let

$P(\lambda), p(\lambda) = \text{The prior distribution function and p.d.f. (or p.m.f.) of the random strength parameter } \lambda$.

$Q(\mu), q(\mu) = \text{The prior distribution function and p.d.f. (or p.m.f.) of the random stress parameter } \mu$.

$f(x/\lambda) = \text{The conditional p.d.f. of the r.v. } X \text{ for a given value of } \lambda$.

$g(y/\mu) = \text{The conditional p.d.f. of the r.v. } Y \text{ for a given value of } \mu$.

$f_x(x) = \text{The unconditional p.d.f. of the r.v. } X$.

$g_y(y) = \text{The unconditional p.d.f. of the r.v. } Y$.

Here $f(x/\lambda) = f_x(x)$ if $\lambda$ is constant and $g(y/\mu) = g_y(y)$ if $\mu$ is constant.
Now

\[ f_X(x) = \int_{-\infty}^{\infty} f(x|\lambda) dP(\lambda) \quad (5.2.2) \]

\[ g_Y(y) = \int_{-\infty}^{\infty} g(y|\mu) dQ(\mu) \quad (5.2.3) \]

Then from (5.2.1)

\[ R(1) = \int_{-\infty}^{\infty} F_X(y_1) g_Y(y_1) dy_1 \quad (5.2.4) \]

\[ R(2) = \int_{-\infty}^{\infty} F_X(y_1) F_Y(ky_1) g_Y(y_1) dy_1 \quad (5.2.5) \]

\section*{5.3 Random Strength Parameter}

Suppose the components under study are manufactured in different lots so that its strength distribution may be represented by distribution with random parameters. But are used in the same environment exerting similar stresses, of course with random fluctuations, i.e. stress distribution is having constant parameter. As already assumed stress-strength are exponential with parameter \( \mu \) and \( \lambda \), so in this case the strength parameter \( \lambda \) is a r.v. whereas the stress parameter \( \mu \) remains constant.

i.e.

\[ f(x|\lambda) = \lambda e^{-\lambda x} \quad \text{and} \quad g_Y(y) = \mu e^{-\mu y} \]

Two types of prior distributions are considered for \( \lambda \):

(a) uniform and (b) two-point

\subsection*{(a) Uniform Prior for \( \lambda \)}

In a situation where each lot is homogeneous within itself but different lots may have different values of \( \lambda \) and taking all the possible sources of the lots together, each value of \( \lambda \)
appears equally frequently, a uniform prior distribution will be suitable for $\lambda$ (Harris and Singpurwalla, 1968).

Let $\lambda$ be uniformly distributed in the range $(a, b)$

$$i.e., \ p(\lambda) = \frac{1}{(b-a)}, \ a < \lambda < b$$

Then the unconditional p.d.f of $X$ is given by

$$f_{\lambda}(x) = \frac{1}{(b-a)} \int_a^b \lambda e^{-\lambda x} d\lambda$$

Hence,

$$R(1) = \int_0^\infty \left[ \int_0^y \frac{1}{b-a} \int_a^b \lambda e^{-\lambda x} d\lambda \ dx \right] \mu e^{-\mu y} dy$$

$$= \frac{\mu}{b-a} \log \frac{b+\mu}{a+\mu}$$

$$R(2) = \int_0^\infty F_X(y_1) F_X(ky_1) g_Y(y_1) dy_1$$

$$= \int_0^\infty \left[ 1 - \frac{1}{b-a} \frac{e^{-a y_1} - e^{-b y_1}}{y_1} \right] \left[ \frac{1}{b-a} \frac{e^{-ak y_1} - e^{-bk y_1}}{ky_1} \right] \mu e^{-\mu y_1} dy_1$$

$$= \frac{\mu}{k(b-a)} \log e \frac{bk + \mu}{ak + \mu} - \frac{\mu}{k(b-a)^2} \int_0^\infty \frac{1}{y^2} \left[ e^{-y(a+ak+\mu)} - e^{-y(a+ak+\mu)} - e^{-y(ak+b+\mu)} + e^{-y(ak+b+\mu)} \right] dy$$

$$= \frac{\mu}{k(b-a)} \log e \frac{bk + \mu}{ak + \mu} - \frac{\mu}{k(b-a)^2} \eta_1$$

where $\eta_1 = \int_0^\infty \frac{1}{y^2} \left[ e^{-y(a+ak+\mu)} - e^{-y(a+ak+\mu)} - e^{-y(a+ak+\mu)} + e^{-y(a+ak+\mu)} \right] dy$

The expression $\eta_1$ can be evaluated numerically.

