Chapter 4

STUDY OF QUANTUM FLUCTUATIONS IN BELINSKII, KHALATNIKOV AND LIFSHITZ COSMOLOGIES

1. Introduction :

It is a well known fact that the present epoch of the universe is described satisfactorily by the Robertson-Walker models which have homogeneous and isotropic space-sections given by t=constant. These models have singularity at t = 0. It is possible that this solution can not give a true description of the universe in its earlier stages of evolution where the temperatures were high and radiation dominating. However, the question arises whether occurrence of singularity in the cosmological models is only because of high degree of symmetry of the solution or it is an inherent property of classical general relativity. Of course to derive the most general solution of Einstein's field equations for non-empty space-times is an impossible task. But that does not obstruct our investigation to analyse the behaviour of the space-time near singularity. This problem has been studied in detail by Belinskii et al [1] and it is shown that there exists a general solution which has a physical singularity at some finite time in the past. Here physical singularity means the divergence of the density of matter and of the invariants of four dimensional curvature tensor.

In the next Section we will study the behaviour of the B-K-L solution near the space-time singularity. The quantization procedure is applied to B-K-L solution in Section 3 and we prove that the quantum fluctuations blow up near the singularity $t = 0$. In Section 4 we generalise the above result for a more general metric which includes B-K-L solution as a special case.

2. The B-K-L space-time :

The Kasner solution [2] corresponding to flat homogeneous but anisotropic empty space is given by

$$ds^2 = dt^2 - t^{2p_1} dx^2 - t^{2p_2} dy^2 - t^{2p_3} dz^2 \qquad (4\text{-}2.1)$$

Here

$$-\tfrac{1}{3} \leqslant p_1 \leqslant 0, \quad 0 \leqslant p_2 \leqslant \tfrac{2}{3}, \quad \tfrac{2}{3} \leqslant p_3 \leqslant 1 \qquad (4\text{-}2.2)$$

and

$$p_1 + p_2 + p_3 = 1, \quad p_1^2 + p_2^2 + p_3^2 = 1 \qquad (4\text{-}2.3)$$

The numbers p_1, p_2, p_3 can be represented in the parametric form

$$p_1(u) = \frac{-u}{1+u+u^2},\ p_2(u) = \frac{1+u}{1+u+u^2},\ p_3(u) = \frac{u(1+u)}{1+u+u^2}$$

(4-2.4)

It is clear that the linear distances along y and z axes increase while along the x-axis it decreases. Here we note that metric (4-2.1) has a singularity at point t = 0 and it remains an approximate solution for small t and in the case of uniformly distributed matter. The generalization of the Kasner metric is given by B-K-L [1] which can be written as

$$ds^2 = dt^2 - \left(a^2\, l_\alpha\, l_\beta + b^2 m_\alpha\, m_\beta + c^2 n_\alpha\, n_\beta\right) dx^\alpha\, dx^\beta$$

(4-2.5)

where

$$a = t^{p_l},\quad b = t^{p_m},\quad c = t^{p_n} \qquad (4\text{-}2.6)$$

Here $\bar{l}, \bar{m}, \bar{n}$ are unit vectors in the directions for the variation of spatial coordinates. We have

$$p_l + p_m + p_n = p_l^2 + p_m^2 + p_n^2 = 1 \qquad (4\text{-}2.7)$$

The Einstein equation for the B-K-L space-time can be written as

$$R_1^{\ 1} = \frac{(\dot{a}\, b\, c)^{\cdot}}{a\, b\, c} = \frac{1}{2a^2b^2c^2}\left[(b^2 - c^2)^2 - a^4\right]$$

$$R_2^{\ 2} = \frac{(a\, \dot{b}\, c)^{\cdot}}{a\, b\, c} = \frac{1}{2a^2b^2c^2}\left[(a^2 - c^2)^2 - b^4\right]$$

$$R_3^{\ 3} = \frac{(a\, b\, \dot{c}\,)^{\cdot}}{a\, b\, c} = \frac{1}{2a^2b^2c^2}\left[(a^2 - b^2)^2 - c^4\right] \qquad (4\text{-}2.8)$$

$$R_o^{\ o} = \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} = 0$$

It may be noted that we used here synchronous reference system. By introducing new variables α, β, γ and τ which are related to a,b,c and t as

