Chapter 2

FUNDAMENTAL CONCEPTS OF RELATIVITY AND QUANTUM THEORY

1. Introduction :

We have mentioned in Chapter 1, that to study quantum effects near the space-time singularity, we will use operator approach. Therefore, in the present Chapter we discuss some important techniques of the operator theory needed in the subsequent chapters. Also, some concepts of classical relativity such as space-time singularity and action principle are reviewed.

In Sections 2 and 3 we will discuss the space-time singularity and action principle in general relativity respectively. Section 4 gives a brief review of the basic concepts of quantum theory wherein we discuss wave functions, operators and expectation values. In Section 5 we derive some important commutator relations and Section 6 deals with time evolution of an operator and the Ehrenfest theorem. The last Section gives mathematical details on the Heisenberg uncertainty principle.

2. Space-time singularity :

The most undesirable feature of the general theory of relativity is the existence of space-time singularities.

The Robertson-Walker models show that universe has a singular origin (i.e. infinite density and infinite curvature) at some finite past. Also, for the extended Schwarzschild metric the curvature scalar goes to infinity at $r=0$. Thus, in both the above models, the first one representing global and the second local scenario, the status of space-time singularity is clear. However, it was argued [1] that the occurrence of singularities in the exact solutions is only because of high degree of symmetry of models and small perturbations in the exact solutions may avoid singularity. Hence, it is important to know whether singularities occur due to symmetries of exact solution or they are a regular feature of the general theory of relativity itself. The Hawking-Penrose singularity theorems [2] proved conclusively that space-time singularities are unavoidable in general theory of relativity under certain reasonable conditions.

Thus, we first analyse the notion of singularity and give an exact definition of the same. The concept of a space-time singularity itself has gone many important changes. First of all one might say that singularity could be characterized by the event at which space-time metric

or curvature components diverge. However, this criterion is not useful because the definition of a manifold precludes all such points. To circumvent the above difficulty space-time singularity should be characterized in a more subtle way and a natural way is to consider geodesic incompleteness property of the space-time manifold.
A manifold is said to be geodesically complete if all the geodesics in it can be extended to infinite values of their affine parameter in both future and past directions. Thus one can say that a manifold is geodesically incomplete if it has a geodesic which is inextendible in at least one direction with a finite range of the affine parameter.
We will define an inextendible space-time to be singular if it possesses at least one incomplete geodesic. We have considered inextendible space-times because sometimes it is possible to remove singularity by the extension of a space-time. A well known example is the Schwarzschild metric with coordinate singularity at $r=2m$. Then, on the basis of the above definition of a space-time singularity we state the Hawking-Penrose theorem as below :

A space-time (M,g) cannot be timelike and null geodesically complete if

(1) $R_{il}\, k^i k^l \geqslant 0$ for all nonspace-like vectors k^i.

(2) The generic condition is satisfied i.e. every non-spacelike geodesic contains a point at which $k_{[i}R_{j]lm[p}k_{s]}k^{l}k^{m} \neq 0$ where k^{i} is tangent to the geodesic,

(3) chronology holds in M, and

(4) there exists in M either a compact achronal set with edge, or a closed trapped surface, or a point p, all past directed null geodesics from which have the expansion θ negative eventually.

The first condition ensures that the energy density measured by any local observer is always non-negative and it is satisfied by all known matter fields. For a detailed discussion of other conditions we refer to [3].

From the above theorem it is proved that the space-time singularity is a generic feature of general relativity and it is not just related with the Robertson-Walker cosmological models which have high symmetry. Of course the singularity theorems do not give clear idea about the nature of singularity or the behaviour of space-time metric near the singularity. However, this theorem established the status of the space-time singularity in a precise way and this will help us to study the physical events near the singularity which is the most unknown fact today.

