CHAPTER - 5

CERTAIN FIXED POINT THEOREM FOR CERTAIN NUMBER OF MAPPINGS

(79-89)
CERTAIN FIXED POINT THEOREM FOR CERTAIN NUMBER OF MAPPINGS

In this chapter we prove two theorems. In theorem one we shall give some fixed point theorems for four mappings in metric space which generalize some result of Paliwal [33].

Paliwal [33] has generalized the previous results of different authors for three maps in complete metric space and established the following theorem:

5.1 FIXED POINT THEOREM OF A METRIC SPACE FOR FOUR MAPPINGS

In this section, first we prove a fixed point theorem for four mappings in metric space.

We prove main result of this section as below:

**THEOREM 5.1.1**: Let $S$ and $T$ be self mappings of complete metric space $(X,d)$ and $P$ and $A$ be continuous self-mappings on $X$ satisfying the following conditions.

$$
[d(SPx,TPy)]^2 \leq a_1 [d(Ax,Ay)]^2 + a_2 d(Ax,SPx)d(Ay,TPy) \\
+ a_3 d(Ax,TPy)d(Ay,SPx) + a_4 d(Ay,SPx)d(Ay,TPy) \\
+ a_5 d(Ax,Ay)d(Ax,SPx) \\
+ a_6 d(Ax,SPx)d(Ay,SPx)
$$

\[5.1.1\]
for all \( x, y \in X \), where \( a_i \geq 0, i = 1, 2, \ldots, 6 \) with \( a_1 + a_2 + a_4 + a_5 + 2a_6 < 1 \) and \( a_1 + a_3 < 1 \) further, assume that either \( SP = PS \) and \( TA = AT \) or \( TP = PT \) and \( SA = AS \); and \( AP = PA \). Then \( S, T, P \) and \( A \) have a unique common fixed point in \( X \).

**Proof.** Let \( x_0 \) be an arbitrary point of \( X \) and we define

\[
A^{2n+1} = SP^{2n}, \quad n = 0, 1, 2, \ldots
\]

\[
A^{2n} = TP^{2n-1}, \quad n = 1, 2.
\]

Now from (5.1.1), we have

\[
[d(A^{2n+1}, A^{2n})]^2 = [d(SP^{2n}, TP^{2n-1})]^2
\]

\[
\leq a_1 [d(A^{2n}, A^{2n+1})]^2
\]

\[
+ a_2 d(A^{2n}, SP^{2n}) d(A^{2n-1}, TP^{2n-1})
\]

\[
+ a_3 d(A^{2n}, TP^{2n-1}) d(A^{2n-1}, SP^{2n})
\]

\[
+ a_4 d(A^{2n-1}, SP^{2n}) d(A^{2n-1}, TP^{2n-1})
\]

\[
+ a_5 d(A^{2n}, A^{2n-1}) d(A^{2n}, SP^{2n})
\]

\[
+ a_6 d(A^{2n}, SP^{2n}) d(A^{2n-1}, TP^{2n-1})
\]

\[
\leq a_1 [d(A^{2n}, A^{2n-1})]^2
\]

\[
+ a_2 d(A^{2n}, A^{2n+1}) d(A^{2n-1}, A^{2n})
\]

\[
+ a_3 d(A^{2n}, A^{2n}) d(A^{2n}, A^{2n+1})
\]

