Chapter 1

Introduction

1.1 Background and Motivation

In this thesis we investigate the hyponormality of trigonometric Toeplitz operators on Hilbert spaces of analytic functions, namely, the Hardy-Hilbert space $H^2(\mathbb{T})$, the Bergman space $A^2(\mathbb{D})$ and the weighted Bergman space $A^2_\alpha(\mathbb{D})$, where $\alpha > -1$, $\mathbb{T} = \partial \mathbb{D}$ and $\mathbb{D}$ is the unit disc of complex plane. A Toeplitz operator on these spaces is a multiplication operator followed by a projection onto the initial space. The main objective of our research work is to address the following question:

"Is every trigonometric Toeplitz operator hyponormal? If not, which trigonometric polynomials induce a hyponormal Toeplitz operator and under what conditions?"

In [4], Cowen first characterized the hyponormality of Toeplitz operators on the Hardy space $H^2$ in terms of the coefficients of the symbol $\varphi$. His work was further extended by Nakazi and Takahashi. In [44], they reformulated Cowen’s result as follows:

“For $\varphi \in L^\infty(\mathbb{T})$, the Toeplitz operator $T_\varphi$ is hyponormal if and only if there exists $k \in H^\infty(\mathbb{T})$, $\|k\|_\infty \leq 1$ such that $\varphi - k\bar{\varphi} \in H^\infty(\mathbb{T})$."

1
This reformulation of Cowen’s characterization gives a way to determine the hyponormality of $T_\varphi$ for arbitrary trigonometric polynomial symbols $\varphi$. But the problem here is that for the hyponormality of $T_\varphi$ one requires to solve a certain functional equation in the unit ball of $H^\infty$ which is not always easy. In [53], Kehe Zhu shows that the hyponormality of Toeplitz operators with trigonometric polynomial symbols on the Hardy space can ultimately be reduced to the classical Schur’s algorithm in function theory. Thus Zhu again reformulated Cowen’s characterization via Schur’s functions $\Phi_n$ for $n \geq 0$. By explicitly using Schur’s functions $\Phi_n$, Zhu has given a general criterion to determine the hyponormality of the Toeplitz operator $T_\varphi$ with polynomial symbol $\varphi(z)$, where $\varphi$ is of the form $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$.

Although Zhu’s Theorem (refer to Theorem 2.4.1) gives a substantial amount of information to determine the hyponormality of $T_\varphi$, the main hurdle here is to find the Schur’s functions $\Phi_n$ beyond $n \geq 2$. As there is no closed form to determine them so the process is quite laborious. In [36], Kim and Lee gave an alternate version of Zhu’s theorem. Similarly in [25], Hwang and Kim studied the hyponormality of $T_\varphi$ for special kinds of trigonometric polynomial symbols $\varphi$ which are called circulant type symbols. Thus attempts are continuously being made to determine conditions under which a trigonometric polynomial will induce a hyponormal Toeplitz operator. We quote here a few significant references [6], [14], [26], [27], [28], [29], [30], [31], [32], [38] and [52].

The results obtained from the study of hyponormality of Toeplitz operators on the Hardy space do not automatically extend to the Bergman space. This is because Cowen’s Theorem does not adapt to the Bergman space. The reason is that Cowen’s theorem is based on a dilation theorem of Sarason [46]. In the Hardy space $H^2$, the functions in $H^2$ are the conjugates of the functions in $zH^2$. But, for the Bergman space $A^2$, $A^2$ is much larger than the conjugates of functions in $zA^2$. And so we
can find no dilation theorem similar to Sarason in $A^2$. In his doctoral thesis, H. Sadraoui \cite{45} was the first to study the hyponormality of trigonometric Toeplitz operators on $A^2$ space via Hankel operators. Hyponormality of Toeplitz operators was also studied in \cite{40}. Then in \cite{23}, \cite{24}, \cite{33} and \cite{39}, the hyponormality of Toeplitz operators was studied by considering some polynomial symbols which are similar to the polynomials considered in the $H^2$ space. Till date there is no concrete characterization in $A^2$ space by which the hyponormality of $T_\varphi$ can be studied for an arbitrary trigonometric polynomial.

