Chapter 5

GLOBALLY HYPERBOLIC SPACE-TIMES AND QUANTUM EFFECTS NEAR SINGULARITY

1. Introduction :

In this Chapter we attempt at a generalisation of the results on quantum effects in the vicinity of space-time singularities obtained in Chapters 3 and 4. The generalisation has been carried out in the following way : In the first place, we consider a very general class of space-times which includes many of the known solutions of Einstein's equations such as collapsing dust ball, Robertson-Walker models, the general cosmological scenario of Belinskii et al. In particular, we do not assume the existence of any Killing vectors and hence no symmetries are present in space-time. Thus, inhomogeneous and anisotropic situations are included. Next, as in Chapter 4 we no longer assume that $S_m = 0$ but the stress-energy tensor is chosen to be that of non-interacting material particles. Since we take the metric potentials, g_{ik} to have a general form, we must expect the occurrence of a fairly general class of space-time singularities as predicted by singularity theorems (see Chapter 2.) Accordingly, we deal with strong curvature singularities, a precise definition of which will be given later. This ensures that singularities we deal here are not fictitious ones arising out of coordinate pathologies but are genuine, in the sense of diverging curvatures. Again, as in the previous Chapters, we quantize only the conformal fluctuations : $g_{ik} \longrightarrow \Omega^2 g_{ik}$, where $0 < \Omega < \infty$.

The state of the system is represented by a wave function Ψ which is allowed to have any general form subject to the conditions such as $\int \Psi \Psi^* dV = 1$ etc. For simplicity of presentation we give the calculations throughout with $\Omega = \Omega(t)$. However, we shall explain later as to how our conclusions can be generalised when $\Omega = \Omega(x^i)$, $i = 0,1,2,3$. In previous cases we have seen that quantum effects turn out to be governed by a non-linear differential equation. Our main task here will be to generate the same for the general scenario mentioned above and then to analyse this equation in the neighbourhood of the space-time singularity.

The next Section describes the general classical space-time and important steps of the quantization procedure. In Section 3 we study the exact situation of homogeneous and isotropic cosmologies in some detail which gives an idea of the structure arising for the general case in the next Section. In Section 4 we discuss the notion of a strong curvature singularity and study the evolution of quantum effects in its vicinity. In the final Section we extend the results to $\Omega = \Omega(x^i)$ and the implications are explained.

2. Classical space-time and quantisation :

First, we specify the general classical background which is supposed to be a solution of Einstein's equations. Choosing the world lines of matter as timelike geodesics and using the proper time along these to define the time coordinate, the metric can be written as,

$$d\bar{s}^2 = dt^2 - 2\bar{g}_{oa}\, dt\, dx^a - \bar{g}_{ab}\, dx^a\, dx^b \qquad (5\text{-}2.1)$$

where a,b = 1,2,3 denote the space coordinates. Without loss of generality, we can take $\bar{g}_{oa} = 0$, i.e. the models have no rotation. In the above, we choose $\bar{g}_{ab} = \bar{g}_{ab}(x^i)$, i.e. the metric coefficients are functions of both space and time coordinates. This class generalises the Belinskii et al.

models considered in Chapter 4, because we assume no conditions for $\bar{g}_{ab}$. In particular, inhomogeneities and anisotropies are included. Even though the local observations seem to imply that universe is homogeneous and isotropic, global extension of the same must be considered as big extrapolation without any evidence. Thus, this may be regarded as a useful generalisation. In fact, (5-2.1) covers a very wide class of space-times which includes basically all interesting non-static globally hyperbolic solutions.

As discussed earlier, the action $\bar{S}_g$ transforms as,

$$\bar{S}_g \longrightarrow \frac{1}{16\pi} \int_{\mathcal{V}} \left[(1+\phi)^2 \bar{R} - 6\phi_i \phi^i \right] \sqrt{-\bar{g}}\, d^4x \tag{5-2.2}$$

Taking now S_m in the form of a system of particles a,b... we have

$$\bar{S}_m = \sum_a \int m_a \, d\bar{s}_a \tag{5-2.3}$$

where the particle masses are m_a, m_b ... etc. Then,

$$\bar{S}_m \longrightarrow \sum_a \int m_a \, \Omega \, d\bar{s}_a = \bar{S}_m + \sum_a \int m_a \phi \, ds_a \tag{5-2.4}$$

Using (5-2.2) and (5-2.4) it is seen that the total action transforms as,

$$\bar{S} \longrightarrow S = \frac{1}{16\pi} \int_{\mathcal{V}} \left[\bar{R}\,(1 + \phi^2) - 6\phi_i\, \phi^i \right] \sqrt{-\bar{g}}\, d^4x \tag{5-2.5}$$