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\]
\[ + a_4 d(Ax_{2n-1}, Ax_{2n+1}) (d(Ax_{2n-1}, Ax_{2n}) \]
\[ + a_5 d(Ax_{2n}, Ax_{2n-1}) d(Ax_{2n}, Ax_{2n+1}) \]
\[ + a_6 d(Ax_{2n}, Ax_{2n+1}) d(Ax_{2n-1}, Ax_{2n+1}) \]
\[ \leq a_1 [d(Ax_{2n}, Ax_{2n-1})]^2 \]
\[ + (a_2 + a_5) d(Ax_{2n}, Ax_{2n+1}) d(Ax_{2n-1}, Ax_{2n+1}) \]
\[ + a_4 d(Ax_{2n-1}, Ax_{2n}) d(Ax_{2n-1}, Ax_{2n}) \]
\[ + d(Ax_{2n}, Ax_{2n+1}) \]
\[ + a_6 d(Ax_{2n}, Ax_{2n+1}) [d(Ax_{2n-1}, Ax_{2n}) \]
\[ + d(Ax_{2n}, Ax_{2n+1}) \]
\[ \leq (a_1 + a_4) [d(Ax_{2n}, Ax_{2n+1})]^2 \]
\[ + a_6 [d(Ax_{2n}, Ax_{2n+1})]^2 + (a_2 + a_5 + a_6) \]
\[ + \left\{ \frac{[d(Ax_{2n}, Ax_{2n+1})]^2 + [d(Ax_{2n-1}, Ax_{2n})]^2}{2} \right\} \]

i.e. \[ [d(Ax_{2n}, Ax_{2n+1})]^2 \leq \frac{a_1 + 1/2(a_2 + 3a_4 + a_5 + a_6)}{1 - 1/2(a_2 + a_4 + a_5 + 3a_6)} [d(Ax_{2n-1}, Ax_{2n})]^2 \]

This implies that

\[ d(Ax_{2n}, Ax_{2n+1}) \leq k d(Ax_{2n-1}, Ax_{2n}) \]

where \[ k^2 = \frac{a_1 + 1/2(a_2 + 3a_4 + a_5 + a_6)}{1 - 1/2(a_2 + a_4 + a_5 + 3a_6)} \leq 1. \]

Similarly \[ d(Ax_{2n-1}, Ax_{2n}) \leq k d(Ax_{2n-2}, Ax_{2n-1}) \]. Therefore
\( \{Ax_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete there exists \( z \in X \), such that \( Ax_n \rightarrow z \) as \( n \rightarrow \infty \).

Since the sequence \( \{SPx_{2n}\} \) and \( \{TPx_{2n-1}\} \) are subsequence of \( \{Ax_n\} \), the have the same limit \( z \). As \( P \) and \( A \) are continuous, we have

\[
A^2x_n \rightarrow Az, \quad ASPx_{2n} \rightarrow Az, \quad ATPx_{2n-1} \rightarrow Az
\]

and

\[
PAX_n \rightarrow Pz, \quad PSPx_{2n} \rightarrow Pz, \quad PTPx_{2n-1} \rightarrow Pz
\]

and

Now by using the commutativity of \( o\{A,T\}, \{A,P\} \) and (5.1.1) we have

\[
[d(SPz, ATPx_{2n-1})]^2 = [d(SPz, PAX_{2n-1})]^2
\]

\[
\leq a_1 [d(Az, A^2x_{2n-1})]^2
\]

\[
+ a_2 d(Az, SPz) d(A^2x_{2n-1}, TPx_{2n-1})
\]

\[
+ a_3 d(Az, TPx_{2n-1}) d(A^2x_{2n-1}, SPz)
\]

\[
+ a_4 d(A^2x_{2n-1}, SPz) + a_5 d(Az, SPz) d(A^2x_{2n-1}, SPz).
\]

Letting \( n \rightarrow \infty \) and using the continuity of \( A \), we obtain

\[
(1-a_5) [d(SPz, Az)]^2 \leq 0
\]

i.e.

\[
SPz = Az.
\]
Similarly by considering \([d(ASP_{2n}, TPz)]^2\), we have considered from (5.1.2) by letting \(n \rightarrow \infty\) that \(TPz = Az\) thus \(SPz = Az = TPz\). Now we shall show that \(Pz = Az\).