$$a = e^{\alpha}, \quad b = e^{\beta}, \quad c = e^{\gamma} \qquad (4\text{-}2.9)$$

$$dt = abc\ d\tau \qquad (4\text{-}2.10)$$

we get

$$2\alpha_{,\tau\tau} = (b^2-c^2)^2 - a^4$$

$$2\beta_{,\tau\tau} = (a^2-c^2)^2 - b^4$$

$$2\gamma_{,\tau\tau} = (a^2-b^2)^2 - c^4 \qquad (4\text{-}2.11)$$

$$\frac{1}{2}(\alpha+\beta+\gamma)_{,\tau\tau} = \alpha_{,\tau}\beta_{,\tau} + \alpha_{,\tau}\gamma_{,\tau} + \beta_{,\tau}\gamma_{,\tau} \qquad (4\text{-}2.12)$$

where the subscript, τ denotes differentiation with respect to that symbol. Here our main purpose is to study the evolution of the model near the singularity $t = 0$. So, if the right hand sides of (4-2.11) are very small so that they can be neglected, then

$$a \sim t^{p_l} \quad , \quad b \sim t^{p_m} \quad , \quad c \sim t^{p_n} \qquad (4\text{-}2.13)$$

which is analogous to the Kasner solution (4-2.1) for a homogeneous empty space. But this situation cannot continue as $t \rightarrow 0$ because some of the terms must be increasing. Suppose the function $a(t)$ contains the negative power of t then the perturbation of the Kasner regime is because of the terms in a^4 since other terms can be dropped as $t \rightarrow 0$. Keeping only increasing terms in the right side of (4-2.11) we have

$$\alpha_{,\tau\tau} = \frac{-1}{2} e^{4\alpha} \quad , \quad \beta_{,\tau\tau} = \gamma_{,\tau\tau} = \frac{1}{2} e^{4\alpha} \qquad (4\text{-}2.14)$$

The solution of (4-2.14) should evolve the metric from the initial state. The initial condition is given by

$$a \sim t^{p_l} , \quad b \sim t^{p_m} , \quad c \sim t^{p_n} \qquad (4-2.15)$$

where $p_l = p_1$, $p_m = p_2$, $p_n = p_3$ and $p_l < 0$

The proportionality coefficients in (4-2.15) can be taken as unity. Therefore, we have

$$a = t^{p_l} , \quad b = t^{p_m} , \quad c = t^{p_n} \qquad (4-2.16)$$

Which implies that $abc = t$ and $\tau = \log t + \text{const.}$ Using the initial condition one can determine α, β, γ and t as below :

$$e^{\alpha} \sim e^{-p_1 \tau}$$

$$e^{\beta} \sim e^{(p_2 + 2p_1)\tau}$$

$$e^{\gamma} \sim e^{(p_3 + 2p_1)\tau}$$

$$t \sim e^{(1 + 2p_1)\tau}$$

That is

$$a \sim t^{p'_l} , \quad b \sim t^{p'_m} , \quad c \sim t^{p'_n}$$

where

$$p'_l = \frac{|p_1|}{1-2|p_1|}, p'_m = -\frac{2|p_1| - p_2}{1-2|p_1|}, p'_n = \frac{p_3 - 2|p_1|}{1-2|p_1|}$$

Thus because of the perturbation one Kasner regime is replaced by another and negative power of t transforms from $\bar{l}$ to $\bar{m}$ direction. That is, if we had $p'_l < 0$ then, now $p'_m < 0$. During this transformation a(t) reaches maximum value and b(t) minimum and then b(t) starts to increase and a(t) decreases, while c(t) continues to decrease. Now we will have the perturbation due to the term b^4 and one can get another Kasner regime. The transformation rule for exponents can be given by reparametrization (4-2.4). If

$$p_l = p_1(u), \quad p_m = p_2(u), \quad p_n = p_3(u)$$

then

$$p'_l = p_2(u - 1), \quad p'_m = p_1(u - 1) \quad p'_n = p_3(u - 1) \tag{4-2.17}$$

The successive changes of negative power between the directions $\bar{l}$ and $\bar{m}$ continue until u becomes less than unity. The value $u < 1$ transforms to $u > 1$ by (4-2.4) and either p_l or p_m is negative. Thus negative exponent will now bounce between $\bar{n}$ and $\bar{l}$ directions or of $\bar{n}$ and $\bar{m}$.

