CHAPTER IV

SOME ADDITIONAL LIMIT THEOREMS ON THE LINEAR EXPLOSIVE REGRESSION MODEL WITH A NON-STATIONARY ERROR FOR TIME SERIES

4.1 INTRODUCTION

Here, our primary concern will be to exploit the potentiality of Theorem 3.4.2, and Theorem 3.5.2, for the purpose of deriving a series of limit theorems on the Stochastic model (3.1.2) for the time series $x$, specified by (3.1.1). In this direction, we are inspired by Venkataraman (1979 (a)), and Venkataraman and Suresh Chandra (1960 (c)) with respect to linear explosive time series.

In this context, it is assumed that the time series error process $\eta = \{ \eta(t); t \geq 1 \}$ is of the type

$$\eta(t) = \sum_{r=0}^{t-1} a(r) \varepsilon(t-r); t \geq 1$$  (4.1.1)

where

$$0 < \left| a(0) \right| + \sum_{r=1}^{\infty} h_0 \left| a(r) \right| < \infty$$  (4.1.2)

for some $h_0 > \frac{1}{2}$. Trivially (4.1.1) allows for the possibility of $\eta(t) = \varepsilon(t)$ for $t \geq 1$, which is of relevance when $1 \leq m < q$, following the notation in Chapter III.

The limit theorems that are proved in this chapter broadly fall into two categories:
(i) Spectral type limit theorems; and,

(ii) Linear type limit theorems.

Sections 4.2 and 4.3 are concerned with spectral limit theorems, relating to the cases: \( m = q \), and, \( 1 \leq m < q \) respectively. The linear type limit theorems relating to the cases: \( m = q \), and, \( 1 \leq m < q \), but with \( \eta(t) = E(t) \), are presented in Section 4.4.

4.2 SPECTRAL TYPE LIMIT THEOREMS ON THE LINEAR EXPLOSIVE REGRESSION MODEL WHEN \( m = q \), AND, \( \eta(t) \) IS SPECIFIED BY (4.1.1).

For integral \( t \), we set

\[
\tilde{\eta}(t) = x(t) - \tilde{b}_1 x(t-1) - \ldots - \tilde{b}_q x(t-q)
- \tilde{X}_0 f_0(t) - \ldots - \tilde{X}_p f_p(t).
\]  
(4.2.1)

By definition, \( \tilde{\eta}(t) = 0 \) for \( t < 0 \). The results that we have in mind for the case \( m = q \), flow from a fundamental interrelation between

\[
\tilde{U}(s) = \sum_{t=1}^{N-|s|} \tilde{\eta}(t) \tilde{\eta}(t+|s|); \ |s| < N
= 0 \quad ; \quad |s| \geq N
\]  
(4.2.2)

and

\[
U(s) = \sum_{t=1}^{N-|s|} \eta(t) \eta(t+|s|); \ |s| < N
= 0 \quad ; \quad |s| \geq N
\]  
(4.2.3)
s being integral. To be precise, our discussion is based on the following basic result on \( \overline{U}(s) \) and \( U(s) \).

**Lemma 4.2.1**  Let the time series \( X \) be specified by (3.1.1), with \( m = q = 1, 2, 3 \); and, \( f_i(t) \) satisfy Assumption 3.1.4-(a), and also the additional Assumption 3.4.1. Then

\[
\overline{U}(s) - U(s) = \sum_{i=1}^{h} B_i(N) L_i(N, s)
\]

\( h \) being a positive integer, and, \( B(N) \) and \( L(N, s) \) satisfying the requirements in (2.1.8).

**Proof:** We recall that, by definition

\[
\eta_l(t) = X(t) - b_1 X(t-1) - \cdots - b_q X(t-q)
\]

\[-\alpha_0 f_0(t) - \cdots - \alpha_p f_p(t); \ t \text{ integral.}
\]

(4.2.4)

On substitutional evaluation, one has from (4.2.1) and (4.2.4) that

\[
\overline{\eta}_l(t) = \eta_l(t) + \sum_{r=1}^{q} (b_r - \overline{b_r}) X(t-r) + \sum_{i=0}^{p} (\alpha_i - \overline{\alpha}_i) f_i(t)
\]

\( ; t \text{ integral.}
\]

(4.2.5)

An appeal to Theorem 3.4.2, (4.2.5) reduces as under:

**Case (i):** \( m = q = 1 \)

\[
\overline{\eta}(t) = \eta(t) + (b_1 - \overline{b_1}) X_1(t-1) + \sum_{i=0}^{p} (\alpha_i - \overline{\alpha}_i) f_i(t)
\]

\( ; t \text{ integral;}
\]
Case (ii) : \( m = q = 2 \)

\[
\tilde{\eta}(t) = \eta(t) - \beta_1^{-1} \tilde{\omega}(12) x_1(t-1) + \beta_1^{-1} (\tilde{\omega}(2) - b_2) x_2(t-1) + \sum_{i=0}^{p} (a_1 - \tilde{x}_1) r_i(t); \text{ t integral};
\]

Case (iii) : \( m = q = 3 \)

\[
\tilde{\eta}(t) = \eta(t) - \beta_1^{-1} (\beta_1 - \beta_2)^{-1} \tilde{\omega}(13) (x_1(t-1) - \beta_2 x_1(t-2)) + \beta_2^{-1} (\beta_1 - \beta_2)^{-1} \tilde{\omega}(23) x_2(t-1) - (\beta_1 \beta_2)^{-1} (\tilde{\omega}(3) - b_3) x_3(t-1) + \sum_{i=0}^{p} (a_1 - \tilde{x}_1) r_i(t). \]

\( \text{t integral} \) \hspace{1cm} (4.2.6)

Through an appeal to Lemma 3.2.8-(a), and, Theorem 3.4.2, we can conclude on routine evaluation, that for the Cases: \( m = q = 1, 2, 3 \), \( \tilde{U}(\alpha) = U(\alpha) \) is additively composed of a finite number of the terms of the type \( B(N) L(N, s) \). \( \square \)

Next, we introduce for \(-\pi \leq \theta \leq +\pi\)

(i) \( \tilde{r}_{U}(\theta) = \sum_{s=-M}^{+M} \tilde{U}(s) \cos \theta s; \)

(ii) \( r_{U}(\theta) = \sum_{s=-M}^{+M} U(s) \cos \theta s; \)

(iii) \( M = \text{integral part of } N \beta; \)

(iv) \( (2\pi_0 + 1)^{-1} \leq \beta < 1; \) (Vide (4.1.2)) \hspace{1cm} (4.2.7)

and, define
\[ f_{\eta}(\theta) = (2\pi)^{-1/2} \sum_{s=-\infty}^{\infty} Q_{\theta}(s) \cos \theta \quad (4.2.8) \]

\( Q_{\theta}(s) \) being defined in the statement of Theorem 3.4.1. Thus settled, we prove:

**THEOREM 4.2.1**: Let the time series \( X \) be specified by (3.1.1), with \( m = q = 1, 2, 3 \), and \( f_{\theta}(t) \) satisfy Assumption 3.1.4-(a) and also the additional Assumption 3.4.1. Let \( \theta_i; i = 1, \ldots, T \) be distinct values from \(-\pi < \theta < +\pi\), such that \( \theta_i + \theta_j \neq 0 \) for \( i \neq j \), and \( \delta_i = \sqrt{2} \) or 1 according as \( \theta_i = 0, \pi \) or not. Then the following statements hold:

(a) \( ((N/2M)^{1/2}(2\pi N)^{-1/2} \cdot f_{\theta_i}(\theta_i) - f_{\eta}(\theta_i)); i = 1, \ldots, T) \)

converges in distribution, as \( N \to \infty \), to a normal vector

\( \left( \delta_i \theta_i; i = 1, \ldots, T \right) \), say, with mean zero, and a diagonal covariance matrix with diagonal elements \( \delta_i^{-2} f_{\eta}(\theta_i); i = 1, \ldots, T \).

(b) \( \left( \delta_i^{-1}(N/2M)^{1/2} \left( \log(2\pi N)^{-1} \cdot f_{\theta_i}(\theta_i) - \log f_{\eta}(\theta_i) \right); i = 1, \ldots, T \right) \)

converges in distribution, as \( N \to \infty \), to a standard normal vector.