1.2 Definitions and Terminologies

Here we give a brief summary of the spaces and operators that are being considered in this work. We include the definitions and significant properties along with the references where these results are discussed in detail. For notational convenience throughout the thesis, $\mathbb{C}$ will always denote the complex plane. The open unit disk in the complex plane $\{z \in \mathbb{C} : |z| < 1\}$ will be denoted by $\mathbb{D}$, and the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ will be denoted by $\mathbb{T}$.

1.2.1 Hardy-Hilbert Space

The Hardy-Hilbert space is the set of all analytic functions whose power series have square summable coefficients. The Hilbert space of functions analytic on the disk is customarily denoted by $H^2$. Some other spaces of analytic functions are the Bergman and Dirichlet spaces.

Details of these spaces are found in references \cite{22}, \cite{43}, \cite{54}.

We begin with a formal definition of the $H^2$ space.
Definition 1.2.1. [43] The Hardy-Hilbert space, denoted by $H^2$, consists of all analytic functions having power series representations with square summable complex coefficients. That is,

$$H^2 = \{ f : f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } \sum_{n=0}^{\infty} |a_n|^2 < \infty \}$$

For $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ in $H^2$, the inner product is defined as

$$\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}$$

Also the norm of the vector $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is

$$\|f\| = \left( \sum_{n=0}^{\infty} |a_n|^2 \right)^{\frac{1}{2}}$$

For $z_0 \in \mathbb{D}$, the function $k_{z_0}$ defined by

$$k_{z_0}(z) = \sum_{n=0}^{\infty} \overline{z_0}^n z^n = \frac{1}{1 - \overline{z_0} z}$$

is called the reproducing kernel for $z_0$ in $H^2$. Clearly, $k_{z_0} \in H^2$ and $\|k_{z_0}\| = (1 - |z_0|^2)^{-\frac{1}{2}}$. Also for $f \in H^2$, $f(z_0) = \langle f, k_{z_0} \rangle$.

The Hardy-Hilbert space can also be viewed as a subspace of the well known Hilbert space $L^2$. $L^2 = L^2(\mathbb{T})$ is the Hilbert space of square-integrable functions on the unit circle $\mathbb{T}$ with respect to Lebesgue measure, normalized so that the measure of the entire circle is 1. The inner product is given by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{0}^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} d\theta$$

where $d\theta$ denotes the ordinary (not normalized) Lebesgue measure on $[0, 2\pi]$. Therefore the norm of the function $f$ in $L^2$ is given by

$$\|f\| = \left( \frac{1}{2\pi} \int_{0}^{2\pi} |f(e^{i\theta})|^2 d\theta \right)^{\frac{1}{2}}.$$

Though the elements of $L^2$ are equivalence classes, we consider them as functions with the understanding that $f$ and $g$ in $L^2$ are equal if $f$ is equal to $g$ almost
everywhere with respect to normalized Lebesgue measure.
For each integer \( n \), let \( e_n(e^{i\theta}) := e^{in\theta} \), regarded as a function on \( \mathbb{T} \). Then \( \{e_n : n \in \mathbb{Z}\} \) forms an orthonormal basis for \( L^2 \). Thus, for \( f \in L^2 \), \( \langle f, e_n \rangle \) denotes the \( n \)th Fourier coefficient of \( f \) and so,
\[
H^2 = \{f \in L^2 : \langle f, e_n \rangle = 0, \text{ for } n \in \mathbb{Z} \text{ and } n < 0\}
\]
It is clear that \( H^2 \) is a closed subspace of \( L^2 \). Thus \( H^2 \) consists of square integrable analytic functions on \( \mathbb{T} \).
If \( L^\infty \) denotes the space of all essentially bounded functions in \( L^2 \), then
\[
H^\infty = \{f \in L^\infty : \langle f, e_n \rangle = 0, \text{ for all } n \in \mathbb{Z} \text{ and } n < 0\}
\]
Equivalently, \( H^\infty \) consists of all functions that are analytic and bounded on the open unit disk \( \mathbb{D} \). The norm of a function \( f \in H^\infty \) is defined by
\[
\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{D}\}
\]