We can foliate the space-time by means of a sequence of space-like hypersurfaces $\mathcal{S}_t$ and without loss of generality we can take the singularity to be at $t = 0$. We can also consider the case when the space-time contains singularities arising out of local collapse, in addition to global cosmological singularities [1], and our considerations can be generalised to include this situation as well. Then (5-2.5) can be written as,

$$S = \frac{1}{16\pi} \int_{\mathcal{V}} \left[(1 + \phi^2)\, h_1(t) - \dot{\phi}^2\, h_2(t) \right] dt \tag{5-2.6}$$

where a dot denotes differentiation with respect to time and we have introduced the notation,

$$h_1(t) = \int_{\mathcal{V} \cap \mathcal{S}_t} \bar{R}\, \sqrt{-\bar{g}}\, d^3x \tag{5-2.7(a)}$$

$$h_2(t) = 6 \int_{\mathcal{V} \cap \mathcal{S}_t} \sqrt{-\bar{g}}\, d^3x \tag{5-2.7(b)}$$

Here we have taken $\phi = \phi(t)$ and $\mathcal{J}_t$ is a member of the sequence of the Cauchy surfaces or partial Cauchy surfaces approaching the singularity. The functions $h_1(t)$ and $h_2(t)$ represent spatial integration carried out on a finite region of space. We shall discuss the behaviour of these functions later.

Then, from (5-2.6), the Lagrangian for the ensemble with the conformal degree of freedom can be written as,

$$L = \frac{1}{16\pi}\left[(1 + \phi^2)\, h_1(t) - \dot{\phi}^2\, h_2(t)\right] \qquad (5\text{-}2.8)$$

The associated conjugate momentum is given by,

$$p = \frac{\partial L}{\partial \dot{\phi}} = -\frac{1}{8\pi}\,\dot{\phi}\, h_2(t) \qquad (5\text{-}2.9)$$

We can now write down the Hamiltonian for the system,

$$H = \dot{\phi}p - L = -\frac{4\pi p^2}{h_2(t)} - \frac{1}{16\pi}(1 + \phi^2)\, h_1(t) \qquad (5\text{-}2.10)$$

after eliminating $\dot{\phi}$ using (5-2.9).

The dispersion from the value $\phi = 0$ basically denotes the quantum effects in our considerations. Again, as earlier, we define,

$$\chi = \langle \phi^2 \rangle - \langle \phi \rangle^2$$
$$\omega = \langle p^2 \rangle - \langle p \rangle^2$$

In particular, we would like to derive the behaviour of χ in the limit of approach to the classical singularity and then interpret the same.

Let $\Psi(\phi, t)$ be a general wave function representing the state of the system. We again assume that the classical state $\phi = 0$ remains, at all times the average for the quantum description. That is, at time t,

$$\langle \phi_t \rangle = \int_{-\infty}^{\infty} \bar{\Psi}(\phi_t)\, \phi_t\, \Psi(\phi_t)\, d\phi_t = 0 \tag{5-2.11}$$

Then, we have

$$0 = \frac{d}{dt} \langle \phi \rangle = i \int \bar{\Psi} [H\phi - \phi H] \Psi \, d\tau \tag{5-2.12}$$

which gives, after using the operator representation $p = (1/i)(\partial/\partial\phi)$ the result.

$$0 = \frac{-8\pi\langle p\rangle}{h_2(t)} \qquad (5\text{-}2.13)$$

Thus, $\langle p\rangle = 0$ at all times and we have

$$\chi = \langle\phi^2\rangle \quad , \omega = \langle p^2\rangle \quad (5\text{-}2.14)$$

We note that the classical metric (5-2.1) obtained as a solution to the Einstein's equations represents the singular situation corresponding to $\phi = 0$. The other states in the quantum ensemble need not satisfy the same and could well be non-singular. If we define an operator by $A = \phi^2$, then $\chi = \langle A\rangle$ and χ can be evolved in time using the evolution equation (2-6.4). Successive applications of (2-6.4) yield a non-linear differential equation for χ which can be analysed to deduce the behaviour of χ. (To be precise, this will be an inequality for the case of a general wave function satisfying $\chi\omega \geq 1/4$).

However, before dealing with the general situation given by (5-2.1), we will first apply the procedure given above to the exact case of homogeneous and isotropic cosmological models. The wave function of the universe will be represented by a completely general wave function $\Psi(\phi, t)$. We hope that an analysis of this case may yield insights for the general situation to be dealt with later.

3. Friedmann-Robertson-Walker Cosmological spacetimes :

Here the assumptions of homogeneity and isotropy globally fix up the geometry of the universe in a unique manner. In the comoving coordinates (t, r, θ, ϕ) the metric is written as,

$$d\bar{s}^2 = dt^2 - Q^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \qquad (5\text{-}3.1)$$

where $Q(t)$ is the scale factor for the universe and $k = \pm 1$ or 0. The picture emerging here is that of three dimensional spacelike hypersurfaces evolving in time and an inevitable curvature singularity at $t = 0$. In this case, the ten Einstein's equations reduce to only two :

$$3\left(\frac{\dot{Q}^2 + k}{Q^2}\right) = 8\pi T^0_0 = 8\pi\rho \qquad (5\text{-}3.2(a))$$

$$2\ \frac{\ddot{Q}}{Q} + \frac{\dot{Q}^2 + k}{Q^2} = 8\pi\ T^1{}_1 = 8\pi\ T^2{}_2 = 8\pi\ T^3{}_3$$

$$= -\ 8\pi\ P \qquad (5\text{-}3.2(b))$$

Here, a dot denotes differentiation with respect to t and, ρ and P denote the density and pressure of the fluid respectively.