Thus the evolution of the metric (in time) can be described by a series of oscillations during which distance along two space axes oscillates while along the third axes decreases monotonically.

3. <u>Quantization of the B-K-L space-time</u> :

From (4-2.5) and (4-2.6)

$$^3(-g) = a^2 b^2 c^2 \vartheta^2 = t^2 \vartheta^2 \qquad (4\text{-}3.1)$$

where $\vartheta = \bar{l}.(\bar{m} \times \bar{n})$ is a function of space coordinates only. As in the previous Chapter here also we quantize the system with a conformal degree of freedom [3,4] . Under such a perturbation, $g_{ij} \longrightarrow \Omega^2(t)\, g_{ij}$, and $0 < \Omega < \infty$. Now we are basically interested in the behaviour of the quantum fluctuations near the classical singular epoch. It is pointed out by Belinskii et al that the matter contribution can be neglected at such an epoch.

Thus using $S_m = 0$ and (3-2.9) the action for the system with conformal degree of freedom reduces to

$$S = -\frac{1}{16\pi} \int_{\mathcal{V}} 6\, \Omega_{i} \Omega^{i} \sqrt{-g}\, d^4 x \qquad (4\text{-}3.2)$$

where $\mathcal{V}$ is a space-time region under consideration and Ω_i has the same meaning as that given in Section 2 of Chapter 3. Then using (4-3.1) and integrating over the three space in (4-3.2) we have

$$S = -\frac{3V}{8\pi}\int t\,\dot{\phi}^2\,dt \qquad (4\text{-}3.3)$$

where V is the volume of three space under consideration. Now using (4-3.3) one can write the Lagrangian as

$$L = -\frac{3V}{8\pi}\,\dot{\phi}^2\,t \qquad (4\text{-}3.4)$$

and accordingly the Hamiltonian H is given by

$$H = \dot{\phi}p - L = -\frac{p^2}{2k_1 t}\,, \quad k_1 = \frac{3V}{4\pi} \qquad (4\text{-}3.5)$$

where $p = \frac{\partial L}{\partial \dot{\phi}}$. Using (4-3.5) one can get

$$[\phi\,, H] = -i\,\frac{p}{k_1 t}\,, \quad [p\,, H] = 0 \qquad (4\text{-}3.6)$$

Here we take $\Psi = \Psi(\phi, t)$ as a wave function of the Universe and $\langle\phi\rangle = 0$. Therefore, the quantum uncertainty in the conformal factor has the form,

$$\chi = \langle \phi^2 \rangle$$

and uncertainty in the momentum p is given by

$$\omega = \langle p^2 \rangle$$

Now, denoting $A = \phi^2$ and using the time evolution equation (2-6.4) we will have

$$\frac{d\chi}{dt} = \frac{-1}{k_1 t} \langle p\phi + \phi p \rangle \qquad (4\text{-}3.7)$$

Applying a second differentiation to (4-3.7) and involving commutators (4-3.6) we obtain

$$\frac{d^2\chi}{dt^2} + \frac{1}{t}\frac{d\chi}{dt} = \frac{2\omega}{k_1^2 t^2} \qquad (4\text{-}3.8)$$

Treating the case $\chi\omega = \frac{1}{4}$, (4-3.8) reduces to the following nonlinear differential equation :

$$t^2 \frac{d^2\chi}{dt^2} + t\frac{d\chi}{dt} = \frac{1}{2\chi k_1^2} \qquad (4\text{-}3.9)$$

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Putting $P = 1/t$ in (4-3.9) it transforms to

$$\chi \frac{d^2\chi}{dP^2} + \frac{\chi}{P}\frac{d\chi}{dP} - \frac{1}{2k_1^2 P^2} = 0 \qquad (4\text{-}3.10)$$

In the limit of approach to the classical singularity $t \longrightarrow 0$ (i.e. $P \longrightarrow \infty$) the last term of (4-3.10) can be neglected. Thus (4-3.10) reduces to

$$\frac{d^2\chi}{dp^2} + \frac{1}{P}\frac{d\chi}{dP} = 0 \qquad (4\text{-}3.11)$$

Putting $R = \frac{d\chi}{dP}$ in (4-3.11) we obtain

$$\frac{dR}{dP} = -\frac{R}{P}$$

which on integration gives

$$RP = C_1 \qquad (4\text{-}3.12)$$

where C_1 being a positive constant of integration.