1.2.2 Toeplitz Operator

A bounded measurable function \( \varphi \) on the unit circle \( \mathbb{T} \) induces in a natural way two operators, one on \( L^2 \) and one on \( H^2 \) as follows:

**Definition 1.2.2.** The **Laurent operator** \( L = L_\varphi \) is the multiplication operator by \( \varphi \) defined as \( L_\varphi f = \varphi \cdot f \) for every \( f \in L^2 \).

**Definition 1.2.3.** The **Toeplitz operator** is the compression of the multiplication operator to the subspace \( H^2 \). That is, for \( \varphi \in L^\infty \), the Toeplitz operator with symbol \( \varphi \) is the operator \( T_\varphi \) defined by \( T_\varphi f = P(\varphi \cdot f) \) for \( f \in H^2 \), where \( P \) is the orthogonal projection of \( L^2 \) onto \( H^2 \).
$T_\varphi$ is linear and since $\varphi \in L^\infty$ so it is also bounded with norm $\|T_\varphi\| = \|\varphi\|_\infty$. If, in particular, $\varphi \in H^\infty$ then $T_\varphi$ is called an **analytic Toeplitz operator**.

The Toeplitz operator $T_\varphi$ is said to be co-analytic if $T_\varphi^*$ is analytic, or equivalently if $\overline{\varphi} \in H^\infty$.

The Toeplitz operator $T_\varphi$ induced by a trigonometric polynomial $\varphi$ defined as $\varphi(z) := \sum_{n=-m}^{N} a_n z^n$, is called a **trigonometric Toeplitz operator**.

We recall here some of the important properties of Toeplitz operators as recorded in [2], [43] and [54].

(a) If $\varphi$ and $\psi$ are in $L^\infty$ then $T_{\alpha \varphi + \psi} = \alpha T_\varphi + T_\psi$ for any scalar $\alpha$. Also, $T_\varphi^* = T_{\overline{\varphi}}$.

(b) For $\varphi$ and $\psi$ are in $L^\infty$, $T_\psi T_\varphi$ is a Toeplitz operator if and only if either $T_\psi$ is co-analytic or $T_\varphi$ is analytic. In both these cases, $T_\psi T_\varphi = T_{\overline{\varphi}}$.

(c) A Toeplitz operator is self adjoint if and only if its symbol is real valued almost everywhere.

(d) Let $\varphi$ and $\psi$ be in $L^\infty$. Then $T_\varphi T_\psi = T_\psi T_\varphi$ if and only if at least one of the following holds:

(i) Both $\varphi$ and $\psi$ are analytic.

(ii) Both $\varphi$ and $\psi$ are co-analytic.

(iii) There exists complex numbers $\alpha$ and $\beta$, not both zero, such that $\alpha \varphi + \beta \psi$ is a constant.

Since $e_n = e^{int} (n \geq 0)$ form an orthonormal basis for $H^2$, so if $\varphi$ is a bounded function on $\mathbb{T}$ with Fourier coefficients $a_n (n \in \mathbb{N})$, then for $n, m \geq 0$ we have

$$< T_\varphi e_n , e_m > = < \varphi e_n , e_m > = \frac{1}{2\pi} \int_{0}^{2\pi} \varphi(e^{i\theta}) e^{in\theta} e^{-im\theta} d\theta$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} \varphi(e^{i\theta}) e^{-i(m-n)\theta} d\theta$$

$$= < \varphi , e_{m-n} > = a_{m-n}$$

Thus the matrix representation of the Toeplitz operator $T_\varphi$ under the standard basis
\{e_n\} has the following form

\[
\begin{pmatrix}
a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\
a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\
a_2 & a_1 & a_0 & a_{-1} & \cdots \\
a_3 & a_2 & a_1 & a_0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