3. The action principle in relativity :

It is convenient to derive field equations from the action principle. The action for the gravitational field is given by the Hilbert action [4] :

$$S = \frac{1}{16\pi G} \int R \sqrt{-g}\, d^4x + S_m \qquad (2\text{-}3.1)$$

Here S_m is the matter action and integration is carried out over all space between two given values of time coordinate x^o. Now we will analyse the above form of the gravitational action. To derive the action principle for gravitational field one has to determine the form of the Lagrangian function L_G. Here it should be noted that Einstein's field equations contain derivatives of the metric tensor g_{ik} up to the second order. Therefore L_G must not have derivatives of g_{ik} higher than first order. Keeping this fact in mind one can think some scalar quantity which contains Christoffel symbols Γ_{ik}^{l} as well as g_{ik}. But there is no scalar quantity which contains both Γ_{ik}^{l} and g_{ik} because the former are not components of a tensor. It seems that Ricci curvature scalar R is the only scalar which contains metric tensor g_{ik} and its derivatives. Of course, the curvature scalar R contains second order derivatives of the metric tensor g_{ik} but we will see that it will not create any problem in deriving field equations from the gravitational action. Thus, we write

$$S = S_g + S_m = \frac{1}{16\pi G}\int (L_G - 2kL_m)\sqrt{-g}\,d^4x \qquad (2\text{-}3.2)$$

where $k = 8\pi G$ and demand that its variation be zero. i.e.

$$\delta S = 0 \qquad (2\text{-}3.3)$$

In the above equations $L_G = R$ is the Lagrangian for the gravitational field where R is the Ricci scalar. Here L_m denotes Lagrangian for other fields. Now, varying the gravitational action S_g and putting $R = g^{ik} R_{ik}$ we have

$$\delta S_g = \frac{1}{16\pi G}\delta\int \sqrt{-g}\; R d^4x$$

$$= \frac{1}{16\pi G}\delta\int \sqrt{-g}\, g^{ik} R_{ik}\, d^4x$$

$$= \frac{1}{16\pi G}\int \sqrt{-g}\, g^{ik}\,\delta R_{ik} d^4x + \frac{1}{16\pi G}\int R_{ik}\delta(\sqrt{-g}g^{ik})d^4x$$

$$(2\text{-}3.4)$$

Now using a geodesic coordinate system and some techniques of tensor calculus we get

$$g^{ik}\,\delta R_{ik} = \nabla_l V^l = \frac{1}{\sqrt{-g}}\,\frac{\partial}{\partial x^l}\,(\sqrt{-g}\,V^l) \qquad (2\text{-}3.5)$$

where

$$V^l = g^{ik}\,\delta\Gamma^l_{ik} - g^{il}\,\delta\Gamma^k_{ik}$$

Hence,

$$\int \sqrt{-g}\,g^{ik}\,\delta R_{ik}\,d^4x = \int \frac{\partial(\sqrt{-g}\,V^l)}{\partial x^l}\,d^4x \qquad (2\text{-}3.6)$$

The above 4-dimensional integral will be reduced to a 3-dimensional integral of $\sqrt{-g}\,V^l$ over the hypersurface surrounding the whole four volume. This integral will vanish because variations of the Christoffel symbols on boundaries are considered to be zero. Thus,

$$\int \sqrt{-g}\,g^{ik}\,\delta R_{ik}\,d^4x = 0 \qquad (2\text{-}3.7)$$

Now, the variation of the second part of S_g gives

$$\delta S_g = \frac{1}{16\pi G}\int R_{ik}\,\delta(\sqrt{-g}\,g^{ik})\,d^4x$$

$$= \frac{1}{16\pi G}\int \sqrt{-g}\left(R_{ik} - \frac{1}{2} g_{ik} R\right)\delta g^{ik}\,d^4x \qquad (2\text{-}3.8)$$

where we have used the well known formula,

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,g_{ik}\,\delta g^{ik} \qquad (2\text{-}3.9)$$

Taking variation of matter action S_m we have

$$\delta S_m = -\delta\int \sqrt{-g}\;L_m d^4x$$

$$= -\int\left[\frac{\partial(\sqrt{-g}\,L_m)}{\partial g^{ik}}\delta g^{ik} + \frac{\partial(\sqrt{-g}\,L_m)}{\partial g^{ik}_{,l}}\,\delta g^{ik}{}_{,l}\right] d^4x$$

(2-3.10)

The integrand in the second term on the r.h.s. of (2-3.10) can be written as

$$\frac{\partial}{\partial x^l}\left[\frac{\partial(\sqrt{-g}\,L_m)}{\partial g^{ik}_{,l}}\,\delta g^{ik}\right] - \frac{\partial}{\partial x^l}\left[\frac{\partial(\sqrt{-g}\,L_m)}{g^{ik}_{,\;l}}\right]\delta g^{ik}$$

(2-3.11)

Now the integration of the first term of (2-3.11) can be written as a surface integral and it will be zero because on the surface boundaries the variation of g^{ik} vanishes.