**PROOF**: Following Lemma 4.2.1, one has

\[ (N/2M)^{1/2} N^{-1} \left| f_{\theta_i}(\theta) - f_{\eta}(\theta) \right| \leq \| X \|_{(M/N)^{1/2}} \sum_{i=1}^{n} \left| \theta_i(N) \right| \left( 2^{1/2} \right) \sum_{s=-M}^{M} \left| L_i(N, s) \right| \right. \]

\[ (4.2.9) \]

By definition of \( L_i(N, s) \), the non-negative term within the square bracket on the R.H.S of (4.2.9) has an expectation
bounded by $h_0$. Thus the L.H.S of (4.2.9) converges in probability, as $N \to \infty$ to zero.

In addition, we know by Vankataaraman (1977) that the statement (a) is valid, when $\tilde{r}_U(\theta_i)$ is replaced by $r_U(\theta_i)$. These remarks together with an appeal to Lemma 1.2.5 yield (a).

Statement (b) follows from (a) through the lines of arguments indicated by Vankataaraman (1972). □

Next, we choose $\beta_0$ and $\beta$ such that

$$(2h_0 + 1)^{-1} \leq \beta_0 < \beta < \frac{1}{2}$$  \hspace{1cm} (4.2.10)

To observe that

$$(N/2M(\beta))^\frac{1}{2} \left( \log \left| \tilde{r}_U(\theta, \beta) \right| - \log \left| \tilde{r}_U(\theta, \beta_0) \right| \right)
= (N/2M(\beta))^\frac{1}{2} \left( \log(2MN)^{-1} |\tilde{r}_U(\theta, \beta)| - \log f_\lambda(\theta) \right)
- (N/2M(\beta))^\frac{1}{2} \left( \log(2MN)^{-1} |\tilde{r}_U(\theta, \beta_0)| - \log f_\lambda(\theta) \right)$$  \hspace{1cm} (4.2.11)

By virtue of Theorem 4.2.1 (b), the choice of $\beta_0$ renders the second term on the R.H.S of (4.2.11) as an $o_p(N)$. We now replace $\theta$ by $\theta_i$ in (4.2.11) for $i = 1, \ldots, t$ and apply Lemma 1.2.5, and also Theorem 4.2.1-(b) to derive the following results.

**Theorem 4.2.2**: Under the assumptions of Theorem 4.2.1, on the time series $\xi_i$,

(a) \hspace{1cm} \left( \frac{N}{2M(\beta)} \right)^\frac{1}{2} \delta_i^{-1} \left( \log |\tilde{r}_U(\theta_i, \beta)| - \log |\tilde{r}_U(\theta_i, \beta_0)| \right); i = 1, \ldots, T
converges in distribution, as \( N \rightarrow \infty \), to a standard normal vector.

\[
(b) \quad \left( \frac{N}{2M(\beta)} \right) \sum_{i=1}^{T} \delta_i^{-2} \left( \log \left| \tilde{r}_U(q_i, \beta) \right| - \log \left| \tilde{r}_U(q_i, \beta_0) \right| \right)^2
\]

converges in distribution, as \( N \rightarrow \infty \), to a chi-square variable with \( T \) degrees of freedom.

When \( \eta(t) = \varepsilon(t) \), \( r_{\eta}(0) \) reduces to \( \sigma^2/2\pi \), and \( U(s) \) to \( W(s) \) defined in (2.1.6)-(i). Further, by Lemma 4.2.1

\[
\left( \frac{N}{2M} \right)^{1/2} \left( N^{-1} \tilde{U}(0) - N^{-1} U(0) \right) \overset{D}{\longrightarrow} 0
\]

as \( N \rightarrow \infty \). This observation together with (2.3.5), (4.2.10), Lemma 1.2.5, and, Theorem 4.2.1-(a) yields the following result, which reads exactly like Theorem 2.3.2.

**THEOREM 4.2.3:** Under the assumptions of Theorem 4.2.1 on the time series \( X \), the following statements hold.

(a) \( \left( (2M)^{-1/2} \left( 2\pi \right)^{-1} \left( \tilde{r}_U(q_i) - \tilde{U}(0) \right) \right); \ i = 1, \ldots, T \)

converges in distribution, as \( N \rightarrow \infty \), to the limiting normal vector specified in the statement of Theorem 2.3.1.

(b) \( \left( \frac{N}{2M} \right) \left( \tilde{U}(0) \right)^{-2} \sum_{i=1}^{T} \delta_i^{-2} \left( \tilde{r}_U(q_i) - \tilde{U}(0) \right)^2 \overset{d}{\longrightarrow} \chi^2(T) \)

as \( N \rightarrow \infty \).
4.3 SPECTRAL TYPE LIMIT THEOREMS ON THE LINEAR EXPLOSIVE REGRESSION MODEL, WHEN $1 \leq m < q$, AND $\eta(t) = \xi(t)$.