The main characteristic of the above matrix (called the Toeplitz matrix) is that the entries on each main diagonal are constant. In fact this property actually characterizes Toeplitz operators. That is, if $T$ is a bounded linear operator on $H^2$ with the above matrix under the standard basis, then $T = T_\varphi$ with $\varphi(t) = \sum_{n \in \mathbb{N}} a_n e^{int}$.

Note that if $T_\varphi$ is an analytic Toeplitz operator, then the matrix of $T_\varphi$ with respect to the basis $\{e^{in\theta}\}_{n=0}^\infty$ is

\[
\begin{pmatrix}
a_0 & 0 & 0 & 0 & \cdots \\
a_1 & a_0 & 0 & 0 & \cdots \\
a_2 & a_1 & a_0 & 0 & \cdots \\
a_3 & a_2 & a_1 & a_0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

where $\varphi(z) = \sum_{n=0}^\infty a_n z^n$.

### 1.2.3 Hankel Operator

Another operator that appears hand in hand with Toeplitz operators is the Hankel operator. To define Hankel operator we need to introduce the flip operator on $L^2$ defined as follows:

**Definition 1.2.4.** The *flip operator* is the operator $J$ mapping $L^2$ into $L^2$ defined by $J(e^{in\theta}) = e^{-i(n+1)\theta}$

Clearly the operator $J$ is self adjoint and unitary.
Definition 1.2.5. [4] For $\varphi \in L^\infty$, the Hankel operator $H_\varphi$ is the operator on $H^2$ given by

$$H_\varphi u = J(I - P)(\varphi \cdot u),$$

where $J$ is the flip operator on $L^2$, and $P$ is the projection of $L^2$ onto $H^2$.

With respect to the standard orthonormal basis for $H^2$, the matrix representation of the Hankel operator $H_\varphi$ is as follows:

$$
\begin{pmatrix}
a_{-1} & a_{-2} & a_{-3} & \cdots \\
a_{-2} & a_{-3} & a_{-4} & \cdots \\
a_{-3} & a_{-4} & a_{-5} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

where $\varphi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$.

The main characteristic of the above matrix is that the entries on each skew-diagonal are constants. In fact, this property determines a Hankel operator on the Hardy space.

Necessary facts about Hankel operators include

(i) $H_{\varphi_1} = H_{\varphi_2}$ if and only if $(I - P)\varphi_1 = (I - P)\varphi_2$;

(ii) $\|H_\varphi\| = \inf \{\|\psi\|_\infty : (I - P)\varphi = (I - P)\psi\}$;

(iii) $H_\varphi^* = H_{\varphi^*}$;

(iv) $H_\varphi U = U^*H_\varphi$, where $U$ is the unilateral shift operator;

(v) Either $H_\varphi$ is one-to-one or $\ker(H_\varphi) = \chi H^2$, where $\chi$ is an inner function.

The closure of the range of $H_\varphi$ is $H^2$ in the former case and $(\chi^*H^2)^\perp$ in the later.

1.2.4 Bergman Space

Let $dA(z)$ be the area measure on $\mathbb{D}$ normalized so that the area of $\mathbb{D}$ is 1. In rectangular and polar coordinates,

$$dA(z) = \frac{1}{\pi} dx dy = \frac{1}{\pi} r dr \theta.$$
Here the Hilbert space $L^2(\mathbb{D})$ is defined with respect to the inner product

$$\langle f, g \rangle = \int_D f(z)\overline{g(z)}dA(z).$$

**Definition 1.2.6.** The Bergman space $A^2 = A^2(\mathbb{D})$ is the closed subspace of $L^2(\mathbb{D})$ consisting of all analytic functions on $\mathbb{D}$.

