Chapter 2

Spectral Gap Problems

The usage of linear algebraic techniques in the computation of spectrum of a bounded self-adjoint operator $A$ on a separable Hilbert space $\mathbb{H}$, are discussed in this chapter. The eigenvalues of truncations of a bounded self-adjoint operator are used to study the behavior of its spectrum.

It was already observed in [19] that the bounds of essential spectrum and the discrete eigenvalues lying outside these bounds, are possible to approximate by this method. The usage of algebraic techniques in this problem was done earlier in [3]. The major problem that is considered here is to predict the gaps that may occur between the bounds of the essential spectrum using the eigenvalues of truncations. An interval $I$ is called spectral gap if there exist real sets $J_1, J_2$ containing the spectrum of $A$ such that $\sup J_1 \leq \inf I < \sup I \leq \inf J_2$. We are interested in the gaps lie between the bounds of essential spectrum of $A$. Also the intervals between these bounds, containing only discrete eigenvalues, are treated as spectral...
gaps. Locating such eigenvalues in between a spectral gap, is another interesting problem, to be handled linear algebraically. Historically, gap related problems have been studied with special attention for Schrodinger operators (see e.g. [26, 34, 35, 59]).

The chapter is organized as follows. We begin with a preliminary section in which a survey of the algebraic and linear algebraic developments in this area due to [3] and [19] are presented. In the second section, the results which predict the existence of spectral gaps, using the eigenvalues of truncations, are proved. A new method to detect the spectral gaps is proposed in the third section, which is an analogue of the study by E.B Davies, Levitin and Shargorodsky (see [34],[35], [51],[52]). Also some computational issues are addressed there.

2.1 The Truncation method

Let A be a bounded self-adjoint operator on the separable Hilbert space $\mathbb{H}$ and let $\{e_1, e_2, \ldots\}$ be an orthonormal basis for $\mathbb{H}$. Denote by $P_n$, the projection of $\mathbb{H}$ onto the finite dimensional subspace, $L_n = \text{span}\{e_1, e_2, \ldots, e_n\}$. Consider the finite dimensional truncations of $A$, that is $A_n = P_nAP_n$. Now if $(a_{i,j}) = (\langle Ae_j, e_i \rangle)$ is the matrix representation of $A$ associated to the orthonormal basis $\{e_1, e_2, \ldots\}$, then the $n \times n$ matrix $(a_{i,j})_{1 \leq i,j \leq n}$ coincides with the matrix representation of $A_n$ restricted to the image of $P_n$.

The following basic question is addressed here. What is the relation between the eigenvalue sequence of the matrices $(a_{i,j})_{1 \leq i,j \leq n}$, and spectrum
of A. Whether the spectrum can be approximated using the eigenvalue sequence of truncations. There are some disappointing examples in which the eigenvalues of truncations give little information about the spectrum. For instance, in the case of the right shift operator on the sequence space $l^2(\mathbb{Z})$, the eigenvalue sequence of the truncations is the constant sequence 1, while the spectrum is the whole closed unit disc. For a self-adjoint example, one can consider the operator $A$ on $l^2(\mathbb{N})$, defined as follows.

$$A(x_n) = (x_{\pi(n)}),$$

where $\pi$ is a suitably chosen permutation on $\mathbb{N}$. The essential properties required for the permutation $\pi$, are discussed in [3], due to which the truncation method fails to approximate the spectrum.

Some developments in this area are reported below. The major contributions are due to W.B. Arveson, who generalized the notion of band limited matrices in [3], and achieved some success in the case of some special class of operators. We brief up the definitions and some results below which will play a very important role in the approximation of spectrum of self-adjoint operators. The notation $A_n$ is used to denote the matrix $(a_{i,j})_{1\leq i,j\leq n}$.

**Definition 2.1.1.** A filtration of a Hilbert space $\mathbb{H}$ is a sequence of finite dimensional subspaces of $\mathbb{H}$, $\{L_n; n \in \mathbb{N}\}$ such that $L_n \subset L_{n+1}$ and closure of $\bigcup_n L_n$ is $\mathbb{H}$.