Now (2-3.10) can be written as

$$\delta\int\sqrt{-g}\; L_m\, d^4x = \int\left\{\frac{\partial(\sqrt{-g}\,L_m)}{\partial g^{ik}} - \frac{\partial}{\partial x^1}\left[\frac{\partial(\sqrt{-g}\,L_m)}{\partial g^{ik}_{,1}}\right]\right\}\delta g^{ik}\, d^4x$$

(2-3.12)

Again defining the energy momentum tensor as

$$T_{ik} = \frac{2}{\sqrt{-g}}\left\{\frac{\partial(\sqrt{-g}\,L_m)}{\partial g^{ik}} - \frac{\partial}{\partial x^1}\left[\frac{\partial(\sqrt{-g}\,L_m)}{\partial g^{ik}_{,1}}\right]\right\} \qquad (2\text{-}3.13)$$

the equation (2-3.12) takes the form

$$\delta S_m = -\delta\int\sqrt{-g}\; L_m\, d^4x$$

$$= -\frac{1}{2}\int T_{ik}\sqrt{-g}\;\delta g^{ik}\, d^4x \qquad (2\text{-}3.14)$$

Now writing $\delta S_g + \delta S_m$ from (2-3.8) and (2-3.14) we get

$$\delta S = \delta S_g + \delta S_m$$

$$= \frac{1}{16\pi G}\int\sqrt{-g}\,(R_{ik} - \tfrac{1}{2}\, g_{ik}R - 8\pi G T_{ik})\, d^4x\;\delta g^{ik} = 0$$

(2-3.15)

Since this equation is valid for an arbitrary variation δg^{ik} (2-3.15) implies

$$R_{ik} - \frac{1}{2} g_{ik} R = 8\pi G T_{ik} \qquad (2\text{-}3.16)$$

which are Einstein's field equations.

4. Basic framework of quantum mechanics :

In this section we would like to review the following important concepts :

(i) Interpretation of the wave function
(ii) Operator concept
(iii) expectation value of an operator.

By using wave mechanics Schrödinger developed wave equation for a particle and its solution is known as the wave function of the particle. Here the wave function $\psi(x,y,z,t)$ describes the probability of finding the particle in some specific region at time $t = t_0$. But the probability cannot be given only by ψ because ψ is a complex function in general and it can take negative values which contradicts the fact that probability must be real and non-negative. The correct interpretation of the wave function was given by Max Born in 1926, by defining the probability of finding the particle in a volume element $\delta x\ \delta y\ \delta z$ about the point (x,y,z) at time $t = t_o$ as

$$\Psi^*(x, y, z, t)\Psi(x, y, z, t)\ \delta x\ \delta y\ \delta z = |\Psi|^2\ \delta x\ \delta y\ \delta z \tag{2-4.1}$$

Now, since the total probability of finding the particle in the whole space is unity, the above statement takes mathematical form

$$\int \Psi^*(x, y, z, t)\Psi(x, y, z, t)\ dV = \int |\Psi|^2\ dV = 1 \tag{2-4.2}$$

where $dV = dxdydz$. Here one can interpret $|\Psi|^2$ as a probability density. The integration is taken over a three dimensional finite volume V. For a treatment on infinite volumes we refer to [5]. Thus, in (2-4.2) the wave function is normalised. Now, suppose that the wave function Ψ_1 of a system is not normalised and

$$\int \Psi_1 \Psi_1^*\ dV = k_1^2 \tag{2-4.3}$$

where k_1 is finite and real. Then by defining $\Psi = \frac{1}{k_1} \Psi_1$ we obtain

$$\int \Psi \Psi^* \, dV = \frac{1}{k^2_1} \int \Psi_1 \Psi_1^* \, dV = 1 \qquad (2\text{-}4.4)$$