Using (5-2.5) and integrating over space gives

$$S = \bar{S} - \frac{3V}{8\pi} \int_{t_1}^{t_2} (\dot{\phi}^2 - \frac{1}{6}\bar{R}\,\phi^2)\ Q^3\ dt \qquad (5\text{-}3.3)$$

where V is the coordinate volume of the space-time region between $r = r_1$ and $r = r_2$. We also have here $\bar{R} = \bar{R}(t)$, $Q = Q(t)$ and $\phi = \phi(t)$. In terms of the scale function $Q(t)$, the scalar curvature $\bar{R}(t)$ can be expressed as,

$$\bar{R} = 6\left[\frac{\ddot{Q}}{Q} + \frac{\dot{Q}^2}{Q^2}\right] = \frac{3\ddot{X}}{X} \qquad (5\text{-}3.4)$$

where we have introduced a new variable $X = Q^2$.

Then (5-3.3) implies that the Lagrangian for the system is given by,

$$L = -\frac{3V}{8\pi}\left[\dot{\phi}^2 \; X^{3/2} - \frac{1}{2}\ddot{X} \; X^{1/2}(1+\phi^2)\right] \quad (5\text{-}3.5)$$

Now, defining the canonically conjugate momentum p as $\partial L/\partial\dot{\phi}$ and the Hamiltonian $H = \dot{\phi}p - L$, we obtain

$$H = k_1 \, X^{-3/2} \; p^2 + k_2 \ddot{X} X^{1/2} \, (1+\phi^2) \qquad (5\text{-}3.6(a))$$

with

$$k_1 = -\frac{2\pi}{3V}, \; k_2 = -\frac{3V}{16\pi} \qquad (5\text{-}3.6(b))$$

A comparision of (5-3.6) with (5-2.10) shows the special forms assumed by the functions $h_1(t)$ and $h_2(t)$ in this particular situation. We will have an occasion to come back to this point later.

The commutators of the variables ϕ and p with the Hamiltonian H are given by,

$$[\phi, H] = i \frac{\partial H}{\partial p} = 2\, i\, k_1\, X^{-3/2}\, p \qquad (5\text{-}3.7(a))$$

$$[p, H] = -i \frac{\partial H}{\partial \phi} = -2\, i\, k_2\, \ddot{X}\, X^{1/2}\, \phi \qquad (5\text{-}3.7(b))$$

Writing $A = \phi^2$, (2-6.4) gives

$$\frac{d\chi}{dt} = 2\, k_1\, X^{-3/2} \langle p\,\phi + \phi\, p \rangle \qquad (5\text{-}3.8)$$

where we have used $\langle \partial A/\partial t \rangle = 0$. Again taking

$$\frac{d\chi}{dt} = \langle B \rangle \text{ , } B = 2k_1\, X^{-3/2}\, (p\phi + \phi p) \qquad (5\text{-}3.9)$$

and another application of (2-6.4) yields

$$i \frac{d}{dt} \langle B \rangle = \langle [B,H] \rangle + i \left\langle \frac{\partial B}{\partial t} \right\rangle = i \frac{d^2\chi}{dt^2} \qquad (5\text{-}3.10)$$

Evaluating individual terms and substituting in (5-3.10) gives the following second order differential equation,

$$\frac{d^2\chi}{dt^2} + \frac{3\dot{X}}{2X} \frac{d\chi}{dt} + \frac{\ddot{X}}{X}\chi = \frac{8k_1^{\,2}\omega}{X^3} \qquad (5\text{-}3.11)$$

If the state of the universe were represented by a Gaussian wave packet, then the mean square deviations χ and ω are related by $\chi\omega = 1/4$, at the initial instance and later one has the inequality $\chi\omega \geqslant 1/4$ as the wave packet 'spreads' with time. Using this relation in (5-3.11) gives

$$\frac{d^2\chi}{dt^2} + \frac{3\dot{X}}{2X}\frac{d\chi}{dt} + \frac{\ddot{X}}{X}\chi \geqslant \frac{2k_1^2}{X^3\chi} \qquad (5\text{-}3.12)$$

The evolution of χ in the past is governed by (5-3.12). To determine this, we recall that $X = Q^2$ is a known function of time which is specified by the Friedman equations (5-3.2). Taking $P = 0$ and $\rho \propto Q^{-3}$, these can be written as

$$\ddot{Q} = -\frac{C}{6Q^2}, \quad \dot{Q}^2 = \frac{C}{3Q} - k, \quad C > 0 \qquad (5\text{-}3.13)$$

where $k = \pm 1, 0$. (We note that taking the pressure $P \neq 0$, which may be the case at the earlier epochs of the universe, would not qualitatively alter the situation). Then (5-3.12) becomes

$$\chi''\left(\frac{C}{3Q} - k\right) - \chi'\left(\frac{3k}{Q} - \frac{5C}{6Q^2}\right) + 2\left(\frac{C}{6Q^3} - \frac{k}{Q^2}\right) \geqslant \frac{2k_1^2}{\chi Q^6}$$