**Example 2.1.1.** A typical example for filtration in a Hilbert space with an orthonormal basis is the following. Let $\{e_n : n \in \mathbb{Z}\}$ be the
bilateral orthonormal basis for \( \mathbb{H} \) and let \( \{L_n\} \) be defined by

\[
L_n = \text{span}\{e_{-n}, e_{-n+1}, \ldots, e_n\}.
\]

Then \( \{L_n; n \in \mathbb{Z}\} \) is a filtration.

**Definition 2.1.2.** Let \( \{L_n : n \in \mathbb{N}\} \) be a filtration. And \( P_n \) be the projection onto \( L_n \). The **degree** of a bounded operator \( A \) on \( \mathbb{H} \) is defined by

\[
\text{deg}(A) = \sup_{n \geq 1} \text{rank}(P_n A - AP_n).
\]

Corresponding to each filtration, a Banach \( \ast \)-algebra of operators can be defined, which is named as Arveson’s class, defined as follows.

**Definition 2.1.3.** \( A \) is an operator in the **Arveson’s class** if

\[
A = \sum_{n=1}^{\infty} A_n,
\]

where \( \text{deg}(A_n) < \infty \) for every \( n \) and convergence is in the operator norm, in such a way that

\[
\sum_{n=1}^{\infty} (1 + \text{deg}(A_n)^{\frac{1}{2}})\|A_n\| < \infty
\]

In case the filtration is the span of finite number of elements in the basis as defined in example (2.1.1), the following gives a concrete description of operators in the Arveson’s class.

**Theorem 2.1.1.** \[3\] Let \( \{L_n; n \in \mathbb{Z}\} \) be the filtration defined in example (2.1.1). Also let \( (a_{i,j}) \) be the matrix representation of a bounded
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operator $A$, with respect to $\{e_n\}$, and for every $k \in \mathbb{Z}$ let

$$d_k = \sup_{i \in \mathbb{Z}} |a_{i+k,i}|$$

be the sup norm of the $k^{th}$ diagonal of $(a_{i,j})$. Then $A$ will be in the Arveson’s class whenever the series $\sum_k |k|^{1/2}d_k$ converges.

In particular, any operator whose matrix representation $(a_{i,j})$ is band-limited, in the sense that $a_{i,j} = 0$ whenever $|i - j|$ is sufficiently large, must be in the Arveson’s class. Before stating the spectral inclusion theorems for arbitrary self-adjoint operators and for operators in the Arveson’s class, recall the notion of essential points and transient points.

**Definition 2.1.4. Essential point:** A real number $\lambda$ is an essential point of $A$, if for every open set $U$ containing $\lambda$,

$$\lim_{n \to \infty} N_n(U) = \infty,$$

where $N_n(U)$ is the number of eigenvalues of $A_n$, in $U$.

**Definition 2.1.5. Transient point:** A real number $\lambda$ is a transient point of $A$ if there is an open set $U$ containing $\lambda$, such that $\sup N_n(U)$ with $n$ varying on the set of all natural numbers, is finite.

**Remark 2.1.1.** It should be noted that a number can be neither transient nor essential.

Denote $\Lambda = \{\lambda \in \mathbb{R}; \lambda = \lim_{n} \lambda_n, \lambda_n \in \sigma(A_n)\}$ and $\Lambda_e$ as the set of all essential points. The following spectral inclusion results for a bounded self-adjoint operator $A$ is of high importance throughout this thesis.
Theorem 2.1.2. \[3\] \( \sigma(A) \subseteq \Lambda \subseteq [m, M] \) and \( \sigma_e(A) \subseteq \Lambda_e \).

Equality in one of the above inclusion for self-adjoint operators in the Arveson’s class, was also proved in [3]. The precise result is the following.

Theorem 2.1.3. \[3\] If \( A \) is a bounded self-adjoint operator in the Arveson’s class, then \( \sigma_e(A) = \Lambda_e \) and every point in \( \Lambda \) is either transient or essential.

Remark 2.1.2. The above two theorems enable us to confine our attention to the limiting set \( \Lambda \) and the essential points \( \Lambda_e \), in the task of computation of spectrum and essential spectrum of a bounded self-adjoint operator respectively. Now the following issues may arise. The limiting set \( \Lambda \) may contain points which do not belong to the spectrum. Such points are called spurious eigenvalues. In the case of an operator in the Arveson’s class, the essential points will give all information about essential spectrum, while the transient points may be misleading. Here we loose only information about eigenvalues of finite multiplicity. But this is very important if such points exist between the lower and upper bounds of essential spectrum, since they lead to the existence of spectral gaps between these bounds.