The set

$$\{e_n(z) = \sqrt{n+1}z^n : n \in \mathbb{Z}, n \geq 0 \text{ and } z \in \mathbb{D}\}$$

acts as an orthonormal basis for $A^2$. The Bergman kernel, to be denoted by $K(z, w)$, is the function

$$K(z, w) = \frac{1}{(1 - z\overline{w})^2}$$

for all $z, w \in \mathbb{D}$. If $P$ denotes the Bergman projection from $L^2(\mathbb{D})$ onto $A^2$ then for any $f \in L^2(\mathbb{D})$ and $z \in \mathbb{D},$

$$Pf(z) = \int_D K(z, w)f(w)dA(w)$$

If $L^\infty(\mathbb{D})$ is the space of bounded area measurable functions on the unit disc $\mathbb{D}$, then for $\varphi \in L^\infty(D)$, the multiplication operator $M_\varphi$, on the Bergman space is defined by $M_\varphi(f) = \varphi.f$, where $f \in A^2(\mathbb{D})$. The Toeplitz operator $T_\varphi$ on $A^2(\mathbb{D})$ is defined as $T_\varphi(f) = P(\varphi.f)$. Similarly, the Hankel operator on $A^2(\mathbb{D})$ is defined by $H_\varphi(f) = J(I - P)(\varphi.f)$.

If $T_\varphi$ is the Toeplitz operator defined on $A^2(\mathbb{D})$ space, then we have

$$T_\varphi(f)(z) = P(\varphi.f)(z) = \int_D \frac{\varphi(w)f(w)}{(1 - z\overline{w})^2}dA(w)$$

**1.2.5 Weighted Bergman space**

Let $\mathbb{D}$ denote the open unit disc in the complex plane. For $-1 < \alpha < \infty$, $L^2(\mathbb{D}, dA_\alpha)$ is the space of functions on $\mathbb{D}$ which are square integrable with respect to the measure $dA_\alpha$. For all $\varphi(z) \in L^\infty(\mathbb{D})$, the multiplication operator $M_\varphi$ on $L^2(\mathbb{D}, dA_\alpha)$ is defined by $M_\varphi(f) = \varphi(z)f(z)$, where $f \in L^2(\mathbb{D}, dA_\alpha)$. The Toeplitz operator $T_\varphi$ on $L^2(\mathbb{D}, dA_\alpha)$ is defined as $T_\varphi(f) = P(\varphi.f)$. Similarly, the Hankel operator on $L^2(\mathbb{D}, dA_\alpha)$ is defined by $H_\varphi(f) = J(I - P)(\varphi.f)$.
\( dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z) \), where \( dA \) denotes the normalized Lebesgue area measure on \( \mathbb{D} \). \( L^2(\mathbb{D}, dA_\alpha) \) is a Hilbert space with the inner product

\[
\langle f, g \rangle_\alpha = \int_{\mathbb{D}} f(z)\overline{g(z)}dA_\alpha(z)
\]

for \( f, g \in L^2(\mathbb{D}, dA_\alpha) \).

**Definition 1.2.7.** The **weighted Bergman space** \( A^2_\alpha \) is the closed subspace of \( L^2(\mathbb{D}, dA_\alpha) \) consisting of analytic functions on \( \mathbb{D} \). If \( \alpha = 0 \), \( A^2_0 \) is the Bergman space.

For any non negative integer \( n \) and \( z \in \mathbb{D} \), let \( e_n(z) = \frac{z^n}{\gamma_n} \) where \( \gamma_n^2 = \frac{\Gamma(n+1)\Gamma(n+\alpha+2)}{\Gamma(n+\alpha+2)} \).

Here \( \Gamma(s) \) stands for the usual Gamma function. Then \( \{e_n\} \) is an orthonormal basis for \( A^2_\alpha(\mathbb{D}) \).