(5-3.14)

where a prime for χ denotes differentiation with respect to Q. Here we will not carry out a detailed analysis of (5-3.14) as we will do a similar job for the general case in the next section. However, we can have an idea of the main features implied by (5-3.14). An observation of the above inequality suggests that a power law expression may be tried for χ. Let us take $\chi = \alpha Q^{\beta}$, then (5-3.14) reduces to

$$\left[\frac{C}{3}\left(\beta^{2}+\frac{3}{2}\beta+1\right)-k\;Q\left(\beta^{2}+2\beta+2\right)\right]Q^{2\beta+3} \geqslant \frac{2k_{1}^{2}}{\alpha^{2}} \tag{5-3.15}$$

Since the right hand side of (5-3.15) is positive, it is clear that β must be negative in order for the inequality to be satisfied in the limit of approach to the singularity, i.e. as $Q \rightarrow 0$. It is not difficult to see that (5-3.15) will be satisfied only when $\beta < -\frac{3}{2}$. Thus, we conclude that as the classical singularity is approached, the quantum uncertainty must diverge atleast as fast as $Q^{-3/2}$ or faster. We also note that effect of the curvature term k becomes negligible in this limit and hence the above conclusion is valid independently of whether the universe is flat, open or closed.

Before closing this discussion, we may recall that our arguments here are independent of the form of the wave function. In fact, as illustrated by the work of Parker and Fulling [2] , the arguments based on a specific form such as a Gaussian wave function might sometimes lead to features like 'particle creation' at late times. They neglected the time dependence of the mass and frequency terms in the Hamiltonian in the vicinity of the singularity and a representation of the state of the system was made in terms of standard harmonic oscillator states. Then, a Gaussian packet of constant width at late times is expressed at early times as a superposition of harmonic oscillator states with an expected particle number which diverges in the limit of the original late time approaching infinity. This feature was attributed by Parker and Fulling to an inappropriate choice of the present time state. It is thus highly desirable to have conclusions regarding quantum evolution of the universe which do not involve restrictive assumptions such as Gaussian form for the wave function of the universe. It is one of the main purpose of the present work to achieve this. However, we should like to add that our approach here to quantisation is basically different as compared to above, in the sense that, whereas Parker and Fulling quantise massive scalar fields in a fixed background of closed Friedmann models, we quantise here a geometric degree of freedom of the spacetime. Again, our conclusions are valid for closed, open or flat ($k = \pm 1, 0$) Friedmann-Robertson-Walker models.

4. General spacetimes :

We would like to consider the general class of inhomogeneous and anisotropic spacetimes as given by (5-2.1). The singularity theorems predict that even such spacetimes must contain singularities in the form of nonspacelike geodesic incompleteness provided very general conditions such as causality, non-negativity of energy and formation of trapped surface etc. are satisfied. However, these are purely existence theorems which provide no information on the nature or structure of these singularities. In general, such singularities need not be physically important or interesting. For example, if one removes a point from the Minkowski spacetime, the resulting spacetime will be geodesically incomplete. However, no physical quantities such as the components of the curvature tensor etc. will diverge at such a singularity and hence no quantum effects will presumably arise. Again, there are examples of non-trivial exact solutions such as the axisymmetric Weyl spacetimes which are singular in the above sense. However the components of the Riemann tensor, when measured in a parallelly propagated frame, remain bounded along curves running into singularity [3] . Again such singularities may not be considered 'genuine' in the above sense. In fact, there exists a whole classification of singularities of various types and genuine curvature singularities are to be factored out from the same. The quantum effects will be presumably important only in the vicinity of the later which we would like to study. Thus, we first make this notion of curvature singularity precise in the next subsection.

4.1 Strong curvature singularities :

Only those singularities may be considered genuine which are physically all embracing curvature singularities in the sense that all observers, whether freely falling or accelerated, falling within the singularity are destroyed by infinite tidal forces. The curvature scalars and the Riemann tensor components should grow unboundedly along the trajectories falling into the singularity. Following Tipler [4] and Clarke and Krolak [5], we can formulate this notion in the following manner :

Let $\lambda(t)$ be a causal geodesic; i.e. $\lambda(t)$ can be timelike or null. Suppose $\lambda(t)$ is incomplete at affine parameter value t = 0. Let K be the tangent vector to $\lambda(t)$ and $\mu(t) = z_1 \wedge z_2 \wedge z_3$ be a volume form defined along $\lambda(t)$ where z_1, z_2, z_3 are linearly independent Jacobi vectors orthogonal to K. (If $\lambda(t)$ is null then $\mu(t)$ is defined as a 2-form). A real-valued map from the space of all such 3-forms can be defined by $\Delta(A \wedge B \wedge C) = \det [A^i, B^i, C^i]$. We denote $\Delta(\mu(t))$ by V(t), which is a volume element along $\lambda(t)$ and is independent of the choice of basis. The singularity at t = 0 is then called a strong curvature singularity if V(t) = 0 in the limit as $t \to 0$ for all possible $\mu(t)$, i.e. for all possible choices of linearly independent Jacobi fields.