Things can be more difficult in the case of an arbitrary bounded self-adjoint operator. There may exist essential points, which are not spectral values. The operator given by the equation (2.1) is of that kind. Anyway the inclusion in Theorem (2.1.2) helps us to determine the spectrum, with an additional assumption of connectedness of the essential spectrum. The details of this claim are given below, which is a brief review of the arti-
2.1. The Truncation method with some slight modifications. This will play a key role in the forthcoming sections.

### 2.1.1 Linear algebraic approach:

Recall that, for a bounded self-adjoint operator $A$, $\sigma(A)$ is contained in the interval $[m, M]$ and $\sigma_e(A)$ in $[\nu, \mu]$ where $m, M, \nu, \mu$, are bounds of $\sigma(A)$ and $\sigma_e(A)$ respectively. The following definitions and preliminary results are needed further.

**Definition 2.1.6.** Consider the singular number $s_k$, $k$ natural number, $s_k(A) = \inf \{ \| A - F \| ; F \in \mathcal{B}(\mathcal{H}), \text{rank} F \leq k - 1 \}$ is the $k^{th}$ approximation number of $A$.

Clearly we have $\| A \| = s_1(A) \geq s_2(A) \geq \ldots \geq 0$

**Theorem 2.1.4.** [37] $\lim_{k \to \infty} s_k(A) = \| A \|_{\text{ess}}$ where $\| A \|_{\text{ess}}$ is the essential norm.

**Theorem 2.1.5.** [19] $\lim_{n \to \infty} s_k(A_n) = s_k(A)$.

**Remark 2.1.3.** For $|A| = (A^*A)^{1/2}$, in case $A$ is a finite matrix, the approximation numbers are the eigenvalues of $|A|$. That is $s_k(A) = \lambda_k(|A|)$, where $\lambda_k(|A|)$ is the $k^{th}$ eigenvalue of $|A|$.

**Theorem 2.1.6.** [37] The set $\sigma(|A|) - [0, \| A \|_{\text{ess}}]$ is at most countable, $\| A \|_{\text{ess}}$ is the only possible accumulation point, and all the points
of the set are eigenvalues with finite multiplicity of $|A|$. Furthermore if

$$
\lambda_1(|A|) \geq \lambda_2(|A|) \geq \ldots \geq \lambda_N(|A|)
$$

are those $N$ eigenvalues ($N$ can be infinity), then

$$
s_k(A) = \begin{cases} 
\lambda_k(|A|), & \text{if } N = \infty \text{ or } 1 \leq k \leq N \\
\|A\|_{\text{ess}}, & \text{if } N < \infty \text{ and } k \geq N + 1 
\end{cases}
$$

(2.2)

**Corollary 1.**

$$
\lim_{n \to \infty} \lambda_k(|A_n|) = \lim_{n \to \infty} s_k(A_n) = s_k(A) = \begin{cases} 
\lambda_k(|A|), & \text{if } N = \infty \text{ or } 1 \leq k \leq N \\
\|A\|_{\text{ess}}, & \text{if } N < \infty \text{ and } k \geq N + 1 
\end{cases}
$$

**Remark 2.1.4.** The above result will play a key role in the approximation of spectrum. Considering the positive operator $A - mI$, it can be deduced that the set $\sigma(A) \cap [\mu, M]$ is at most countable and that consists of eigenvalues of finite multiplicity by Theorem (2.1.6). Also $\mu$ is the only possible accumulation point. Let these eigenvalues be

$$
\lambda_R^+(A) \leq \ldots \leq \lambda_2^+(A) \leq \lambda_1^+(A).
$$

Similarly by considering the operator $MI - A$, it can be observed that $\sigma(A) \cap [m, \nu)$ consists of at most countably many eigenvalues of finite multiplicity with only possible accumulation point $\nu$. Let

$$
\lambda_I^-(A) \leq \lambda_2^-(A) \leq \ldots \leq \lambda_S^-(A)
$$

be those eigenvalues. Also the numbers $R$ and $S$ can be infinity. Arrange
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the eigenvalues of $A_n$ as

$$\lambda_1(A_n) \geq \lambda_2(A_n) \geq \ldots \geq \lambda_n(A_n).$$

From here onwards, the above notations will be used.