The reproducing kernel of \( A^2_\alpha(\mathbb{D}) \) is given by

\[
K^{(o)}_z(w) = \frac{1}{(1 - \bar{z}w)^{2+\alpha}} \quad \text{for} \quad z, w \in \mathbb{D}
\]

The orthogonal projection \( P_\alpha \) of \( L^2(\mathbb{D}, dA_\alpha) \) onto \( A^2_\alpha(\mathbb{D}) \) is given by

\[
(P_\alpha f)(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - z\bar{w})^{2+\alpha}}dA_\alpha(w),
\]

for \( f \in L^2(\mathbb{D}, dA_\alpha) \).

If \( L^\infty(\mathbb{D}) \) denotes the space of all essentially bounded, measurable functions then for \( \varphi \in L^\infty(\mathbb{D}) \) the multiplication operator \( M_\varphi \) on \( A^2_\alpha(\mathbb{D}) \) is defined by \( M_\varphi(f) = \varphi.f \).

The Toeplitz operator \( T_\varphi \) with symbol \( \varphi \) is defined on \( A^2_\alpha(\mathbb{D}) \) by \( T_\varphi f = P_\alpha(\varphi.f) \).

Thus we have

\[
T_\varphi f(z) = \int_{\mathbb{D}} \frac{\varphi(w)f(w)}{(1 - z\bar{w})^{2+\alpha}}dA_\alpha(w),
\]

\( f \in A^2_\alpha(\mathbb{D}) \) and \( z \in \mathbb{D} \).

Similarly, the Hankel operator \( H_\varphi \) on \( A^2_\alpha(\mathbb{D}) \) is defined by \( H_\varphi f = J(I - P_\alpha)(\varphi.f) \).

As \( \varphi \in L^\infty(\mathbb{D}) \), the operators \( T_\varphi \) and \( H_\varphi \) are bounded. For details on these results, we may refer \cite{54}.
1.2.6 Hyponormal Operator

**Definition 1.2.8.** A bounded linear operator $T$ on a complex Hilbert space $H$ is said to be hyponormal if its self commutator $[T^*, T] := T^*T - TT^*$ is positive semi definite. That is, $T$ is hyponormal if $T^*T - TT^* \geq 0$.

The operator $T$ is said to be normal if $T$ commutes with $T^*$, that is $T^*T = TT^*$.

$T$ is said to be subnormal if it has a normal extension.

It is well known from results discussed in [8] and [16] that every normal operator is subnormal, and every subnormal operator is hyponormal. The converse however is not true. For example, the unilateral shift operator on the $\ell^2$ space is subnormal but not normal. Here $\ell^2$ denotes the space of all square integrable complex sequences.

Again for $0 < a < b < 1$, if $\{\alpha_n\}$ denotes the weight sequence defined as $\alpha_0 = a$, $\alpha_1 = b$ and $\alpha_n = 1 (n \geq 2)$, then the weighted shift operator $T$ defined on $\ell^2$ as

$$T(x_0, x_1, x_2, \ldots) = (0, \alpha_0x_0, \alpha_1x_1, \alpha_2x_2, \ldots)$$

is hyponormal but not subnormal.

Further, if $T$ is a hyponormal operator, then so is $T - \lambda I$ for $\lambda \in \mathbb{C}$. However, powers of hyponormal operators need not be hyponormal.

1.3 Chapterwise brief outline

The thesis comprises of five chapters and has been organized as follows. The first chapter is introductory in nature. It includes a brief background leading to the problem in hand. The operators and spaces referred to in the sequel are also defined in this chapter.

Our main work begins with chapter 2. Throughout the thesis we have worked on trigonometric Toeplitz operators, denoted by $T_\varphi$, acting on different spaces of analytic functions. First we consider $T_\varphi$ acting on the Hardy space $H^2(\mathbb{T})$. Subsequently
we consider $T_\varphi$ acting on the Bergman space $A^2(\mathbb{D})$. And finally we consider $T_\varphi$ acting on the weighted Bergman space $A^2_\alpha(\mathbb{D})$. Our objective is to determine necessary and sufficient conditions for $T_\varphi$ to be hyponormal.