This definition effectively captures the notion that all objects falling into a strong curvature singularity are crushed to zero volume. Specifically, necessary and sufficient conditions for the occurrence of strong curvature singularities are derived in [5] which are shown to involve tetrad components of the Riemann, Ricci and Weyl tensors and also the divergence of their integrals along nonspacelike geodesics running into the singularity. In particular, it is shown in [5] that an incomplete geodesic $\gamma(t)$ cannot define a strong curvature singularity unless either the Weyl or Ricci tensor components diverge sufficiently fast and a sufficient condition for the same to occur is $R_{44} \geqslant K/t^2$ for some fixed constant K, along a timelike geodesic.

To fix the ideas, we would like to consider the case $R_{44} = K/t^2$ in some detail. It is sometimes convenient to define a length scale y associated with the volume $V(t)$ by defining $y^3 = V$. The propagation equation for $y(t)$ near the singularity along $\gamma(t)$ is the Raychaudhuri equation [3] which is a second order linear ordinary differential equation,

$$\frac{d^2y}{dt^2} + \frac{1}{3}(R_{44} + 2\sigma^2) = 0 \qquad (5\text{-}4.1)$$

Here $\sigma^2 = \sigma_{ij}\sigma^{ij}$ corresponds to the trace of the shear tensor σ_{ij}. (For the null case, similar equation holds with $\frac{1}{3}$ replaced by $\frac{1}{2}$. We omit these details). Writing $F(t) = \frac{1}{3}(R_{44} + 2\sigma^2)$ and ignoring the effects of the shear tensor (which will any way enhance the focussing effect that we are considering) we can take $F(t) = A/t^2$

where $A > 0$ is some fixed constant. It is clear that $\gamma(t)$ defines a strong curvature singularity and (5-4.1) can be integrated to give,

$$\left(\frac{dy}{dt}\right)_t = \left(\frac{dy}{dt}\right)_{t_1} + \int_t^{t_1} y(t)\, F(t)\, dt \qquad (5\text{-}4.2)$$

where $t < t_1$. If we try a solution of the form $y = t^{\alpha}$ with $F(t)$ of the form above, we obtain the condition

$$A = \alpha - \alpha^2 \qquad (5\text{-}4.3)$$

Since $V(t) \rightarrow 0$ in the limit of singularity $t = 0$, we must have $\alpha > 0$. Again $A > 0$ and (5-4.3) implies that $0 < \alpha < 1$. Further, solving (5-4.3) for α gives that in order for α to be real A must satisfy the condition $A \leqslant \frac{1}{4}$. The solution for y is then given by,

$$y = t^{\left[1 \pm (1 - 4A)^{1/2}\right]/2} \qquad (5\text{-}4.4)$$

Thus, depending on the value of A it is seen that $y \sim t^{\alpha}$ with $\frac{1}{2} \leqslant \alpha \leq 1$. The volume $V(t)$ then goes to zero near the singularity at least as fast as $t^{3/2}$. We will recall these remarks at the end of the next subsection while discussing the behaviour of the quantum fluctuations near the singularity. From now on we assume that all singularities considered here are the strong curvature singularities in the sense defined above.

4.2 Evolution of quantum effects:

We now return to the general scenario described by (5-2.1). In order to evolve the quantum effects using (2-6.4), we first work out the commutators $[\phi, H]$ and $[p, H]$ using the Hamiltonian given by (5-2.10) :

$$[\phi, H] = i \frac{\partial H}{\partial p} = - i \frac{8\pi p}{h_2(t)} \quad (5\text{-}4.5(a))$$

$$[p, H] = -i \frac{\partial H}{\partial \phi} = i \frac{\phi h_1(t)}{8\pi} \quad (5\text{-}4.5(b))$$

Then,

$$[\phi^2, H] = - i \frac{8\pi}{h_2(t)} (p\phi + \phi p)$$

and we get,

$$\frac{d\chi}{dt} = \frac{-8\pi}{h_2(t)} \langle p\phi + \phi p \rangle \quad (5\text{-}4.6)$$

Defining a new operator B by $\langle B\rangle = d\chi/dt$ we would like to use again (2-6.4). Thus

$$[B, H] = \frac{-8\pi}{h_2(t)} [p\phi + \phi p, H]$$

and again using the relations (5-4.5a,b) this can be evaluated as

$$[B, H] = -\frac{2ih_1(t)}{h_2(t)} \phi^2 + \frac{128\pi^2 i}{h_2{}^2(t)} p^2 \qquad (5\text{-}4.7)$$

Next,

$$\frac{\partial B}{\partial t} = \frac{8\pi}{h_2{}^2(t)} \dot{h}_2(t)(p\phi + \phi p)$$

and using (5-4.6)

$$\left\langle\frac{\partial B}{\partial t}\right\rangle = -\frac{\dot{h}_2(t)}{h_2(t)} \frac{d\chi}{dt} \qquad (5\text{-}4.8)$$