On invoking Lemma 3.5.2, the representation for $\tilde{\eta}(t)$, to be called $\tilde{\xi}(t)$ henceforth, given by (4.2.5) reduces as follows:

Case (i) : $m = 1$, $q = 2$,

$$
\tilde{\xi}(t) = \xi(t) + (b_1 - \tilde{b}_1)X_2(t-1) - \tilde{b}(12)X_1(t-2) + \sum_{i=0}^{p} (\alpha_i - \tilde{\alpha}_i) r_i(t).
$$

Case (ii) : $m = 1$, $q = 3$,

$$
\tilde{\xi}(t) = \xi(t) + (b_1 - \tilde{b}_1)(X_2(t-1) + \tilde{\phi}_1 X_2(t-2)) + (b_2 - \tilde{b}_2)X_2(t-2) + \tilde{b}(13)X_1(t-3) + \sum_{i=0}^{p} (\alpha_i - \tilde{\alpha}_i) r_i(t).
$$

Case (iii) : $m = 2$, $q = 3$

$$
\tilde{\xi}(t) = \xi(t) + (b_1 - \tilde{b}_1)X_3(t-1) + (\tilde{\phi}_1 - \phi_2)\tilde{b}(23)X_2(t-2) - (\phi_1 - \phi_2)^{-1} \tilde{b}(13)(X_1(t-2) - \phi_2 X_1(t-3)) + \sum_{i=0}^{p} (\alpha_i - \tilde{\alpha}_i) r_i(t) \tag{4.3.1}
$$

In parallel to Lemma 4.2.1, the following result is derived.

**Lemma 4.3.1** : Let the time series $X$ be specified by (3.1.1), with $1 \leq m < q = 2, 3$; and $f_i(t)$ satisfy Assumption 3.1.4-(b), and, also the additional assumptions 3.4.1, and 3.5.1. Then $\tilde{U}(a) - U(a)$ is of the form
\[
\sum_{s=1}^{N} B_i(N) L_i(N, s) + I(m, q), \text{ (say),}
\]

where

(a) \[ I(m, q) = (b_1 - b_1) \mu_{m+1}(|s|-1) \sum_{t=1}^{N-|s|} \varepsilon^2(t) \]

for \( m = 1, q = 2; \)

(b) \[ I(m, q) = (b_1 - b_1) \left( \mu_{m+1}(|s|-1) + \mu_{m+1}(|s|-2) \right) \sum_{t=1}^{N-s} \varepsilon^2(t) \]

\[ + (b_2 - b_2) \mu_{m+1}(|s|-2) \sum_{t=1}^{N-|s|} \varepsilon^2(t) \]

for \( m = 1, q = 3; \)

(c) \[ I(m, q) = (b_1 - b_1) \mu_{m+1}(|s|-1) \sum_{t=1}^{N-|s|} \varepsilon^2(t) \]

for \( m = 2, q = 3; \)

...in defining that \( \mu_{m+1}(r) = 0 \) for \( r < 0; \) \( \mu_{m+1}(0) = 1, \) and, for \( r \geq 1 \)

\[
\prod_{i=m+1}^{q} (1 - p_i \varepsilon r^{-1}) \mu_{m+1}(r) = 0.
\]

NOTE 4.3.1: The proof of this lemma follows, on routine evaluation, from an application of Lemma 3.2.8-(b), Theorem 3.5.2, and, on recalling earlier details presented in Section 3.2.

It is obvious that

(i) \[ \sum_{s=-m}^{N} \left| \mu_{m+1}(|s|-1) \right| \leq \sum_{u=0}^{\infty} \left| \mu_{m+1}(u) \right| \leq h_0; \]

(ii) \[ E \left( N^{-1} \sum_{t=1}^{N} \varepsilon^2(t) \right) \leq h_0 \] (4.3.2)
We also recall from Assumption 1.1.5-(f) that $R^{-\frac{1}{2}}(N) Y^2 \leq \alpha$, and infer from Theorem 3.5.1 that $R^{-\frac{1}{2}}(N) \left| b_{1}\right| \leq \frac{\delta_{1}}{2}$ are bounded in probability. These remarks yield through Lemma 4.3.1 that when $1 \leq m < q = 2, 3$, and, $\eta(t) = \epsilon(t)$

$$(N/2m)^{\frac{1}{2}} N^{-1} \left( r_{u}(a) - r_{y}(a) \right) \xrightarrow{P} 0$$

as $N \rightarrow \infty$.  