Now we prove the following result from [19] which is the major tool that is used frequently in this thesis.

**Theorem 2.1.7.** For every fixed integer $k$ we have

$$\lim_{n \to \infty} \lambda_k(A_n) = \begin{cases} 
\lambda_k^+(A), & \text{if } R = \infty \text{ or } 1 \leq k \leq R \\
\mu, & \text{if } R < \infty \text{ and } k \geq R + 1 
\end{cases}$$

$$\lim_{n \to \infty} \lambda_{n+1-k}(A_n) = \begin{cases} 
\lambda_k^-(A), & \text{if } S = \infty \text{ or } 1 \leq k \leq S \\
\nu, & \text{if } S < \infty \text{ and } k \geq S + 1 
\end{cases}$$

In particular,

$$\lim_{k \to \infty} \lim_{n \to \infty} \lambda_k(A_n) = \mu \text{ and } \lim_{k \to \infty} \lim_{n \to \infty} \lambda_{n+1-k}(A_n) = \nu.$$

**Proof.** The following observations are made first.

$$|A - mI| = A - mI, \ P_n(A - mI)P_n = A_n - mI_n, \text{ and } |A_n - mI_n| = A_n - mI_n.$$ 

Hence from the above corollary, we have

$$\lim_{n \to \infty} \lambda_k(A_n - mI_n) = \begin{cases} 
\lambda_k(A - mI), & \text{if } R = \infty \text{ or } 1 \leq k \leq R \\
\|A - mI\|_{ess}, & \text{if } R < \infty \text{ and } k \geq R + 1 
\end{cases}$$

(2.3)
Similarly, by considering the operator $MI - A$, we get

$$\lim_{n \to \infty} \lambda_k(MI_n - A_n) = \begin{cases} 
\lambda_k(MI - A), & \text{if } S = \infty \text{ or } 1 \leq k \leq S \\
\|MI - A\|_{ess}, & \text{if } S < \infty \text{ and } k \geq S + 1 
\end{cases} \tag{2.4}$$

Also we have the following identities

$$\|A - mI\|_{ess} = \mu - m, \quad \|MI - A\|_{ess} = M - \nu. \tag{2.5}$$

$$\lambda_k(A_n - mI_n) = \lambda_k(A_n) - m, \quad \lambda_k(MI_n - A_n) = M - \lambda_{n+1-k}(A_n). \tag{2.6}$$

$$\lambda_k(A - mI) = \lambda^+_k(A) - m, \quad \lambda_k(MI - A) = M - \lambda^-_k(A). \tag{2.7}$$

Substituting them in equations (2.3) and (2.4), we get

$$\lim_{n \to \infty} \lambda_k(A_n) = \begin{cases} 
\lambda^+_k(A), & \text{if } R = \infty \text{ or } 1 \leq k \leq R \\
\mu, & \text{if } R < \infty \text{ and } k \geq R + 1 
\end{cases}$$

$$\lim_{n \to \infty} \lambda_{n+1-k}(A_n) = \begin{cases} 
\lambda^-_k(A), & \text{if } S = \infty \text{ or } 1 \leq k \leq S \\
\nu, & \text{if } S < \infty \text{ and } k \geq S + 1 
\end{cases}$$

Hence the proof.

**Remark 2.1.5.** The above results are also true if we replace $A_n$ by some other sequence $A_{1n}$ with the property that

$$\|A_n - A_{1n}\| \to 0 \text{ as } n \to \infty$$

In order to justify this, we need only to recall an important inequality concerning the eigenvalues of self-adjoint matrices $A, B$ (see page no. 63).
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of [7])

\[ |\lambda_k(A) - \lambda_k(B)| \leq \|A - B\|. \tag{2.8} \]

**Remark 2.1.6.** By Theorem (2.1.7), all the discrete spectral values lying outside the bounds of essential spectrum and the upper and lower bounds of the essential spectrum can be approximated. Note that, the theorem points out exactly the particular sequence that converges to a discrete spectral value. But how fast does the convergence take place, is still not known. Looking at some concrete situations, one may hope for a better rate of convergence.

Even the rate of convergence is not estimated, it can be proved that the order of convergence is the same as the order of convergence of approximation numbers. The following theorem gives a vague idea about the rate of convergence.