Transforming (4-3.12) into old variable t we get

$$\frac{d\chi}{dt}\, t = -C_1$$

Which on integration gives

$$\chi = -C_1 \log t + C_2, \quad C_1 > 0 \qquad (4\text{-}3.13)$$

In the case of $\chi(\omega) > 1/4$ in place of (4-3.9) we have $t^2\chi'' + t\chi' > 1/2\,\chi k_1^{\,2}$ where prime denotes differentiation with respect to t and instead of (4-3.11) we get

$$\chi \frac{d^2\chi}{dP^2} + \frac{\chi}{P}\,\frac{d\chi}{dP} > 0 \qquad (4\text{-}3.14)$$

leading to the inequality

$$\chi > -C_1 \log t + C_2 \qquad (4\text{-}3.15)$$

Thus (4-3.13) and (4-3.15) show that the quantum uncertainty χ must diverge logarithmically in limit of approach to the singularity.

4. A more general space-time :

In this Section we generalise the results of the previous Section [5] . In fact, the results can be generalised for a wide class of space-times, namely for a globally hyperbolic universe [6] for which the space-time has topology $S^3 \times R$ where S^3 is a space-like hypersurface and the space-time metric takes a very general form

$$ds^2 = dt^2 - h_{ab}(x^i)\, d x^a\, d x^b \qquad (4\text{-}4.1)$$

where $i=0,1,2,3$ and $a,b = 1,2,3$. Clearly metric (4-4.1) contains arbitrary function h_{ab} and hence includes inhomogeneities and anisotropies of the matter distribution. The metric of Belinskii et al. discussed in Section 3 is then included here as a special case. As before we quantize the conformal degree of freedom $\Omega(t)$ for the metric tensor. Then under a conformal perturbation $\Omega(t)$ classical action $\bar{S}$ transforms to the new action S given by

$$S = -\frac{3}{8\pi} \int_{\mathcal{V}} \Omega_{i}\, \Omega^{i} \sqrt{-g}\, d^4x \qquad (4\text{-}4.2)$$

$\mathcal{V}$ being the space-time region under consideration. Here also we assume that the contribution of matter is negligible near the space-time singularity. Hence, we take scalar curvature $\bar{R} = 0$ and the matter action $S_m = 0$. Thus using this fact we can write (4-4.2) as

$$S = \frac{-3}{8\pi} \int \dot{\phi}^2 \, h(t) \, dt \qquad (4\text{-}4.3)$$

where $h(t) = \int \sqrt{-g} \, d^3x$ and the dot denotes differentiation with respect to time. We would like to define a new time transformation and simplify the action (4-4.3) further. Defining a new variable τ by

$$d\tau = \frac{dt}{h(t)} \qquad (4\text{-}4.4)$$

we get,

$$\left(\frac{d\phi}{dt}\right)^2 \cdot h(t)dt = \left(\frac{d\phi}{d\tau}\right)^2 \left(\frac{d\tau}{dt}\right)^2 h^2(t) \, d\tau$$

$$= \left(\frac{d\phi}{d\tau}\right)^2 \, d\tau \qquad (4\text{-}4.5)$$

Substituting (4-4.5) in (4-4.3) we have

$$S = -\frac{3}{8\pi} \int \phi'^2 \, d\tau \tag{4-4.6}$$

where a prime denotes differentiation with respect to τ.
The Hamiltonian H for the system is written as

$$H = -\frac{2}{3}\pi p^2 \tag{4-4.7}$$

where

$$p = \frac{\partial L}{\partial \phi'} = -\frac{3\phi'}{4\pi}$$

To get the time evolution of the quantum uncertainty χ we must work out the commutators of the operators ϕ and p with H. These are given by

$$[\phi , H] = i\frac{\partial H}{\partial P} = -i\left(\frac{4}{3}\pi p\right) \tag{4-4.8}$$

$$[p , H] = -i\frac{\partial H}{\partial \phi} = 0 \tag{4-4.9}$$

Then using standard time evolution equation for operators and (4-4.8) we get

$$\frac{d\chi}{d\tau} = -\frac{4}{3}\pi \langle p\phi + \phi p\rangle \tag{4-4.10}$$