Thus, it is always possible to normalise the wave function if the integral (2-4.3) converges to some finite number k_1^2. Here it should be noted that the equation (2-4.2) is written for some specific value of time $t = t_o$. But the equation (2-4.4) must be true for any arbitrary time because the total probability of finding the particle in the entire space at any arbitrary instance is unity. Therefore (2-4.2) is independent of time, i.e.

$$\frac{\partial}{\partial t} \int_V |\Psi|^2 \, dV = 0 \qquad (2\text{-}4.5)$$

where V is a fixed volume of the entire space. (For a mathematically rigorous derivation see [5].)

Next, we will discuss the operator concept. An operator acts on a function f and gives a new function f'.

An operator A is said to be linear if the following conditions hold :

$$\text{(i)} \quad A(\Psi_1 + \Psi_2) = A\Psi_1 + A\Psi_2$$

$$\text{(ii)} \quad A(\lambda_1 \Psi_1) = \lambda_1 (A\Psi_1) \qquad (2\text{-}4.6)$$

where Ψ_1 and Ψ_2 are arbitrary functions and λ_1 is an arbitrary constant. It can be seen easily that the addition of two linear operators gives a new linear operator. The commutator of two linear operators is defined by

$$[A,B] = AB - BA \qquad (2\text{-}4.7)$$

The right hand side of the above equation is not in general zero. In quantum mechanics linear operators and their commutation relations are important because every physical observable can be represented by a linear operator. According to the Schrödinger representation the position co-ordinate and time are represented by q_j and t respectively whereas the conjugate momentum p_j is represented by the operator $\frac{\hbar}{i} \frac{\partial}{\partial q_j}$.

The relation

$$\left[q_j , \frac{\hbar}{i} \frac{\partial}{\partial q_k}\right]\Psi = i\hbar\delta_{jk}\Psi \qquad (2\text{-}4.8)$$

can be varified easily. This relation is similar to the Poisson bracket $\{q_j , p_k\} = \delta_{jk}$ in classical mechanics. That is in quantum mechanics physical quantities take the form of a linear operator and we have commutators rather than Poisson brackets. Here we state some basic properties of commutators which we need later. Using the definition of a cummutator one can derive the following relations for arbitrary operators A,B and C :

$$[A,B] = - [B,A] \qquad (2\text{-}4.9(a))$$

$$[A,B + C] = [A,B] + [A,C] \qquad (2\text{-}4.9(b))$$

$$[A, BC] = [A,B]\, C + B\, [A,C] \qquad (2\text{-}4.9(c))$$

Let us now consider the notion of expectation values. We have seen earlier that in quantum mechanics the wave function gives probabilistic description of the position of material particles. Suppose that measurement is made for the x-coordinate of a particle. The measurement is repeated many times for the same particle. Now $\Psi^*\Psi\, dV$ is the probability of finding the particle in a volume dV around some point $r = (x,y,z)$, where Ψ is a normalised wave function. It follows that the average of a very large number of such experiments for the measurement of x-coordinate will be equal to

$$\frac{\int_V \Psi x \Psi^* \, dV}{\int_V |\Psi|^2 \, dV} = \langle x \rangle = \int_V x |\Psi|^2 \, dV \qquad (2\text{-}4.10)$$

Ψ being normalised. In the above formula the expectation value of x is denoted by $\langle x \rangle$.

We can derive similar relations for the measurement of y and z coordinates of a particle and accordingly the expectation value for the position vector $\bar{r}$ of the particle is given by

$$\langle \bar{r} \rangle = \int_V \Psi^* \bar{r} \Psi \, dV \qquad (2\text{-}4.11)$$