**Theorem 2.1.8.** If \(s_k(A_n) - s_k(A) = O(\theta_n)\), where \(\theta_n\) goes to 0 as \(n\) tends to \(\infty\), then

\[
\lambda_k(A_n) = \begin{cases} 
\lambda_k^+ (A) + O(\theta_n), & \text{if } R = \infty \text{ or } 1 \leq k \leq R \\
\mu + O(\theta_n), & \text{if } R < \infty \text{ and } k \geq R + 1
\end{cases}
\]

\[
\lambda_{n+1-k}(A_n) = \begin{cases} 
\lambda_k^- (A) + O(\theta_n), & \text{if } S = \infty \text{ or } 1 \leq k \leq S \\
\nu + O(\theta_n), & \text{if } S < \infty \text{ and } k \geq S + 1
\end{cases}
\]

where \(R\) and \(S\) are the same notations used in Theorem(2.1.7).

**Proof.** Let \(N\) be the number of eigenvalues lying in \(\sigma(|A|) - [0, \|A\|_{\text{ess}}]\). From equation (2.2), and the remarks that was made just before Theorem
we have the following identity.

\[
\lambda_k(|A_n|) - \lambda_k(|A|) = \begin{cases} 
\lambda_k(|A_n|) - \lambda_k(|A|), & \text{if } N = \infty \text{ or } 1 \leq k \leq N \\
\lambda_k(|A_n|) - ||A||_{ess}, & \text{if } N < \infty \text{ and } k \geq N + 1
\end{cases}
\]

Since by hypothesis, \( s_k(A_n) - s_k(A) = O(\theta_n) \), we get

\[
\lambda_k(|A_n|) - \lambda_k(|A|) = O(\theta_n), \quad \text{if } N = \infty \text{ or } 1 \leq k \leq N,
\]

\[
\lambda_k(|A_n|) - ||A||_{ess} = O(\theta_n), \quad \text{if } N < \infty \text{ and } k \geq N + 1.
\]

Applying this to the positive operators \( A - mI \), and \( MI - A \), with the notations used in Theorem (2.1.7), we get the following conclusions.

\[
\lambda_k(A_n - mI_n) = \begin{cases} 
\lambda_k(A - mI) + O(\theta_n), & \text{if } R = \infty \text{ or } 1 \leq k \leq R \\
||A - mI||_{ess} + O(\theta_n), & \text{if } R < \infty \text{ and } k \geq R + 1
\end{cases}
\]

and

\[
\lambda_k(MI_n - A_n) = \begin{cases} 
\lambda_k(MI - A) + O(\theta_n), & \text{if } S = \infty \text{ or } 1 \leq k \leq S \\
||MI - A||_{ess} + O(\theta_n), & \text{if } S < \infty \text{ and } k \geq S + 1
\end{cases}
\]

Using the identities (2.5), (2.6) and (2.7), we get the desired conclusions

\[
\lambda_k(A_n) = \begin{cases} 
\lambda_k^+(A) + O(\theta_n), & \text{if } R = \infty \text{ or } 1 \leq k \leq R \\
\mu + O(\theta_n), & \text{if } R < \infty \text{ and } k \geq R + 1
\end{cases}
\]

\[
\lambda_{n+1-k}(A_n) = \begin{cases} 
\lambda_k^-(A) + O(\theta_n), & \text{if } S = \infty \text{ or } 1 \leq k \leq S \\
\nu + O(\theta_n), & \text{if } S < \infty \text{ and } k \geq S + 1
\end{cases}
\]

Hence the proof.
The above theorem is the first result regarding the rate of convergence in the approximations done in Theorem (2.1.7). So far there is no evidence of remainder estimation and the error estimation in these approximations in the case of an arbitrary self-adjoint operator to the best of our knowledge. The subsequent theorem taken from [19] denies the existence of spurious eigenvalues under the assumption of connectedness of essential spectrum.

**Theorem 2.1.9.** [19] If $A$ is a self-adjoint operator and if $\sigma_e(A)$ is connected, then $\sigma(A) = \Lambda$.

**Remark 2.1.7.** It is worthwhile to notice that the connectedness of essential spectrum enables us to compute the spectrum using finite dimensional truncations. Thus, if we can not determine the spectrum fully by the truncations, then the essential spectrum is not connected. In short, if there is a spurious eigenvalue, then there exists a gap in the essential spectrum.