Substituting (5-4.7) and (5-4.8) in (2-6.4) gives the required differential equation for χ:

$$\frac{d^2\chi}{dt^2} + \frac{\dot{h}_2(t)}{h_2(t)} \frac{d\chi}{dt} + 2 \frac{h_1(t)}{h_2(t)}\chi = \frac{128\pi^2\omega}{h_2^2(t)} \qquad (5\text{-}4.9)$$

Again, the quantum state of the universe is represented by a general wave function $\Psi = \Psi(\phi, t)$ and χ and ω satisfy $\chi\omega \geqslant 1/4$. When Ψ has the wave packet form then the equality holds and $\chi\omega = 1/4$. In the following we assume equality which is easily generalised as we will see later. It is convenient to write $t = 1/r$ to get,

$$h_1(t) = h_1(1/r) = g_1(r)$$

$$h_2(t) = h_2(1/r) = g_2(r) \qquad (5\text{-}4.10)$$

$$\dot{h}_2(t) = g_2'(r)\,\frac{dr}{dt} = -\,r^2\,g_2'(r)$$

where a prime denotes differentiation with respect to r. With this substitution,(5-4.9) is written as,

$$g_2^{\,2}(r)\,\frac{\chi''}{\chi} + g_2^{\,2}(r)\left(\frac{2}{r} + \frac{g_2'(r)}{g_2(r)}\right)\frac{\chi'}{\chi} + \frac{2g_1(r)\,g_2(r)}{r^4} = \frac{32\pi^2}{\chi^2 r^4}$$

(5-4.11)

It is now necessary to consider the nature of the functions $h_1(t)$ and $h_2(t)$ as defined by (5-2.7a,b) and appearing in (5-4.11). For the Friedmann-Robertson-Walker case considered in Section 3, they have the form

$$h_1(t) = 3V \ddot{X} X^{1/2} = 6VQ(\dot{Q}^2 + Q\ddot{Q}) \qquad (5\text{-}4.12a)$$

$$h_2(t) = 6VQ^3 \qquad (5\text{-}4.12b)$$

For $k = 0$ models, $Q \sim t^{2/3}$ and we get $h_1(t)$ to be a constant and $h_2(t) \sim t^2$. Thus, the product $h_1(t) \cdot h_2(t)$ vanishes in the limit of approach to the singularity. A similar behaviour will be seen for the case of collapsing dust ball to be discussed later. In fact, the function $h_2(t)$ is a special case of the volume element $V(t)$ defined in Section 4.1 along a timelike geodesic. This is because the three-form $\mu(t)$ can be regarded as the metric determinant when defined using a geodesically comoving coordinate system. Thus, $h_2(t) \longrightarrow 0$ in the limit of approach to the singularity. Again, we expect $h_1(t)$ to remain atleast finite in this limit. Hence, we assume that the term $2g_1(r)\, g_2(r)/r^4$ vanishes in the limit as $t \longrightarrow 0$ and we ignore the same. The resulting equation from (5.4.11) can be simplified considerably if we define a new variable τ by,

$$\tau = -\int \frac{dr}{r^2 g_2(r)} \qquad (5\text{-}4.13)$$

and the non-linear differential equation (5-4.11) reduces to the form

$$\chi \frac{d^2\chi}{d\tau^2} = A \qquad (5\text{-}4.14)$$

where $A = 32\pi^2$. The equation (5-4.14) again is a non-linear equation; however it is possible to analyse the same using a method similar to that used in [6]. Writing $d\chi/d\tau = R$ we can integrate (5-4.14) to obtain,

$$h_2(t)\ \frac{d\chi}{dt} = \pm\ (2A \log\chi + c_1)^{1/2} \qquad (5\text{-}4.15)$$

At the initial epoch $t = t_1$ the situation is highly classical however, as $t \longrightarrow 0$, the action S is of the order $\hbar$ and in general χ increases. Thus, in general χ is a decreasing function of t and we choose the negative sign in (5-4.15). Writing $z = (2A \log\chi + c_1)^{1/2}$ and using a series expansion for e^{z^2} as in Chapter 3, one can again integrate (5-4.15) to obtain finally,

$$\left| \int \frac{dt}{h_2(t)} \right| \leqslant (A \log\chi + c_1)^{1/2}\,\chi \qquad (5\text{-}4.16)$$

As pointed out above, clearly $h_2(t) \longrightarrow 0$ in the limit of approach to the classically singular epoch. The integral on the left will diverge if $h_2(t)$ goes to zero atleast as fast as t. For the case of homogeneous and isotropic cosmologies, $h_2(t) \sim t^2$ as seen above. Further, for all known important examples of exact solutions such as the collapsing dust ball, Bianchi models or the general cosmological scenario of Belinskii et al this condition is satisfied. Finally, we considered in Section 4.1 a general class of strong curvature singularities where $F(t) \sim 1/t^2$ and it was shown there that $V(t)$, which is proportional to $h_2(t)$, goes as $t^{3/2}$ near the singularity. Thus, we conclude that for almost all physically reasonable cases of strong curvature singularities, the integral on the left of (5-4.16) must diverge. Then (5-4.16) implies that the quantum uncertainty χ must diverge in the limit of approach to the classical singularity.