To apply a second differentiation we write $B = \frac{-4}{3}\pi(p\phi + \phi p)$ and then

$$i\frac{d^2\chi}{d\tau^2} = i\frac{d\langle B\rangle}{d\tau} = \langle[B , H]\rangle + i\left\langle\frac{\partial B}{\partial \tau}\right\rangle \qquad (4\text{-}4.11)$$

A simple calculation shows that

$$[B , H] = i\frac{32\pi^2 p^2}{9}, \frac{\partial B}{\partial \tau} = 0 \qquad (4\text{-}4.12)$$

Substituting (4-4.12) in (4-4.11) we get

$$\chi'' = \frac{32}{9}\pi^2\omega \qquad (4\text{-}4.13)$$

Here we use the same form of χ and ω as given in Section 3. In general, we have $\chi\omega \geqslant 1/4$ when the wave function has a general form. However, to begin with we first take Gaussian wave packet and put $\chi\omega = 1/4$ in (4-4.13) to get

$$\chi\chi'' = A \qquad (4\text{-}4.14)$$

where $A = \frac{8}{9}\pi^2 > 0$. Multiplying (4-4.14) by $2\chi'$ and integrating we get

$$\chi' = \frac{d\chi}{dt} h(t) = \pm (2A \log\chi + C_1)^{1/2} \qquad (4\text{-}4.15)$$

We shall choose here the negative sign because in general one expects χ to be a decreasing function of time which is large near the singularity and decreases to almost zero at the present epoch. Therefore

$$\frac{d\chi}{dt}\, h(t) = -\,(B \log\chi + C_1)^{1/2} \qquad (4\text{-}4.16)$$

Note that $h(t) > 0$, $2A = B$. Now, substituting $z = (2A \log\chi + C_1)^{1/2}$ in (4-4.16) we get

$$\int \frac{dt}{h(t)} = -\, D \int e^{z^2/B}\, dz$$

Taking modulus on both sides of the above equation;

$$\left| \int \frac{dt}{h(t)} \right| = D \int e^{z^2/B}\, dz \qquad (4\text{-}4.17)$$

Then using a series expansion for $e^{z^2/B}$ and integrating on the right hand side in (4-4.17) we obtain the following inequality

$$\left| \int \frac{dt}{h(t)} \right| < \frac{2}{B}\, (B \log\chi + C_1)^{1/2} \chi \qquad (4\text{-}4.18)$$

In the case of a general wave function, that is $\chi\omega \geqslant 1/4$ we get, in the place of (4-4.14) the equation $\chi\chi'' \geqslant A$, which implies

$$\chi' \leqslant -(B \log\chi + C_1)^{1/2} \qquad .(4\text{-}4.19)$$

An analysis, similar to that following (4-4.15) then leads to (4-4.18) again for the behaviour of the quantum uncertainty χ .

It is interesting to compare the above results with the scenario of Belinskii et al. considered in Section 3, which is a particular case of the situation considered here. There we have $h(t) = Vt$ and (4-4.18) reduces to

$$V^{-1}\,|\log t| \leqslant \frac{2}{B}\,(B \log\chi + C_1)^{1/2}\chi \qquad (4\text{-}4.20)$$

The left hand side in (4-4.20) diverges logarithmically in the limit of the classical singularity at $t = 0$ and hence quantum uncertainty χ must diverge in this limit. Now to deduce the behaviour of χ from (4-4.18), the function $h(t)$ must vanish at least as fast as t if χ were to diverge. This means that the space integral of $\sqrt{-g}$ must converge fast enough as the singularity approached.

It is clear that this condition is satisfied in important cases of physically meaningful curvature singularities such as those occuring in the collapsing dust ball, Friedmann-Robertson-Walker models, Bianchi universes and finally the general cosmological scenario of Belinskii et al, as well. In fact, there has been considerable discussion on the nature and classification of singularities [7] and it is believed that the singularity occuring in nature must be strong curvature type where again above condition must hold. Here we do not discuss these issues further but only point out that whenever the focussing effects are strong enough near the singularity then our considerations show that for a very general class of space-times given by (4-4.1) the quantum effects will dominate in the sense that χ diverges and non-classical non-singular states can occur with finite probability [8]. Further generalizations such as $S_m \neq 0$ or a general perturbation $\phi = \phi(x,y,z,t)$ will be taken up in the next chapter.

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