CHAPTER - 6

COMMON FIXED POINT ON MENGGER SPACES

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COMMON FIXED POINTS ON MANGER SPACES

In this chapter, we divided two sections. In section one we prove a common fixed point theorem using the notion of asymptotic regularity for commuting mappings on Menger space.

Bharucha-Reich (1976) underlined the need of the research regarding the study of fixed points of mappings on probabilistic metric spaces (PM-spaces). Sehgal and Bharucha-Reid (1972) introduced the notion of contractions and local contractions to the setting of menger spaces and obtained some fixed point theorems in a sub class of probabilistic metric spaces. Subsequently a number of fixed point theorems were proved in PM-spaces and Menger spaces, see for instance M.Stojakovic [85], Singh and Pant ([80],[81],[82]) etc.

In section two we prove some theorems on common fixed point for four mappings using compatible concept which generalizes the result of Dediic and Sarapa [12a] on Menger space.
Singh and Pant [83] generalized commuting maps by calling self maps $A$ and $S$ of a probabilistic metric space $(X,F)$ a weakly commuting pair iff $F(ASx,SAx;c) \geq F(Ax,Sx;c)$ for all $x \in X$ and $c > 0$. This has enabled establish some common fixed point theorems on complete Menger spaces by using compatible concept.

For topological preliminaries on Menger space Schwizer and Sklar [85a] is an excellent reference.

Before presenting our result, we also need the following definitions and lemma.

**Definition 6.1.** Let $(X,F',t)$ be a Menger space. The value of $F'$ at $(u,v) \in X \times X$ is represented by $F_{u,v}$. The functions $F_{u,v}$ are assumed to satisfy the following conditions:

(a) \[ F_{u,v}(x) = 1 \text{ for all } x > 0, \text{ iff } u = v; \]
(b) \[ F_{u,v}(0) = 0; \]
(c) \[ F_{u,v}(x) = 1 \text{ and } F_{v,w}(y) = 1 \text{ then } F_{u,w}(x+y) = 1; \]
(d') \[ F_{u,w}(x+y) \geq \tau[F_{u,v}(x),F_{v,w}(y)] \text{ for all } u,v,w \in X \text{ and } x \geq 0, y \geq 0. \]

**Definition 6.2.** Let $A$, $P$ and $Q$ be mappings from $X$ to itself. A sequence $\{x_n\}$ of elements of $X$ will be called asymptotically $(A,P,Q)$-regular if for each $c > 0$ and $\lambda \in (0,1)$ such that
DEFINITION 6.3. Two self mappings \( A \) and \( S \) of a Menger space \((X,F,\min)\) will be called compatible if for each \( \varepsilon > 0 \) and \( \lambda \in (0,1) \), there exists an integer \( N = N(\varepsilon, \lambda) \) such that

\[
F(ASx_n, SAx_n; \varepsilon) > 1 - \lambda
\]

for all \( n \geq N \) whenever \( \{x_n\} \) is a sequence in \( X \) such that

\[
F(Ax_n, z, \varepsilon) > 1 - \lambda \text{ and } F(Sx_n, z, \varepsilon) > 1 - \lambda
\]

for some \( z \) in \( X \) and for all \( n \geq N \).

LEMMA 6.1. Let \( \{y_n\} \) be a sequence in a Menger space \((X,F',t)\), where \( t \) is continuous and satisfies \( t(x,x) = x \) for every \( x \in [0,1] \). If there exists a constant \( h \in (0,1) \) such that

\[
F(y_n, hx) \geq F(y_{n+1}, y_{n+1})^h, \quad n = 1, 2, \ldots
\]

for all \( x > 0 \) then \( \{y_n\} \) is a Cauchy sequence in \( X \).