Thus, for any function f(r) the expectation value is given by

$$\langle f(r) \rangle = \int_V \Psi^* f(r) \Psi \, dV \qquad (2\text{-}4.12)$$

This equation gives the expectation values of physical quantities which are only functions of the position coordinates. However, to define expectation value of the momentum and energy operators, one has to put them in terms of r and t. We assume that this is possible by using the differential operator representations of energy and momentum which are given by

$$E \longrightarrow i\hbar\frac{\partial}{\partial t}\,, \quad p \longrightarrow -i\hbar\nabla \qquad (2\text{-}4.13)$$

where $\nabla = \bar{i}\,\frac{\partial}{\partial x} + \bar{j}\,\frac{\partial}{\partial y} + \bar{k}\,\frac{\partial}{\partial z}$. Thus,

$$\langle E\rangle = \int_V \Psi^* \, i\hbar \frac{\partial\Psi}{\partial t}\, dV$$

$$\langle p\rangle = \int_V \Psi^* (-i\hbar)\nabla\Psi \, dV$$

The last equation may be written explicitly as follows :

$$\langle p_x\rangle = -i\hbar\int \Psi^* \frac{\partial\Psi}{\partial x}\, dV$$

$$\langle p_y\rangle = -i\hbar\int \Psi^* \frac{\partial\Psi}{\partial y}\, dV \qquad (2\text{-}4.14)$$

$$\langle p_z\rangle = -i\hbar\int \Psi^* \frac{\partial\Psi}{\partial z}\, dV$$

5. Commutator relations :

We have defined commutators in Section 4. Here we will derive some important commutator relations. If q_j denotes the position of a system and p_j its conjugate momentum, where $j=1,----,n$, one can easily give the following commutators

$$[q_j, q_k] = 0$$
$$[p_j, p_k] = 0 \qquad (2\text{-}5.1)$$

$$[q_j, p_k] = i\hbar\delta_{jk} \qquad (2\text{-}5.2)$$

The commutators given in (2-5.1) are obvious whereas the equation (2-5.2) can be derived by representing p_k in operator form $\frac{\hbar}{i}\frac{\partial}{\partial q_k}$ and then operating on a arbitrary wave function Ψ. Some important commutators of q's and p's with arbitrary functions $F(q_1,---,q_n)$ and $G(p_1,-----,p_n)$ are given by

$$[p_j, F(q_1,---,q_n)] = \frac{\hbar}{i}\frac{\partial F}{\partial q_j} \qquad (2\text{-}5.3)$$

$$[q_j, G(p_1,---,p_n)] = -\frac{\hbar}{i}\frac{\partial G}{\partial p_j} \qquad (2\text{-}5.4)$$

The relation (2-5.3) can be derived by putting p_j into operator form $\frac{i}{\hbar} \frac{\partial}{\partial q_j}$ and then operating the commutator on an arbitrary wave function Ψ . To derive the second commutator one has to put q_j in the momentum representation. If $\phi(p_1, ----, p_n)$ is the wave function of momentum space corresponding to $\Psi(q_1, ----, q_n)$ then $q_j \Psi(q_1, ----, q_n) \longrightarrow \frac{\hbar}{i} \frac{\partial}{\partial p_j} \phi(p_1, ----, p_n)$.

Now, we know that the Hamiltonian H of a system is a function of q_j and p_j and one can write similar relations as below :

$$[p_j, H] = -i\hbar \frac{\partial H}{\partial q_j}$$

$$[q_j, H] = i\hbar \frac{\partial H}{\partial p_j} \qquad (2\text{-}5.5)$$

The above commutators also hold for any other operator $A(q_1, -----, q_n, p_1, ----, p_n)$ as well.

6. Ehrenfest's theorem :

Before proving this theorem we discuss time evolution of quantum mechanical operators [6]. In classical mechanics the rate of change of any physical variable

$F(q_j, p_j, t)$ is given by

$$\frac{dF}{dt} = \{F, H\} + \frac{\partial F}{\partial t} \qquad (2\text{-}6.1)$$

A similar result can be derived in quantum mechanics. If the state of a system is described by the normalized wave function Ψ and A is an operator representing some physical quantity then as we have defined earlier,

$$\langle A \rangle = \int_V \Psi^* A \Psi \, dV$$

It is clear that the expectation value is a function of time and the rate of change of $\langle A \rangle$ with respect to time is given by

$$\frac{d}{dt}\langle A \rangle = \int_V \Psi^* \frac{\partial A}{\partial t} \Psi + \frac{\partial \Psi^*}{\partial t} A \Psi + \Psi^* A \frac{\partial \Psi}{\partial t} \, dV \qquad (2\text{-}6.2)$$