**Remark 2.1.8.** The converse of the above observation need not be true. That is the existence of a spectral gap does not lead to the existence of a spurious eigenvalue. For example, if we take $A$ to be the projection operator on to some closed subspace of $\mathbb{H}$, then the eigenvalues of truncations are 0 and 1 only. There we have $\Lambda = \sigma(A) = \{0, 1\}$. Hence no spurious eigenvalues, but still there is a gap.

In summary, the upper and lower bounds of the essential spectrum can be computed by using the sequence of eigenvalues of finite dimensional truncations. Also the discrete eigenvalues lying below and above these bounds can be computed. The above results pinpointing the par-
particular sequence of eigenvalues that converges to a particular eigenvalue of the operator. Now the remaining part is the computation of essential spectrum. The problem is whether it is possible to locate the gaps in the essential spectrum using these truncations. If it is possible, then the spectrum is fully determined up to some discrete eigenvalues that may have trapped between these gaps.

2.2 Gaps in the essential spectrum

The following theorem is an attempt to predict the existence of spectral gaps, using the finite dimensional truncations. The notation \#S is used to denote the number of elements in the set S.

**Theorem 2.2.1.** Let \( A \) be a bounded self-adjoint operator and \( \lambda_{n1}(A_n) \geq \lambda_{n2}(A_n) \geq \ldots \geq \lambda_{nn}(A_n) \) be the eigenvalues of \( A_n \) arranged in decreasing order. For each positive integer \( n \), let \( \{w_{nk} : k = 1, 2, \ldots n\} \) be a set of numbers such that \( 0 \leq w_{nk} \leq 1 \) and \( \sum_{k=1}^{n} w_{nk} = 1 \). If there exists a \( \delta > 0 \) and \( K > 0 \) such that

\[
\# \left\{ \lambda_{nj} ; \left| \sum_{k=1}^{n} w_{nk} \lambda_{nk} - \lambda_{nj} \right| < \delta \right\} < K \tag{2.9}
\]

and in addition if \( \sigma_e(A) \) and \( \sigma(A) \) has the same upper and lower bounds, then \( \sigma_e(A) \) has a gap.

**Proof.** Consider the set \( S = \left\{ \sum_{k=1}^{n} w_{nk} \lambda_{nk}, n = 1, 2, 3 \ldots \right\} \). Observe
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that \( \lambda_{nn} \leq \sum_{k=1}^{n} w_{nk} \lambda_{nk} \leq \lambda_{n1} \). Also since each \( \lambda_{nj} \)'s lying in the interval \([m, M]\), the set \( S \) is contained in the interval \([m, M] = [\nu, \mu] \).

Case 1. Assume that \( S \) is a finite set, say \( S = \{a_1, a_2, a_3 \ldots a_m\} \). In this case, the value of the sum \( \sum_{k=1}^{n} w_{nk} \lambda_{nk} \) equals some of the numbers \( a_i \)'s for infinitely many \( n \). Let \( a_1, a_2, a_3 \ldots a_p \) be those numbers. That is

\[
\sum_{k=1}^{n} w_{nk} \lambda_{nk} = a_i \text{ for infinitely many } n \text{ and } i = 1, 2, \ldots p.
\]

From this and by the condition (2.9), for each \( i = 1, 2, \ldots p \), we have

\[
N_n(a_i - \delta, a_i + \delta) = \# \{ \lambda_{nj}; |a_i - \lambda_{nj}| < \delta \} < K \text{ for infinitely many } n.
\]

Hence \( N_n(a_i - \delta, a_i + \delta) \) will not go to infinity as \( n \) goes to infinity. Therefore no number in the interval \((a_i - \delta, a_i + \delta)\) is an essential point. Since the essential spectrum is contained the set of all essential points, by Theorem (2.1.2), there is no essential spectral values in this interval. Also since each \( a_i \) lies between the bounds of essential spectrum, we can choose an appropriate \( \epsilon > 0 \) such that \((a_i - \epsilon, a_i + \epsilon)\) lies between the bounds and contained in the interval \((a_i - \delta, a_i + \delta)\). Then the interval \((a_i - \epsilon, a_i + \epsilon)\) is a spectral gap.