This result leads to:

**THEOREM 4.3.1**: Let (i) the time series $X$ be specified by (3.1.1) with $1 \leq m < q = 2, 3$, and, $\eta(t) = \epsilon(t)$; (ii) $r_{1}(t)$ satisfy Assumption 3.1.4-(b), and, also the additional Assumptions 3.4.4 and 3.5.1, and; (iii) $\theta_{i}$ and $\delta_{i} ; i = 1, \ldots, T$ be chosen as in statement of Theorem 4.2.1. Then the statements in Theorem 4.2.1, and, Theorem 4.2.2 hold with $r_{u}(a)$ as $\sigma^{2}/2\pi$.

Further, by virtue of Lemma 4.2.1, (4.2.12) continuous to be valid. In consequence, we have the following parameter free spectral theorem, in parallel to Theorem 2.3.2, and, Theorem 4.2.3.

**THEOREM 4.3.2**: Under the assumptions of Theorem 4.3.1 on the time series $X$, the statements in Theorem 4.2.3 continue to hold.

### 4.4 Linear Type Limit Theorems on the Linear Explosive Regression Model, When $\eta(t) = \epsilon(t)$.

Let us define, for integral $s$, that

(i) $\overline{r}_{1}(s) = \overline{u}(s) - b_{1} \overline{u}(s-1) - \ldots - b_{q} \overline{u}(s-q)$;
(ii) \( L_1(s) = U(s) - b_1 U(s-1) - \ldots - b_q U(s-q) \); 
(iii) \( \tilde{L}_1(s) = \tilde{U}(s) - \tilde{b}_1 \tilde{U}(s-1) - \ldots - \tilde{b}_q \tilde{U}(s-q) \).

From Lemma 4.2.1, and Lemma 4.3.1, it can be deduced, on routine evaluation, that 

\[
N^{-\frac{3}{2}} (\tilde{L}_1(s) - L_1(s)) \xrightarrow{D} 0 \tag{4.4.2}
\]

as \( N \to \infty \), under the following cases:

(i) \( m = q = 1, 2, 3 \); 
(ii) \( m = 1; q = 2 \); 
(iii) \( m = 1, 2; q = 3 \). \tag{4.4.3}

Further the assumption \( \gamma(t) = E(t) \) implies that 

\[
(N^{-\frac{1}{2}} \sigma^{-2} U(s); \ s = 1, \ldots, T) \tag{4.4.4}
\]

converges in distribution, as \( N \to \infty \), to a standard normal vector \( (\Phi_0(s); s = 1, \ldots, T) \). (Vide: Lemma 2.2.3).

These observations together with an application of Lemma 1.2.5 yield the following basic result.

THEOREM 4.4.1: Let the time series \( X \) be specified by (3.1.1) with \( \gamma(t) = E(t) \). Then the following statements hold.

(a) Let \( m = q = 1, 2, 3 \). Then under assumption 3.1.4-(a), 
Assumption 3.4.1 on \( f_1(t) \) 

\[
(N^{-\frac{1}{2}} \tilde{U}_1(s); s = q + 1, \ldots, q + T) \]
converges in distribution, as $N \to \infty$, to a normal vector

$$\mathcal{S}_{11}(s); s = q + 1, \ldots, q + T$$

so, where

$$\mathcal{S}_{11}(s) = \sigma^2 (\mathcal{S}_{0}(s) - b_1 \mathcal{S}_{0}(s-1) - \cdots - b_q \mathcal{S}_{0}(s-q)); s \geq q+1.$$ 

(b) Let $1 \leq m < q = 2, 3$. Then under assumption 3.1.4-(b); and the additional assumptions 3.4.1, and, 3.5.1 on $r_i(t)$,

$$(N^{-1/2} \nu_i(s); s = q + 1, \ldots, q + T)$$

converges in distribution, as $N \to \infty$, to the normal vector

$$\mathcal{S}_{11}(s); s = q + 1, \ldots, q + T$$

specified in (a).