However, the state function Ψ must satisfy the time dependent Schrödinger equation,

$$\frac{\partial \Psi}{\partial t} = \frac{-i}{\hbar} H \Psi$$

$$\frac{\partial \Psi^*}{\partial t} = \frac{i}{\hbar} (H\Psi)^* = \frac{i}{\hbar} \Psi^* H \qquad (2\text{-}6.3)$$

where the Hamiltonian operator is hermitian. Now, substituting the values from (2-6.3) into (2-6.2) we obtain

$$\frac{d}{dt}\langle A\rangle = \int\left[\Psi^* \frac{\partial A}{\partial t}\Psi + \frac{i}{\hbar}\Psi^* HA\Psi - \Psi^* A\frac{i}{\hbar}H\Psi\right]dV$$

$$= \int\Psi^* \frac{\partial A}{\partial t}\Psi\, dV + \frac{i}{\hbar}\int\Psi^* \{HA - AH\}\Psi\, dV$$

$$= \left\langle\frac{\partial A}{\partial t}\right\rangle + \frac{i}{\hbar}\left\langle[H , A]\right\rangle$$

Thus, we can write,

$$i\hbar\frac{d}{dt}\langle A\rangle = \left\langle[A, H]\right\rangle + i\hbar\left\langle\frac{\partial A}{\partial t}\right\rangle \qquad (2\text{-}6.4)$$

If B commutes with H, then

$$[B, H] = 0 \qquad (2\text{-}6.5)$$

and when t does not occur explicitly in B, then $\frac{\partial B}{\partial t} = 0$.
Hence,

$$\frac{d}{dt}\langle B\rangle = 0 \qquad (2\text{-}6.6)$$

or the expectation value of an operator B is a constant of motion and B is said to be conserved. Now, we give below the proof of Ehrenfest's theorem :

If $(q_1, \text{----}, q_n)$ give the position coordinates and $(p_1, \text{---}, p_n)$ are conjugate momenta then their expectation values satisfy the following :

$$\frac{d}{dt}\langle q_j\rangle = \left\langle \frac{\partial H}{\partial p_j}\right\rangle , \qquad j = 1, \text{---}, n$$

$$\frac{d}{dt}\langle p_j\rangle = -\left\langle \frac{\partial H}{\partial q_j}\right\rangle , \qquad j = 1, \text{---}, n \tag{2-6.7}$$

Now, to prove the above relations first one has to apply formula (2-6.4) on the coordinates q_j and momenta p_j. From this it is easy to see that

$$i\hbar\frac{d}{dt}\langle q_j\rangle = \left\langle [q_j, H]\right\rangle$$

$$i\hbar\frac{d}{dt}\langle p_j\rangle = \left\langle [p_j, H]\right\rangle \tag{2-6.8}$$

Putting the values of commutators on the right hand side of (2-6.8) from (2-5.5) we get

$$\left.\begin{aligned} \frac{d}{dt}\langle q_j\rangle &= \left\langle \frac{\partial H}{\partial p_j}\right\rangle , \\ \frac{d}{dt}\langle p_j\rangle &= -\left\langle \frac{\partial H}{\partial q_j}\right\rangle , \end{aligned}\right] \quad j = 1, \text{---}, n$$

In particular, if the particle is in some force field then

$$H = \frac{p^2}{2m} + V(r)$$

Then, in this case the Ehrenfest equations are written as

$$\frac{d}{dt} \langle \bar{r} \rangle = \frac{\langle \bar{p} \rangle}{m} \qquad (2\text{-}6.9)$$

$$\frac{d}{dt} \langle \bar{p} \rangle = \langle \bar{F} \rangle = -\langle \nabla V \rangle \qquad (2\text{-}6.10)$$

i.e.

$$\langle \bar{F} \rangle = m \frac{d^2}{dt^2} \langle \bar{r} \rangle \qquad (2\text{-}6.11)$$

The above equation has the same form as the Newtonian equation in classical mechanics. Thus, Ehrenfest theorem gives laws of motion for the expectation values of the position and momentum operators.