This result shows that the quantum effects, which were negligible initially, grow and become very important in the vicinity of spacetime singularity, as governed by (5-4.16) whose implications will be considered later.

It is useful to compare the above result with the general cosmological scenario of Belinskii et al (Chapter 3) which is a special case of the situation considered here. There we have $h_2(t) = Vt$, where V comes out of integration on space. Then (5-4.16) reduces to,

$$V^{-1} \; |\log t| \leq (A \log \chi + C_1)^{1/2} \chi \quad (5\text{-}4.17)$$

The left hand side in (5-4.17) diverges logarithmically near the singularity at $t = 0$ and hence χ diverges.

Finally, we note that it is not difficult to generalise the situation when Ψ has a general form and $\chi\omega \geq 1/4$. Then, in place of (5-4.14), we get $\chi d^2\chi/d\tau^2 \geq A$. A similar procedure can again be carried out to arrive at (5-4.16).

4.3 <u>Gravitational Collapse Scenario</u>:

Here we would like to point out the relation of the results of Section 4.2 with the earlier work [6] on homogeneous and spherically symmetric collapse. We also indicate another important subcase relevant to inhomogeneous collapse.

The metric for the collapse of a homogeneous dust ball is given by

$$ds^2 = dt^2 - Q^2(t) \left[\frac{dr^2}{1 - \alpha r^2} + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \right] \qquad (5\text{-}4.18)$$

Here (r, θ, ϕ) are the comoving coordinates for dust particles and the parameter α is related to the initial density ρ_o of the object by $\alpha = 8\pi G \rho_o/3$. The functions $h_1(t)$ and $h_2(t)$ are then worked out to be,

$$h_1(t) = 8\pi \rho_o \, V \qquad (5\text{-}4.19a)$$

$$h_2(t) = 6Q^3 \, V \qquad (5\text{-}4.19b)$$

In fact $Q = \frac{3}{2} \alpha \, t^{2/3}$ in this case and we again get $h_2(t) \sim t^2$ in the limit as $t \rightarrow 0$. Using (5-4.19 a,b) we can now substitute in (5-4.9) and writing $X = Q^2$ we get,

$$\frac{d^2\chi}{dt^2} + \frac{3}{2} \frac{\dot{X}}{X} \frac{d\chi}{dt} + \frac{\alpha}{X^{3/2}} \chi = \frac{2\omega}{k_1^2 X^3} \qquad (5\text{-}4.20)$$

where $k_1 = -3V/4\pi$. This is actually equation (29) of [6] and thus the special case of homogeneous collapse is contained as a special case in the present results.

Next, considerable attention has been paid recently to the case of inhomogeneous collapse and in particular on Tolman-Bondi spacetimes in connection with the issues of naked singularities and cosmic censorship [7] . As far as the strengths of the singularities are concerned, those arising in the spherically symmetric and homogeneous case discussed above may be considered to be the strongest. In cases where inhomogeneities and other perturbations from symmetry are important, the singularities should be weaker in general. The rate of convergence of the quantity $h_2(t) = \int \sqrt{-g}\, d^3x$ in a way characterises this strength and it should converge slower than t^2 for general case. The form

$$h_2(t) = A\, t^{\alpha} \quad , \alpha > 0 \qquad (5\text{-}4.21)$$

then represents a general class of singularities weaker than the homogeneous collapse case, reducing to the same for $\alpha = 2$. Then (5-4.9) can be solved again using procedures similar to those in Section 4.2 and we obtain,

$$\frac{1}{t^{\alpha-1}} + C_1 \leq \frac{2}{K_1} (K_1 \log \chi + C_2)^{1/2} \chi \quad (5\text{-}4.22)$$

Thus, whenever $\alpha > 1$, the quantum uncertainty must diverge in the limit of approach to the singularity $t \to 0$.

5. Discussion and extensions:

In this final section we discuss several extensions and possible implications which we believe are suggested more or less directly by the present work.

Through out we have presented the calculations with the assumption $\phi = \phi(t)$. However, it is not difficult to generalise the conclusions to the situation when the fluctuation ϕ is both a function of space and time coordinates. So, far the expectation values $\bar{\chi} = \langle \bar{\phi}^2 \rangle$ are evaluated over all possible configurations (constants) and given by,

$$\langle \bar{\phi}^2 \rangle = \int_{-\infty}^{\infty} \Psi \, \bar{\phi}^2 \, \bar{\Psi} \, d\bar{\phi} \qquad (5\text{-}5.1)$$

when $\bar{\phi}$ belongs to the real line. Suppose now, for example, that the state of the system is described by a discrete set of variables $\phi^{(1)}, \phi^{(2)}, \ldots, \phi^{(n)}$ rather than a single variable $\bar{\phi}$. The average value of an observable F at a time t is then given by

$$\langle F \rangle = \int \ldots \int \Psi(\phi^{(1)}, \ldots, \phi^{(n)}) \, F \, \bar{\Psi}(\phi^{(1)}, \ldots, \phi^{(n)}) d\phi^{(1)} \ldots d\phi^{(n)} \quad (5\text{-}5.2)$$