Case 2. Now we consider the case when \( S \) is an infinite set. Here \( S \) has at least one limit point in \( \mathbb{R} \). If \( w_0 \) is a limit point of the set \( S \), then we have \( \nu \leq w_0 \leq \mu \).

Now the interval \((w_0 - \delta/2, w_0 + \delta/2)\) will contain infinitely many
points from the set $S$. Corresponding to these points, there are infinitely many $A_n$’s for which the number of eigenvalues in $\left( w_0 - \delta/2, w_0 + \delta/2 \right)$ is bounded by $K$ due to (2.9). Hence the sequence $N_n \left( w_0 - \frac{\delta}{2}, w_0 + \frac{\delta}{2} \right)$ will not go to infinity, since a subsequence is bounded by $K$. Hence no point in the interval $\left( w_0 - \delta/2, w_0 + \delta/2 \right)$ is an essential point. Since the essential spectrum is contained the set of all essential points, by Theorem (2.1.2), $\left( w_0 - \delta/2, w_0 + \delta/2 \right)$ contains no essential spectral values. Hence, as in the case 1, we can choose an $\epsilon > 0$, such that the interval $(w_0 - \epsilon, w_0 + \epsilon)$ is a spectral gap between the bounds of the essential spectrum and the proof is completed.  

\begin{proof}

Remark 2.2.1. The proof of the above theorem gives some information regarding the gap size. Since the interval $\left( w_0 - \delta/2, w_0 + \delta/2 \right)$ contains no essential spectral values, it is a spectral gap if it lies between the bounds of the essential spectrum. In that case the gap size may be greater than $\delta$. In the case 1, it could be greater than $2\delta$.

Remark 2.2.2. There is the possibility for the presence of discrete eigenvalues inside the spectral gaps detected using the above theorem.

Remark 2.2.3. The special case which is more interesting is when we choose $w_{nk} = \frac{1}{n}$, for all $n$. In that case, we are actually looking at the averages of eigenvalues of truncations and these averages can be computed using the trace at each level.

In the Theorem (2.2.1), the weighted mean of the eigenvalues at each level and its deviation is analyzed. Now some special choices of the weighting method are discussed below to predict the existence of spectral gaps, using the Theorem (2.2.1).
2.2. Gaps in the essential spectrum

Special Choice I
Here is an instance where these weights $w_{nk}$ arises naturally associated with a self-adjoint operator on a Hilbert space. Let $A_n = \sum_{k=1}^{n} \lambda_{n,k} Q_{n,k}$ be the spectral resolution of $A_n$. Define $w_{nk} = \langle Q_{n,k}e_1, e_1 \rangle$. Then $0 \leq w_{nk} \leq 1$ and $\sum_{k=1}^{n} w_{nk} = 1$. Now

$$\sum_{k=1}^{n} w_{nk} \lambda_{nk} = \sum_{k=1}^{n} \lambda_{nk} \langle Q_{n,k}e_1, e_1 \rangle = \langle A_n e_1, e_1 \rangle = \langle Ae_1, e_1 \rangle = a_{11}.$$  

Therefore by Theorem (2.2.1), if there exists a $\delta > 0$ and a $K > 0$, such that

$$\# \{ \lambda_{nj}; |a_{11} - \lambda_{nj}| < \delta \} < K$$

then there exists a gap in the essential spectrum of $A$. That means the spectral gap prediction is done by looking at the first entry in the matrix representation of $A$. That is, if the first entry in the matrix representation of $A$, is not an essential point, then there exists a gap in the essential spectrum.

Remark 2.2.4. All points of the form $\langle Ae_i, e_i \rangle = a_{ii}$ are in the numerical range which lies between the bounds of the essential spectrum, in the case that the bounds coincide with the bounds of the spectrum. Hence in that case, if $a_{ii}$ is not an essential point for some $i$, then that will lead to the existence of a spectral gap. That means if any one of the diagonal entries in the matrix representation of $A$ is not an essential point, then there exists a gap in the essential spectrum as indicated in the above special choice of $w_{nk}$.
The following is an example where the first entry $a_{11}$ is a transient point and the spectral gap prediction is valid.