LEMMA 6.2. Let \( A, B, S \) and \( T \) be self maps of a Menger space \((X,F, \min)\) such that \( A(X) \subseteq T(X) \) and \( B(X) \subseteq S(X) \). If there exists a constant \( k \in (0,1) \) such that

\[
F(Ax, By; k\varepsilon) \geq \min \{F(Sx, Ty; \varepsilon), F(Sx, Ax; \varepsilon), \}
\]

\[
F(Ty, Bs; \varepsilon), F(Ty, Ax; 2\varepsilon), F(Sx, By, 2\varepsilon)\}
\]

(6.1)
for all \( x, y \in X \) and for each \( \varepsilon > 0 \)

suppose that an arbitrary point \( x_0 \) in \( X \) such that the sequen

\( \{y_n\} \) formed by

\[ T_{x_{2n+1}} = A_{x_{2n}} = y_{2n} \quad \text{and} \quad S_{x_{2n+2}} = B_{x_{2n+1}} = y_{2n+1} \quad (6.1) \]

for all \( n \in N_0 = N \cup \{0\} \)

then \( \{y_n\} \) is a Cauchy sequence in \( X \).

PROOF. Since \( A(X) \subseteq T(X) \) and \( B(X) \subseteq S(X) \).

We can form the sequence \( \{y_n\} \) as was noted above.

By (6.1) and (6.2)

\[
F(y_{2n}, y_{2n+1} ; k\varepsilon) = F(Ax_{2n}, B_{x_{2n+1}} ; k\varepsilon)
\]

\[
= \min \{ F(y_{2n-1}, y_{2n} ; \varepsilon) , F(y_{2n-1}, y_{2n+1} ; \varepsilon) \}
\]

which implies

\[
F(y_{2n}, y_{2n+1} ; k\varepsilon) \leq F(y_{2n-1}, y_{2n} ; \varepsilon)
\]

as

\[
F(y_{2n-1}, y_{2n+1} ; 2\varepsilon) \geq \min \{ F(y_{2n-1}, y_{2n} ; \varepsilon) , F(y_{2n}, y_{2n+1} ; \varepsilon) \}
\]

Similarly
\[ F(y_{2n-1}, y_{2n}; \lambda \varepsilon) \geq F(y_{2n-2}, y_{2n-1}; \varepsilon). \]

So in general

\[ F(y_n, y_{n+1}; \lambda \varepsilon) \geq F(y_{n-1}, y_n; \varepsilon) \]

and it follows that

\[ F(y_{n-1}, y_n; \frac{1-k}{k^n} \varepsilon) \geq F(y_{n-2}, y_{n-1}; \frac{1-k}{k^{n-1}} \varepsilon) F(y_{n-1}, y_n; \frac{1-k}{k^n} \varepsilon) \]

Since \( k \in (0,1) \) we conclude that for every \( \varepsilon > 0 \) and every \( \lambda \in (0,1) \) there exists \( n \geq N_1(\varepsilon, \lambda) \in \mathbb{N} \)

such that

\[ F(y_{n-1}, y_n; \frac{1-k}{k^n} \varepsilon) > 1-\lambda \text{ for } n \geq N_1. \]

Let us prove by induction, m, that for \( n \geq N_1 \) and every \( m \in \mathbb{N} \) we have

\[ F(y_n, y_{n+m}; \varepsilon) > 1-\lambda \quad (6.3) \]

For \( m = 1 \), we have

\[ F(y_n, y_{n+1}; \varepsilon) \geq F(y_{n-1}, y_n; \frac{\varepsilon}{\lambda}) F(y_{n-1}, y_n; \frac{1-k}{k} \varepsilon) > 1-\lambda \]

for \( n \geq N_1 \). Suppose that \( (6.3) \) holds true for some \( m \). Then by Menger's triable inequality, we get

\[ F(y_n, y_{n+m+1}; \varepsilon) \geq F(y_{n-1}, y_{n+m}; \varepsilon) \]

\[ \geq \min \{ F(y_{n-1}, y_n; \frac{1-k}{k} \varepsilon), F(y_n, y_{n+m}; \varepsilon) \} \]

\[ \geq \min \{ 1-\lambda, 1-\lambda \} = 1-\lambda \]
for \( n \geq N_1 \). Thus \( \{y_n\} \) is a Cauchy sequence.