Using the commutativity of \(\{S, P\}\) and \(\{T, A\}\), we have

\[
[d(PSP_{2n}, ATP_{2n-1})]^2 = [d(SPP_{2n}, TPA_{2n-1})]^2
\]

Further, using the commutativity of \(\{S, A\}\) and \(\{T, P\}\), we have

\[
(1 - a_1 - a_3) [d(Pz, Az)]^2 = 0
\]

which implies that \(Pz = Az\). Hence by (5.2), we obtain

\[
SPz = Az = Pz = TPz \quad (5.1.3)
\]

Similarly using the commutativity of \(\{S, A\}\) and \(\{T, P\}\), we have same result. Now we have
\[ (1-a_1-a_3)[d((Az, z))]^2 = 0 \]

which implies that Az = z.

Similarly by considering \(d((SPx_{2n}, TPx_{2n-1}))^2 \)

we obtain by letting \(n \to \infty \) that \(z = Pz \).

Thus \(z = Sz = Tz = Pz \) uniqueness of common fixed point following easily by \((5.1.1)\).

This complete the proof of the theorem.

5.2 COMMON FIXED POINTS OF TWO MAPPINGS SATISFYING A NEW CONTRACTIVE TYPE CONDITION

The intent of this section is to obtain some common fixed point theorems for two self mappings in a complete metric space satisfying a new contractive type condition.
In 1977, Fisher [16] established the following result.

**THEOREM A.** Let \( S \) is a continuous mapping and \( T \) is a mapping of the complete metric space \((X,d)\) into itself satisfying the inequality

\[
d(Sx,Tsv) \leq \alpha d(x,SY) + \beta [d(x,Sx) \\
+ d(Sy,Tsy)] + \gamma [d(x,Tsy)+d(Sy,Sx)]
\]

(5.2.1)

for all \( x,y \) in \( X \), where \( \alpha, \beta, \gamma \geq 0 \) and \( \alpha + 2\beta + 2\gamma < 1 \), then \( S \) and \( T \) have a unique common fixed point.

In this section we generalize the above result by removing the condition of the continuity of a mapping.

We prove main result of this section as below:

**THEOREM 5.2.1.** Let \((X,d)\) be a complete metric space and \( S \) and \( T \) be self mappings of \( X \) satisfying the condition,

\[
d(Sx,Tsy) \leq \alpha d(x,SY) + \beta [d(x,Sx) \\
+ d(Sy,Tsy)] + \gamma [d(x,Tsy)+d(Sy,Sx)]
\]

(5.2.2)

for all \( x,y \) in \( X \), where \( \alpha, \beta, \gamma \geq 0 \) and \( \alpha + 2\beta + 2\gamma < 1 \).

Then \( S \) and \( T \) have a unique common fixed point.

**PROOF.** Let \( x_0 \) be an arbitrary common fixed point, we define the
sequence \( \{x_n\} \) of points of \( X \), such that

\[
x_{2n+1} = Sx_{2n}, \quad x_{2n+2} = Tx_{2n+1}, \quad n = 0, 1, 2, \ldots
\]

By (2), we have

\[
d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1}) = d(Sx_{2n}, TSx_{2n})
\]

\[
\leq \alpha d(x_{2n}, x_{2n+1}) + \beta [d(x_{2n}, Sx_{2n}) + d(Sx_{2n}, TSx_{2n})]
\]

\[
+ \gamma [d(x_{2n}, TSx_{2n}) + d(Sx_{2n}, x_{2n})]
\]

\[
\leq \alpha d(x_{2n}, x_{2n+1}) + \beta (d(x_{2n}, x_{2n+1})
\]

\[
+ d(x_{2n+1}, x_{2n})] + \gamma d(x_{2n}, x_{2n+2})
\]

which implies

\[
d(x_{2n+1}, x_{2n+2}) \leq \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} d(x_{2n}, x_{2n+1})
\]

i.e.

\[
d(x_{2n+1}, x_{2n+2}) \leq k d(x_{2n}, x_{2n+1})
\]

where \( k = \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} < 1 \).

Similarly, we can show that

\[
d(x_{2n}, x_{2n+1}) \leq kd(x_{2n-1}, x_{2n}).
\]

It follows that \( \{x_n\} \) is a Cauchy sequence. Now from the completeness of \( X \), there exists some \( z \in X \) such that \( \lim_{n \to \infty} x_n = z \).