Then the reliability $R_2$ for a 2-cascade system, from the equation (3.2.8), is given by
\[ R_2 = R(1) + R(2) \]

**Table 5.1 (cf. Appendix)** gives a few values of \( R_1 \) and \( R_2 \) for different values of parameters \( \mu, a, b \) and attenuation factor \( k \).

**(b) Two-Point prior for \( \lambda \)**

In a situation where it is known that \( \lambda \) can take two only values \( \lambda_1 \) and \( \lambda_2 \) (say) with probabilities \( p \) and \((1 - p)\), respectively, a two point prior distribution for \( \lambda \) is appropriate (Harris and Singpurwalla, 1968).

Let \( \lambda \) have a two-point distribution, given by,

\[ \Pr(\lambda = \lambda_1) = p(\lambda_1) \quad \text{and} \quad \Pr(\lambda = \lambda_2) = p(\lambda_2) \]

Then

\[ f_x(x) = \sum_{i=1}^{2} f(x/\lambda_i) p(\lambda_i) = p\lambda_1 e^{-\lambda_1 x} + (1 - p)\lambda_2 e^{-\lambda_2 x} \quad (5.3.1) \]

Hence

\[
R(1) = \left( \frac{p}{\lambda_1 + \mu} + \frac{1-p}{\lambda_2 + \mu} \right) \mu \\
R(2) = \frac{p\mu}{\mu + \lambda_1 k} - \frac{p^2 \mu}{\mu + \lambda_1 + \lambda_2 k} - \frac{\mu p(1-p)}{\mu + \lambda_1 + \lambda_2 + \lambda_k} + \frac{\mu(1-p)}{\mu + \lambda_1 + \lambda_2 k} - \frac{\mu(1-p)^2}{\mu + \lambda_1 + \lambda_2 + \lambda_2 k}
\]

Then \( R_2 = R(1) + R(2) \)

A few values of \( R_1 \) and \( R_2 \) are tabulated in **Table 5.2** (cf. Appendix) for different values of parameters \( \mu, p, \lambda_1, \lambda_2 \) and attenuation factor \( k \).

**5.4 Random Stress Parameter**

The situation may be opposite of that considered in Section 5.3. That is, the components might have come from the same lot or otherwise also the strength parameters
may not vary from one lot to another. In other words, the parameter of component’s strength distribution remains constant. But as discussed earlier the stress on the component may not suitably be represented by a constant parameter distribution. So we shall consider a stress distribution with random parameters. Let, as in Section 5.3, \( X \) and \( Y \) be both exponential with parameter \( \lambda \) and \( \mu \) i.e. stress and strength are exponential with mean \( 1/\mu \) and \( 1/\lambda \) respectively. But now \( \mu \) is a random variable and \( \lambda \) remains constant.

i.e. \( g(y/\mu) = \mu e^{-\mu y} \) and \( f_X(x) = \lambda e^{-\lambda x} \)

For \( \mu \) also the prior distributions considered are uniform and two-point distributions, respectively, in the following sub-sections.

(a) Uniform Prior for \( \mu \)

Similar situations, as that in case of \( \lambda \), may be envisaged for the use of this distribution as the prior distribution for \( \mu \) also. For example, the components may be working in such an environment where the values taken by \( \mu \) in a given range are equally likely, and then we can assume a uniform distribution for \( \mu \).

Let \( \mu \) be distributed uniformly in the range \((c,d)\), then

\[
q(\mu) = \frac{1}{d-c}, \quad c \leq \mu \leq d
\]

Then,

\[
g_y(y) = \frac{1}{d-c} \int_c^d \mu e^{-\mu y} d\mu
\]

Hence,

\[
R(1) = 1 - \frac{\lambda}{d-c} \log \left( \frac{d + \lambda}{c + \lambda} \right)
\]

\[
R(2) = \frac{\lambda}{d-c} \left[ k \log \left( \frac{d + \lambda k + \lambda}{c + \lambda k + \lambda} \right) + \log \left( \frac{d + \lambda k + \lambda}{c + \lambda k + \lambda} \right) - k \log \left( \frac{d + \lambda k}{c + \lambda k} \right) \right]
\]

Then \( R_2 = R(1) + R(2) \)
For different values of the parameters the reliabilities of $R_1$ and $R_2$ are tabulated in Table 5.3 (cf. Appendix).