By definition 6.3, it is easy to prove the proposition stated below:

**PROPOSITION 6.1.** Let \((X,F,t)\) is a Menger space, where \( t = \text{min} \).

Let \( A, S \) be compatible pair defined as above.

1. \( F(SAz, ASz; \varepsilon) > 1-\lambda \), if \( F(Az, Sz; \varepsilon) > 1-\lambda \) for some \( z \) in \( X \) and for all \( \varepsilon > 0 \) and \( \lambda \in (0,1) \)

2. Suppose that for each \( \varepsilon > 0 \) and \( \lambda \in (0,1) \), then there exists an integer \( N = N(\varepsilon, \lambda) \) such that

   \[ F(Ax_n, z; \varepsilon) > 1-\lambda \text{ and } F(Sx_n, z; \varepsilon) > 1-\lambda \]

   for \( z \) in \( X \) and for all \( n \geq N \), then

2.a) \( F(ASx_n, Sz; \varepsilon) > 1-\lambda \) and

   \[ F(SSx_n, Sz; \varepsilon) > 1-\lambda, \text{ if } S \text{ is continuous at } z. \]

2.b) \( F(AzSz; \varepsilon) < 1-\lambda \) and \( F(ASz, ASz; \varepsilon) > 1-\lambda \), if \( A \) and \( S \) are continuous.

Next we assume that \((X,F,t)\) is a complete Menger space, where \( t = \text{min} \).
6.1 COMMON FIXED POINT THEOREM FOR COMMUTING MAPPINGS IN PROBABILISTIC METRIC SPACES

In this section, first we prove a common fixed point theorem for commuting mappings in probabilistic metric spaces.

We prove main result of this section as below:

THEOREM 6.2.1. Let \((X,F,\min)\) be complete menger space. Suppose that \(A, B, S\) and \(T\) be self maps of \(X\) satisfying the following conditions:

(6.2.1) \(A(X) \subseteq T(X)\) and \(B(X) \subseteq S(X)\)

(6.2.2) \(S\) and \(T\) are continuous

(6.2.3) the pairs \((A,S)\) and \((B,T)\) are compatible

(6.2.4) \(F(Ax,By;k\epsilon) \geq \min\{F(Sx,Ty;\epsilon), F(Sx,Ax;\epsilon),\)

\(F(Ty,By;\epsilon), F(Ty,Ax,2\epsilon), F(Sx,By;2\epsilon)\}\)

for all \(x,y \in X, k \in (0,1)\) and \(\epsilon > 0\).

Then \(A, B, S\) and \(T\) have a unique common fixed point in \(X\).

PROOF. By lemma 6.2 the sequence \(\{y_n\}\) defined by (6.2) is a Cauchy sequence in \(X\). Since space \((X,F,\min)\) is complete which means that'

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\[
\lim_{n \to \infty} y_n = \lim_{n \to \infty} Ax_{2n} = \lim_{n \to \infty} Bx_{2n+1} = \lim_{n \to \infty} Sx_{2n}
\]
\[
= \lim_{n \to \infty} Tx_{2n+1} = z \in X.
\]

Since the pairs A, S and B, T are compatible and S and T are continuous, proposition (2.a) imply for every \(\varepsilon > 0\) and \(\lambda \in (0,1)\) there exists a integer \(N_2 = N_2 (\varepsilon, \lambda)\) such that

\[
F(ASx_{2n}, Sz; \frac{1-k}{2k}) > 1-\lambda
\]
\[
F(SSx_{2n}, Sz; \frac{1-k}{2k}) > 1-\lambda
\]
\[
(6.2.5) \quad F(BTx_{2n+1}, Tz; \frac{1-k}{2k}) > 1-\lambda \quad \text{and}
\]
\[
F(TTx_{2n+1}, Tz; \frac{1-k}{2k}) > 1-\lambda \quad \text{for } n \geq N_2.
\]