Now we shall show that \( z \) is the common fixed point of \( S \) and \( T \), for if possible we assume that \( z \neq Sz \), then we have
\[ d(z, Sz) \leq d(z, x_{2n+2}) + d(x_{2n+2}, Sz) \]
\[ \leq d(z, x_{2n+2}) + d(Sz, TSz) \]
\[ \leq d(z, x_{2n+2}) + a d(z, Sz) + \beta [d(z, Sz) \]
\[ + d(Sx_{2n}, TSz) + d(Sz, Sz) + d(TSx_{2n}, Sz) \].

On letting \( n \to \infty \), we have
\[ d(z, Sz) \leq (\beta + \gamma) d(z, Sz) \]
a contradiction, hence \( z = Sz \), next we assume that \( z = Tz \). Then we have
\[ d(z, Tz) \leq d(z, x_{2n+1}) + d(x_{2n+1}, Tz) \]
\[ \leq d(z, x_{2n+1}) + d(Sx_{2n+1}, TSz) \]
\[ \leq d(z, x_{2n+1}) + a d(x_{2n+1}, Sz) + \beta [d(x_{2n+1}, Sz) + d(Sz, TSz) \]
\[ + d(Sz, Ts) + d(Sx_{2n}, Sz) \].

On letting \( n \to \infty \) and using \( z = Sz \), we have
\[ d(z, Tz) \leq (\beta + \gamma) d(z, Tz) \]
Which implies that \( z = Tz \) i.e. \( z \) is the common fixed point of \( S \) and \( T \).

The unicity of the common fixed point easily follows from (5.2.2). This completes the proof, obviously, we may prove...
the following:

**COROLLARY 5.2.1.** Let \((X,d)\) be a complete metric space and let \(T\) be a self mapping of \(X\) satisfying the inequality

\[
d(Tx,T^2y) \leq \alpha d(x,Ty) + \beta [d(x,Tx) + d(Ty,T^2y)] + \gamma [d(x,T^2y) + d(Ty,Tx)]
\]

for all \(x,y\) in \(X\), where \(\alpha, \beta, \gamma \geq 0\).

Then \(T\) has a unique fixed point.

Finally, we furnish an example to discuss the validity and generality of our theorem.

**EXAMPLE 5.3.1.** Let \(X = [0,1]\) with usual metric and let \(S\) and \(T\) be two self-maps of \(X\), such that

\[
Sx = \begin{cases} 
x/2, & x \in [0,1/2) \\
x/4, & x \in [1/2,1]
\end{cases}
\]

\[
Tx = \begin{cases} 
x/(x+2), & x \in [0,1] \\
1/2, & x = 1.
\end{cases}
\]

By setting \(\alpha = \beta = \gamma = 1/10\), it is easy to see that \(S\) and \(T\) satisfy the conditions of Theorem 5.2.1 for every pair of points.

\(x=0, y = 1; x=1, y = 0; x = 1/2, y = 0; x = 0, y = 1/2; x = 1/2.\)
\(y = 1\) and \(x = 1, y = 1/2\) and obviously 0 is the unique common fixed point of \(S\) and \(T\).

**Remark 5.2.1**

(a) On setting \(\alpha = \beta = \gamma = 1/10, x = 0, y = 1\) in the above example \(S\) and \(T\) do not satisfy the condition (5.2.1)

\[
\begin{align*}
\text{Define } T_x &= \begin{cases} 
  x/2, & x \in [0, 1/2) \\
  x/4, & x \in [1/2, 1]
\end{cases}
\end{align*}
\]

(b) By setting \(\alpha = \beta = \gamma = 1/10, x = 1, y = 0\); \(T\) satisfies all the conditions of (5.2.3) and 0 is the unique fixed point of \(T\).

But by setting \(\alpha = \beta = \gamma = 1/10, x = 1, y = 0\); \(T\) does not satisfy the theorem of Fisher [16].

Above remark show that our results are genuine generalization of the results of Fisher [16].