By virtue of Lemma 4.2.1, Lemma 4.3.1, and, Theorem 3.5.1, we can infer that, under all the cases: $m = q = 1, 2, 3; 1 \leq m < q = 2, 3$

$$N^{-1} (\widetilde{U}(s) - U(s)) \overset{P}{\to} 0$$

as $N \to \infty$. However, the assumption $\eta(t) = \mathcal{E}(t)$ implies that

$N^{-1} U(s) \overset{P}{\to} 0$, for $s > 1$, and, to $\sigma^{-2}$ for $s = 0$, as $N \to \infty$.

These observations together with another application of Theorem 3.5.1 yield that

$$N^{-1/2} (\widetilde{P}_1(s) - \widetilde{L}_1(s)) = (F^{-1/2}(N) N^{-1/2}) \sum_{i=1}^{q} F^{-1/2}(N) (\widetilde{b}_i - b_i) N^{-1/2} U(s-1) \overset{P}{\to} 0$$

as $N \to \infty$, for $s \geq q + 1$. These remarks enable us to validate the following results on the basis of Lemma 1.2.5, Theorem 3.4.1, Theorem 3.5.1, and, Theorem 4.4.1.
THEOREM 4.4.2: Let the time series $\mathbf{X}$ be specified by (3.1.1), with $m = q = 1, 2, 3; \eta(t) = \mathcal{E}(t)$; and, $f_i(t)$ satisfy Assumption 3.1.4-(a), and, Assumption 3.4.1. Then the following statements hold.

(a) $\left( N^{-\frac{1}{2}} \tilde{L}_1(s); s = q + 1, \ldots, q + T \right)$ converges in distribution, as $N \to \infty$, to the normal vector $(\xi_1(s); s = q + 1, \ldots, q + T)$.

(b) Let $s_i; i = 1, \ldots, T$ be positive integers such that $s_i - s_{i-1} \geq q + 1$ for $i = 1, \ldots, T; (s_0 = 0)$. Then

$$N \left( \sum_{i=1}^{q} b_{i, i}^2 + R \right) \left( \tilde{U}(0) \right)^{-2} \sum_{i=1}^{T} \tilde{L}_1^2(s_i) \xrightarrow{d} \chi^2(T)$$

as $N \to \infty$.

THEOREM 4.4.3: Let the time series $\mathbf{X}$ be specified by (3.1.1), with $1 \leq m \leq q = 2, 3; \eta(t) = \mathcal{E}(t)$; and, $f_i(t)$ satisfy Assumption 3.1.4-(b), and, the additional Assumptions 3.4.1 and 3.5.1. Then the statements (a) and (b) of Theorem 4.4.2 continue to hold.

4.5 DISCUSSION

Reduced to essentials, we have discussed in Chapters I to IV, three versions of the stochastic model of the type

$$x(t) = c_1 x(t-1) - \ldots - c_h x(t-h) = f(t) + \mathcal{E}(t); t \geq 1 \quad (4.5.1)$$
with the explicit solution

\[ X(t) = \sum_{r=0}^{t-1} d(r) (f(t-r) + \varepsilon(t-r)) \]  \hspace{1cm} (4.5.2)

where \( d(r) = 0 \) for \( r < 0 \); \( d(0) = 1 \); and for \( r > 1 \)

\[ d(r) = c_1 d(r-1) - \ldots - c_h d(r-h) = 0 . \]  \hspace{1cm} (4.5.3)

These versions are conditioned by three distinct hypotheses listed below which relate to the zeroes \( \Gamma_1, \ldots, \Gamma_h \) of the polynomial

\[ c(z) = z^h - c_1 z^{h-1} - \ldots - c_h \]  \hspace{1cm} (4.5.4)

and to the regression components \( f_i(t) \).

Hypothesis I \( (H_1) \): \[ \bar{\Gamma}_r < 1; \ r = 1, \ldots, h; \] and \( f_i(t) \) satisfy Assumptions 1.1.5, and, 1.4.1.

Hypothesis II \( (H_2) \): \[ \bar{\Gamma}_1 > \bar{\Gamma}_2 > \ldots > \bar{\Gamma}_h > 1; \] \( f_i(t) \) satisfy Assumptions 3.1.4-(a) and 3.4.1.

Hypothesis III \( (H_3) \): \[ \bar{\Gamma}_1 > \ldots > \bar{\Gamma}_m > 1 > \bar{\Gamma}_r \] \( r = m+1, \ldots, h, \) \( (1 \leq m < h) \); and, \( f_i(t) \) satisfy Assumptions 3.1.4-(b), 3.4.1 and 3.5.1.