Now, by (6.2.4), we have

\[
F(ASx_{2n}, BTx_{2n+1}; \varepsilon) = \min \{ F(SSx_{2n}, TTx_{2n+1}; \varepsilon), \quad \\
F(SSx_{2n}, ASx_{2n}; \varepsilon), F(TTx_{2n+1}, BTx_{2n+1}; \varepsilon), \\
F(TTx_{2n+1}, ASx_{2n}; \varepsilon), F(SSx_{2n}, BTx_{2n+1}; \varepsilon) \}
\]

which implies

\[
(6.2.6) \quad F(ASx_{2n}, BTx_{2n+1}; \varepsilon) = \min \{ F(SSx_{2n}, TTx_{2n+1}; \varepsilon), \quad \\
F(SSx_{2n}, ASx_{2n}; \varepsilon), F(TTx_{2n+1}, BTx_{2n+1}; \varepsilon) \}
\]
as

\[
F(TTx_{2n+1}, ASx_{2n}; \varepsilon) = \min \{ F(TTx_{2n+1}, BTx_{2n+1}; \varepsilon), \\
F(BTx_{2n+1}, ASx_{2n}; \varepsilon) \}
\]

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and
\[ F(Sx_{2n},STx_{2n+1};2e) \geq \min \{ F(SSx_{2n},ASx_{2n};e), F(ASx_{2n},STx_{2n+1};e) \} \]

Then
\[ F(ASx_{2n},BSTx_{2n+1};ke) \geq \min \{ F(SSx_{2n},TTx_{2n+1};e), F(SSx_{2n},Sz;\frac{1+k}{2k}-e) \]
\[ F(Sz,ASx_{2n};\frac{1-k}{2k}-e), F(TTx_{2n+1},Tz;\frac{1+k}{2k}-e), \]
\[ F(Tz,BTx_{2n+1};\frac{1-k}{2k}) \}
\[ > 1-\lambda, \quad \text{by (6.2.5)} \]

that is,
\[ Sz = Tz. \]

Further,
\[ F(Az,TTx_{2n+1};ke) \geq \min \{ F(Sz,TTx_{2n+1};e), F(Sz,Az;e), \]
\[ F(TTx_{2n+1},BTx_{2n+1};e), F(TTx_{2n+1},Az;2e), \]
\[ F(Sz,BTx_{2n+1};2e) \} \]
\[ = \min \{ F(Tz,TTx_{2n+1};e), F(Tz,Az;e), \]
\[ F(TTx_{2n+1},BTx_{2n+1};e) \}
\[ = \min \{ FTz,TTx_{2n+1};\frac{1+k}{2k}-e), F(Tz,Az;e) \]
\[ F(TTx_{2n+1},Tz;\frac{1+k}{2k}-e), F(Tz,BTx_{2n+1};\frac{1-k}{2k}) \}
\[ > 1-\lambda, \quad \text{By (1),} \]

so that
\[ Az = Tz. \]
Similarly \( Bz = Sz \).

To prove \( Az = z \) let \( Uz (c, \lambda) \) be any neighbourhood of \( z \). Since \( Bx_{2n+1} ; Tx_{2n+1} \to z \), there exist an integer \( N_2 (c, \lambda) \) such that \( n \geq N_2 \) implies

\[
(6.2.7) \quad F(Bx_{2n+1}, z; \frac{1-k}{2k}) > 1-\lambda \text{ and } F(Tx_{2n+1}, z; \frac{1+k}{2k}) > 1-\lambda.
\]

Then by (6.2.4)

\[
F(Az, Bx_{2n+1}; c) \leq \min \{ F(Sz, Tx_{2n+1}; c), F(Sz, Az; c), F(Tx_{2n+1}, Bx_{2n+1}; c), F(Tx_{2n+1}, Az; 2c), F(Sz, Bx_{2n+1}; 2c) \},
\]

\[
= \min \{ F(Az, Tx_{2n+1}; c), F(Tx_{2n+1}, Bx_{2n+1}; c) \}
\]

\[
= \min \{ F(Az, Tx_{2n+1}; c), F(Tx_{2n+1}, z; \frac{1+k}{2k}), F(z, Bx_{2n+1}; \frac{1-k}{2k}) \}
\]

\[
> 1-\lambda \text{ by } (6.2.7)
\]

which implies \( Az = z \).