(b) **Two-Point prior for $\mu$**

Similar to Section 5.3(b), if it is known that $\mu$ can take only two values $\mu_1$ and $\mu_2$ with probabilities $q$ and $(1 - q)$, respectively, then we have two-point prior distributions for $\mu$ given as,

\[
\Pr(\mu = \mu_1) = q(\mu_1) \quad \text{and} \quad \Pr(\mu = \mu_2) = q(\mu_2)
\]

Then,

\[
g_y(y) = \sum_{j=1}^{2} g(y \mid \mu_j) q(\mu_j) = q\mu_1 e^{-\mu_1 y} + (1 - q)\mu_2 e^{-\mu_2 y}
\]

Hence,

\[
R(1) = \frac{q\mu_1}{\mu_1 + \lambda} + \frac{(1 - q)\mu_2}{\mu_2 + \lambda}
\]

\[
R(2) = q\mu_1 \left( \frac{1}{\mu_1 + \lambda k} - \frac{1}{\mu_1 + \lambda + \lambda k} \right) + (1 - q)\mu_2 \left( \frac{1}{\mu_2 + \lambda k} - \frac{1}{\mu_2 + \lambda + \lambda k} \right)
\]

and as usual $R_2 = R(1) + R(2)$

Table 5.4 (cf. Appendix) gives a few values of $R_1$ and $R_2$ for different values of parameters $\mu_1, \mu_2, \lambda, q$ and attenuation factor $k$.

### 5.5 Graphical Representations

A few graphs of $R_1, R_2$ are drawn in **Fig. 5.1(a), Fig. 5.1(b), Fig. 5.2(a), Fig. 5.2(b), Fig. 5.3(a), Fig. 5.3(b), Fig. 5.4(a) and Fig. 5.4(b)** for different parametric values involved. From these graphs one can read directly the values of reliabilities $R_1, R_2$ for intermediate
values of $\mu, \lambda$ and $k$. In Fig. 5.1(a), Fig. 5.1(b), Fig. 5.2(a) and Fig. 5.2(b) reliabilities are steadily increasing with $\mu$ and $k$ whereas in Fig. 5.3(a), Fig. 5.3(b), Fig. 5.4(a) and Fig. 5.4(b) it is decreasing with increasing $\lambda$ and $k$.

Fig. 5.1(a) Graph of $R_1, R_2$ for exponential Stress-Strength: Strength parameter $\lambda$ is random and uniformly distributed in the range $[a, b)$.

Fig. 5.1(b) Graph of $R_1, R_2$ for exponential Stress-Strength: Strength parameter $\lambda$ is random and uniformly distributed in the range $[a, b)$.

Fig. 5.2(a) Graph of $R_1, R_2$ for exponential Stress-Strength: Strength parameter $\lambda$ is random and has a two-point distribution.

Fig. 5.2(b) Graph of $R_1, R_2$ for exponential Stress-Strength: Strength parameter $\lambda$ is random and has a two-point distribution.
Fig. 5.3(a) Graph of $R_1, R_2$ for exponential Stress-Strength: Stress parameter $\mu$ is random and uniformly distributed in the range $(c, d)$

Fig. 5.3(b) Graph of $R_1, R_2$ for exponential Stress-Strength: Stress parameter $\mu$ is random and uniformly distributed in the range $(c, d)$

Fig. 5.4(a) Graph of $R_1, R_2$ for exponential Stress-Strength: Stress parameter $\mu$ is random and has a two-point distribution

Fig. 5.4(b) Graph of $R_1, R_2$ for exponential Stress-Strength: Stress parameter $\mu$ is random and has a two-point distribution
5.6 Results and Discussions

A few values of $R_1$ and $R_2$ are tabulated for the 1st case when strength parameter is random but stress parameter is a constant. Uniform distributions are considered as a prior distribution for $\lambda$ in Table 5.1 (cf. Appendix) for different values of $\mu, a, b$ and attenuation factor $k$. From the table we have seen that reliabilities are steadily increasing with $\mu$ and $k$ but decrease with increasing value of $a$ and $b$. Similarly in Table 5.2 (cf. Appendix), two-point distributions are considered as the prior distribution for $\lambda$ for the 1st case we have tabulated some values of $R_1$ and $R_2$ for different values of $\mu, p, \lambda_1, \lambda_2$ and attenuation factor $k$. Here also we have seen that the reliabilities are steadily increasing with $\mu$ and $k$.

For different values of $c, d, k$ and $\lambda$ we have tabulated the values of $R_1$ and $R_2$ in Table 5.3 (cf. Appendix) in the 2nd case when stress parameter is random but strength parameter is a constant and uniform distributions are considered as a prior distribution for $\mu$. The values of the reliability are on expected line. From the table we see that, increase in the values of $\lambda$ and $k$ decrease the reliability but reliabilities are steadily increasing with $c$ and $d$. Similarly, when two-point distributions are considered as the prior distribution for $\mu$ in the 2nd case we have tabulated some values of $R_1$ and $R_2$ for different values of $\mu_1, \mu_2, \lambda, q$ and attenuation factor $k$ in Table 5.4 (cf. Appendix). Here also we have seen that the reliabilities are decreasing with increasing values of $q, \lambda$ and $k$.

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