Let \( c_r^* ; \ r = 1, \ldots, h; \) and \( \lambda_i^* \; i = 0, \ldots, p \) be the least square estimators of \( c_r ; \ r = 1, \ldots, h; \) and \( \lambda_i; \ i = 0, \ldots, p \) which minimise the sum of squares.
\[
\sum_{t=1}^{N-h} (x(t+h) - c_1 x(t+h-1) - \ldots - c_h x(t))
- \alpha_0 r_0 (t+h) - \ldots - \alpha_p r_p (t+h)^2
\]  
(4.5.5)

with respect to these parameters.

Let

(i) \[ E^*(t) = x(t) - \sum_{r=1}^{h} c_r x(t-r) - \sum_{i=0}^{p} \alpha_i^* r_i (t); \ t \text{ integral} \]

(ii) \[ W^*(s) = \sum_{t=1}^{N-|s|} E^*(t) E^*(t+|s|); \ |s| < N \]

\[ = 0 \quad \text{for} \ |s| > N \]

(iii) \[ F^*_W(\theta) = \sum_{s=-M}^{M} W^*(s) \cos \theta s; \ -\pi \leq \theta \leq +\pi \]

(iv) \[ M = \text{Integral part of} \ N^p; \ 0 < \beta^p < 1. \]  
(4.5.6)

To steer clear of redundant details, we make the

Hypothesis IV (H): \[ r_0(t) = \cos \psi(t); \ r_1(t) = \sin \psi t, \ \psi \]

being known; \( 0 < \psi < \pi \).

A careful perusal of spectral type limit Theorems 2.3.2,
Theorem 4.2.3, and Theorem 4.3.2 will enable us to derive the
following unified result.

THEOREM 4.5.1: Let the time series \( x \) be specified by (4.5.2),
and the zeroes \( T_r^c; r = 1, \ldots, n \) of the polynomial \( C(z) \) have
either of the placements covered by the hypotheses \( H_1, H_2, \) and
$H_3$. Let $\theta_i; i = 1, \ldots, T$ be distinct values from $-\pi < \theta_i < +\pi$, such that $\theta_i + \theta_j \neq 0$ for $i \neq j$, and, $\delta_i = \sqrt{2}$ or 1 according as $\theta_i = 0, \pi$ or not. Then, under $H$, as $N \to \infty$,

$$(N/2m)^{\frac{1}{2}} (u^*(0))^{-2} \sum_{i=1}^{T} \delta_i^{-2} (f^*_u (\theta_i)) - u^*(0))^2 \overset{d}{\to} \chi^2(T).$$

This unified result is rendered possible because, under $H$ the respective requirements on $f_1(t)$ as specified in hypotheses $H_1, H_2$ and $H_3$ are satisfied under appropriate placement of $\gamma^*_r; r = 1, \ldots, h$.

Such a unified result cannot hold when $f_1(t) = t^i$, since under placement $|\gamma^*_r| < ?; r = 1, \ldots, h$, assumption 1.4.1 is violated.

In essence a unified result is rendered possible, if, and, only if the respective requirements imposed on $f_1(t)$ under $H_1, H_2$, and, $H_3$ are satisfied by the particular choice of $f_1(t)$. Thus a unified approach to the time series is rendered difficult, unlike the case of simple linear explosive stochastic model considered by [Vankataraman (1979 (a))], and, [Parthasarathy (1980)].

Based upon the linear type limit Theorem 2.2.2, Theorem 4.4.2, and, Theorem 4.4.3 we can derive the following unified result.
THEOREM 4.5.2: Under the hypotheses of Theorem 4.5.1

\[ N \left( \sum_{i=1}^{n} \left( c_i^* + 1 \right) \left( w^*(0) \right)^{-2} \frac{1}{T} \sum_{i=1}^{T} \left( L_i^* \left( s_i \right) \right)^2 \right) \xrightarrow{d} \chi^2(T) \]

as \( N \to \infty \), where

(i) \( L_i^* \left( s \right) = w^*(s) - c_i^* \cdot w^*(s-1) - \ldots - c_n^* \cdot w^*(s-n) \);

(ii) \( s_i - s_{i-1} \geq h + 1 \); \( i = 1, \ldots, T \), with \( s_0 = 0 \).