Thus \( Az = Bz = Sz = Tz = z \) i.e. \( z \) is a common fixed point of \( A, B, S \text{ and } T \).

The uniqueness of the common fixed point follows easily from
The following theorem follows easily from Theorem 6.2.1.

**THEOREM 6.2.2.** Let \(A, B, S\) and \(T\) be self map of a complete Menger space \((X,F,\text{min})\) satisfying the condition (6), (7), (8), and (13),

\[
\text{(6.2.8)} \quad F(Ax,By;ke) \leq \min \{F(Sx,Ty;c), F(Sx,Ax;c), F(Ty,By;c)\}
\]

for all \(x,y \in X, k \in (0,1)\) and \(c > 0\). Then \(A, B, S\) and \(T\) have a unique common fixed point in \(X\).

Throughout this section let \((M,d)\) be a complete metric space. It is easy to prove that \((M,F,\text{min})\) is a Menger space, if we take \(F(x,y;\varepsilon) = H(\varepsilon-d(x,y))\) for all \(x,y \in M\). Since the condition (6.1) \(\rightarrow\) the condition,

\[
\text{(6.2.9)} \quad F(Ax,By;ke) \leq \min \{F(Sx,Ty;c), F(Sx,Ax;c), F(Ty,By;c),
\]

\[\quad F(Ty,Ax;c), F(Sx,By;c)\}.\]

It is easy to prove that the condition (6.2.9) is equivalent to the condition

\[
\text{(6.2.10)} \quad d(Ax,By) \leq k \max\{d(Sx,Ty), d(Sx,Ax), d(Ty,By),
\]

\[\quad d(Ty,Ax), d(Sx,By)\}
\]

for every \(k \in (0,1)\) and all \(x,y \in M\).
Further the condition (13) is equivalent to the condition

\[ (6.2.11) \quad d(Ax, By) \leq k \max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By)\} \]

for every \( k \in (0, 1) \) and all \( x, y \) in \( M \).

Since the condition (6.2.9) is equivalent the condition

\[ (6.2.12) \quad d(Ax, By) \leq k \max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \]

\[ \frac{1}{2}d(Ty, Ax), \frac{1}{2}d(Sx, By)\} \]

\[ \leq k \max\{d(Sx, By), d(Sx, Ax), d(Ty, By), \]

\[ \frac{1}{2}(d(Ty, Ax) + d(Sx, By))\} \]

for every \( k \in (0, 1) \) and all \( x, y \in M \).

The following results follow from Theorem 6.2.1 and Theorem 6.2.2.

**Corollary 6.2.1.** Let \( A, B, S \) and \( T \) be self maps of a complete metric space \((M, d)\) satisfying the condition (6.2.1), (6.2.2), (6.2.3) and (6.2.10), then \( A, B, S \) and \( T \) have a unique common fixed point in \( M \).

**Corollary 6.2.2.** Let \( A, B, S \) and \( T \) be self maps of a complete metric space \((M, d)\) satisfying the condition (6.2.1), (6.2.2), (6.2.3) and (6.2.11); then \( A, B, S \) and \( T \) have a unique common fixed
COROLLARY (6.2.3). Let $A, B, S$ and $T$ be self maps of a complete metric space $(m, d)$ satisfying the condition (6.2.1), (6.2.2), (6.2.3) and (6.2.12); then $A, B, S$ and $T$ have a unique common fixed point in $M$.

REMARK 6.2.1. The theorem 3.2 generalizes the result of Dedic and Sarapa [12a].

REMARK 6.2.2. The theorem 4.1 generalizes the result of Das and Naik [11].

REMARKS 6.2.3. The theorem 4.3 is a result of Jungck [3].