Now, for a transition to the situation $\phi = \phi(x^i)$, a continuum limit is to be taken in (5-5.2), as is normally done in evaluating path integrals [8] , and we have

$$\langle F \rangle = \int \Psi(\phi) \, F \, \bar{\Psi}(\phi) \, \mathcal{D}\phi(x^i) \qquad (5\text{-}5.3)$$

Here $\mathcal{D}$ denotes a measure on the space of all the functions ϕ on a given hypersurface $\mathcal{G}_t$. We can think of $\phi(x^i)$ as a 'field' with infinitely many degrees of freedom. The infinitely many numbers $\phi(x^i)$, obtained by varying x^i, characterise the state of the system.

We have earlier evaluated the action S and defined the functions $h_1(t)$, $h_2(t)$ etc. by integrating over a finite range of variables in space. This was done at a constant time t, integrating over a compact region $\mathcal{V}$ in the spacelike hypersurface $\mathcal{G}_t$. A limit was taken then, as the sequence of hypersurfaces approach the singular epoch. Define now,

$$\bar{\phi} = \min_{\mathcal{G}_t \cap \mathcal{V}} \phi(t, x^\mu) \qquad (5\text{-}5.4)$$

Evaluating such minimums over all possible functions at a given time t, we have an entire range of constants given by $-1 < \bar{\phi} < \infty$. Clearly, we always have,

$$\bar{\phi} \leq \phi\,(x^i) \qquad (5\text{-}5.5)$$

When the state is described by a finite set of variables $\phi^{(1)}, \ldots, \phi^{(n)}$, then it is clear from (5-5.2) that whenever $F \leq G$ for two functions F and G, then $\langle F \rangle \leq \langle G \rangle$. In the continuum limit given by (5-5.3) there is generally an ambiguity concerning the question of measure. However, in this case also, we would expect that for any reasonable definition of the measure $\mathcal{D}$ we must have $\langle F \rangle \leq \langle G \rangle$ whenever $F \leq G$ pointwise. Then from (5-5.5) because $\bar{\phi}^2 \leq \phi^2$, we obtain that $\bar{\chi} \leq \chi$ where $\bar{\chi}$ and χ are the expectation values of $\bar{\phi}^2$ and ϕ^2 respectively. Since we have shown earlier that $\bar{\chi}$ diverges in the limit of approach to the classical singularity, it follows that χ also must diverge in the same limit. This recovers our earlier result on quantum effects when $\phi = \phi(x^i)$.

When $\phi = \phi(x^i)$, it also appears possible to use the canonical formalism for quantisation of fields [8].

The Lagrangian here is written as

$$L \equiv \int_{-\infty}^{\infty} d^3x \; \mathcal{L}\left(\phi, \frac{\partial\phi}{\partial x^{\mu}}\right) \qquad (5\text{-}5.6)$$

where $\mathcal{L}$ is the Lagrangian density. The Hamiltonian density is then defined by $\mathcal{H} = \pi\dot{\phi} - \mathcal{L}$ where $\pi(t, x^{\mu})$ are the associated momenta. The operators can then be evolved in time using equations similar to (2-6.4).

It should be noted that our considerations here apply mainly to the case of evolving spacetimes, i.e. when g_{ik} depend on time as well. This assumption is contained in the statement that the functions h_1 and h_2 defined by (5-2.8 a,b) are explicit functions of time. This excludes important static situations such as that of a Schwarzschild space-time. For example, our conclusions regarding the diverging quantum uncertainty in the case of the collapsing dust ball apply only for the observer who is moving with the dust particles to collapse into the singularity. Whereas the interior metric is given by (5-4.18) for which all the conclusions on quantum uncertainties hold, the exterior metric will be given by the Schwarzschild solution. The curvature singularity there is described by the condition $r = 0$, the event horizon being at $r = 2m$. One needs then somewhat different techniques to analyse the quantum effects near the singularity at $r = 0$. Infact, in the next chapter we will discuss quantum effects near the Schwarzschild singularity $r = 0$ in detail.

REFERENCES

1. P.S. Joshi and J.V. Narlikar — 1986 — Int. Jour. Mod. Phys. 1, 243.

2. L. Parker and S.A. Fulling — 1973 — Phys. Rev. D 7, 2357.

3. A. Held (Ed.) — 1980 — In General Relativity and Gravitation, Plenum Press, New York.

4. F.J. Tipler — 1977 — Phys. Lett. A 64, 8.

5. C.J.S. Clarke and A. Krolak — 1986 — J. Geom. Phys. 2, 127.

6. P.S. Joshi and S.S. Joshi — 1987 — Gen. Rel. and Grav. 19, 1033.

7. D. Christodoulou — 1984 — Commu. Math. Phys. 93, 171.

8. R.P. Feynman and A. Hibbs — 1965 — 'Quantum Mechanics and Path Integrals', McGraw-Hill Book Co., New York.