The second and third chapters deal with $T_\varphi$ acting on the Hardy space $H^2(\mathbb{T})$. In our work we mainly refer to Kehe Zhu’s reformulation of Cowen’s theorem in terms of the Schur functions $\Phi_n$. In chapter 2 we make explicit evaluations of $\Phi_0$, $\Phi_1$, $\Phi_2$ and $\Phi_3$. These results are then used in chapter 3 to determine hyponormality conditions of $T_\varphi$ for situations where the coefficients of $\varphi$ satisfy partial symmetry conditions.

In chapter 4, we investigate hyponormality of $T_\varphi$ in the Bergman space $A^2(\mathbb{D})$. In this case, J. Lee [39] gave the following necessary condition for the hyponormality of $T_\varphi$:

**Theorem 1.3.1.** Let $\varphi(z) = \bar{g(z)} + f(z)$, where $f(z) = a_1z + a_2z^2$ and $g(z) = a_{-1}z + a_{-2}z^2$. If $T_\varphi$ is hyponormal, then

(i) $2(|a_2|^2 - |a_{-2}|^2) \geq 3(|a_{-1}|^2 - |a_1|^2)$

(ii) $(\frac{1}{2}(|a_1|^2 - |a_{-1}|^2) + \frac{1}{3}(|a_2|^2 - |a_{-2}|^2)) (\frac{1}{3}(|a_1|^2 - |a_{-1}|^2) + (|a_2|^2 - |a_{-2}|^2))$

$\geq \frac{4}{9}|\bar{a}_1a_2 - \bar{a}_{-1}a_{-2}|^2$.

First we show that these conditions are not sufficient to guarantee hyponormality of $T_\varphi$. Subsequently, in Theorems 4.3.2 and 4.3.3 we give conditions which make $T_\varphi$ hyponormal. Comparing these results with those obtained for the Hardy space $H^2(\mathbb{T})$ we have generated examples to bring out the fact that the hyponormality conditions of $T_\varphi$ on $H^2(\mathbb{T})$ do not naturally extend to $A^2(\mathbb{D})$.

The Chapter 5 contains necessary and sufficient conditions for the hyponormality of $T_\varphi$ in the weighted Bergman space. In particular, we have shown that the following result holds for specific values of $m$ and $N$:

**Theorem 1.3.2.** Let $\varphi(z) = \bar{g(z)} + f(z)$, where $f(z) = a_mz^m + a_Nz^N$, $g(z) = a_{-m}z^m + a_{-N}z^N$ ($1 \leq m < N$). If $a_m\bar{a}_N = a_{-m}\bar{a}_{-N}$ and $\alpha > -1$, then $T_\varphi$ on $A^2_\alpha(\mathbb{D})$
is hyponormal

\[ \iff \begin{cases} \frac{\Pi_{j=0}^{N-1} (\alpha+2+j)}{\Pi_{j=0}^{N-1} (m+1+j)} (|a_m|^2 - |a_m|^2) \leq (|a_N|^2 - |a_N|^2) \text{ if } |a_N| \leq |a_N| \\ N^2(|a_N|^2 - |a_N|^2) \leq m^2(|a_m|^2 - |a_m|^2) \text{ if } |a_N| \leq |a_N| \end{cases} \]

Though Toeplitz operators on spaces of analytic functions are quite well understood and the volume of work published in this area of research is immense, yet not much is known about the behavior of hyponormal Toeplitz operators. In our present work we attempt to carry forward the ongoing research; to plug some of the holes in the existing literature; to make particular case studies to gain better insight; with the aim of answering at least some of the queries regarding the various aspects of hyponormal Toeplitz operators.