7. Uncertainty principle :

In this Section we will give mathematical derivation of the uncertainty principle which can be stated as follows : If A and B are two arbitrary operators and satisfy the equation $[A,B] = i\hbar$ then $\Delta A \cdot \Delta B \geqslant \frac{\hbar}{2}$

The uncertainty in measurement of any variable is defined as

$$(\Delta A)^2 = \langle (A - \langle A \rangle)^2 \rangle = \langle A^2 \rangle - \langle 2A \langle A \rangle \rangle + \langle \langle A \rangle^2 \rangle$$

$$= \langle A^2 \rangle - \langle A \rangle^2$$

Similarly

$$(\Delta B)^2 = \langle (B - \langle B \rangle)^2 \rangle = \langle B^2 \rangle - \langle B \rangle^2$$

Now if we write

$$\alpha = A - \langle A \rangle \qquad \beta = B - \langle B \rangle$$

then

$$(\Delta A)^2 (\Delta B)^2 = \int \psi^* \alpha^2 \psi \, dV \int \psi^* \beta^2 \psi \, dV$$

$$= \int (\psi\alpha)^* \alpha \psi \, dV \int (\psi\beta)^* \beta \psi \, dV$$

In the above we have made use of the fact that $\alpha^* = \alpha$, $\beta^* = \beta$.

Therefore

$$(\Delta A)^2 (\Delta B)^2 = \int (\alpha^* \psi^*)(\alpha \psi) dV \int (\beta^* \psi^*)(\beta \psi) dV \qquad (2\text{-}7.1)$$

Now if Ψ_1 and Ψ_2 are two quadratically intergrable functions then the Schwarz inequality is given by

$$\int|\Psi_1|^2 \, dV \int|\Psi_2|^2 \, dV \geqslant \left|\int\Psi_1^{*}\Psi_2 \, dV\right|^2 \qquad (2\text{-}7.2)$$

Using this inequality in equation (2-7.1) we have

$$(\Delta A)^2 (\Delta B)^2 \geqslant \left|\int (\alpha^{*}\Psi^{*}) (\beta\Psi) \, dV\right|^2 \qquad (2\text{-}7.3)$$

Putting $\alpha\Psi = \Psi_1$ and $\beta\Psi = \Psi_2$ in (2-7.3) we get

$$(\Delta A)^2 (\Delta B)^2 \geqslant \left|\int\Psi_1^{*}\Psi_2 \, dV\right|^2 \qquad (2\text{-}7.4)$$

Since the square of modulus of a complex quantity is greater than or equal to square of its imaginary part, we get,

$$\left|\int\Psi_1^{*}\Psi_2 \, dV\right|^2 \geqslant \frac{1}{4}\left|\int\Psi_1^{*}\Psi_2 \, dV - \int\Psi_2^{*}\Psi_1 \, dV\right|^2$$

$$\geqslant \frac{1}{4}\left|\int (\Psi^{*}\alpha\beta\Psi - \Psi^{*}\beta\alpha\Psi) \, dV\right|^2$$

$$\geqslant \frac{1}{4}\left|\int \Psi^{*}(\alpha\beta - \beta\alpha)\Psi \, dV\right|^2 \qquad (2\text{-}7.5)$$

Clearly,

$$\alpha\beta - \beta\alpha = AB - BA = i\hbar$$

Which implies that,

$$\left|\int \Psi_1^{*} \Psi_2 \, dV \right| \geqslant \frac{\hbar^2}{4} \left|\int \Psi^{*} \Psi \, dV \right|^2$$

$$\geqslant \frac{\hbar^2}{4}$$

Thus, we have

$$(\Delta A)^2 (\Delta B)^2 \geqslant \frac{\hbar^2}{4}$$

$$(\Delta A)(\Delta B) \geqslant \frac{\hbar}{2} \qquad (2\text{-}7.6)$$

This is a general statement of Heisenberg's principle. In particular if $A = x$ and $B = p$ where x and p are the coordinates and momentum respectively then uncertainty in p and x will satisfy

$$\Delta x \, \Delta p \geqslant \frac{\hbar}{2} \qquad (2\text{-}7.7)$$

Thus we have seen that in quantum mechanics any physical quantity does not have an exact value and the product of uncertainty in measurement of two variables is always greater than $\frac{\hbar}{2}$.

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