**Example 2.2.1.** Define a bounded self-adjoint operator $A$ on $l^2(\mathbb{N})$, as follows.

$$A(x_n) = (x_{n-1} + x_{n+1}) + (v_n x_n), x_0 = 0;$$

where the periodic sequence $(v_n) = (1, 2, 3, 1, 2, 3, \ldots)$. The matrix representation of $A$, associated to the standard orthonormal basis, is tridiagonal. The diagonal entries are the entries in the periodic sequence $(v_n)$ and upper and lower diagonal will be 1. In the next chapter, we will see that such matrices can be identified as the block Toeplitz operator with corresponding matrix valued symbol given by

$$\tilde{f}(\theta) = \begin{bmatrix} 1 & 1 & e^{i\theta} \\ 1 & 2 & 1 \\ e^{-i\theta} & 1 & 3 \end{bmatrix}.$$ 

By our special choice above, Theorem (2.2.1) guarantees that if $\langle A(e_1), e_1 \rangle = 1$ is a transient point, then $\sigma_e(A)$ has a gap. The proof for the fact that 1 is a transient point, is given in the Example (3.4.1).

The operator considered in the above example comes as a discrete version of Schrodinger operator, which arises naturally in many practical problems. In general, the discrete Schrodinger operator is defined on $l^2(\mathbb{Z})$ as follows.

$$A(y) = (y_{n-1} + y_{n+1}) + (v_n y_n) \text{ for every } y = (\ldots, y_1, y_2, y_3, \ldots) \in l^2(\mathbb{Z})$$

where $v = (\ldots, v_1, v_2, \ldots)$ is a fixed bounded sequence. The corresponding
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truncations are

\[
(A)_{2n+1} = \begin{bmatrix}
v_{-n} & 1 & 0 & 0 \\
1 & v_{-n+1} & 1 & 0 & 0 \\
0 & 1 & . & 1 & 0 & 0 \\
0 & 0 & 1 & . & 1 & 0 & 0 \\
. & . & . & . & . & . & . \\
0 & 0 & 1 & . & 1 & 0 \\
. & . & . & . & . & . & . \\
0 & 0 & 1 & . & 1 \\
. & . & . & . & . & . & . \\
0 & 0 & 1 & v_n \\
\end{bmatrix}
\]

We consider the cases where the bounds of \(\sigma(A)\) and \(\sigma_e(A)\) coincide. Let \(a_n\) be the averages of the \(2n+1\) terms \((v_{-n} \ldots v_1, v_2, \ldots v_n)\) of the sequence \(v\). If we choose \(w_{nk} = \frac{1}{n}\), by Theorem (2.2.1), if there exists a \(\delta > 0\) such that

\[
\# \{\lambda_{nj}; |a_n - \lambda_{nj}| < \delta\} < K
\]

for some fixed \(K\), then there exists a gap in the essential spectrum.

**Remark 2.2.5.** The Borg-type theorems will be proved in the next chapter, which will ensure that if the potential function is periodic and non constant, then the operator will have gaps in the essential spectrum [39]. For eg. if \(x = (\ldots a, b, a, b, a, b, \ldots)\), with \(a < b\), then the interval \((a, b)\) is a gap.

**Special Choice II**

By invoking Theorem (2.1.7), there exist sequences of eigenvalues of trun-
cations, which converges to the bounds of the essential spectrum. That is

there exist $\lambda_{n_l}, \lambda_{n_m}$ such that $\lim_{n_l \to \infty} \lambda_{n_l} = \nu$, and $\lim_{n_m \to \infty} \lambda_{n_m} = \mu$.

Now fix a number $t \in (0, 1)$, and define $w_{nk}$ as follows.

\[
\begin{align*}
    w_{nk} &= \begin{cases} 
        t, & \text{if } k = l, \\
        1 - t, & \text{if } k = m, \\
        0, & \text{otherwise,}
    \end{cases}
\end{align*}
\]

Then we have the following conclusions. If there exists a $\delta > 0$ and $K > 0$ such that

\[
\# \{ \lambda_{n_j}; |t\lambda_{nl} + (1 - t)\lambda_{nm} - \lambda_{nj}| < \delta \} < K,
\]

then there is a gap in the essential spectrum $\sigma_e(A)$.

Proof of the above assertion is only a repetition of the arguments used in the proof of Theorem (2.2.1). Theorem (2.2.1) can not be applied directly because the crucial assumption that $\sigma(A)$ and $\sigma_e(A)$ have the same bounds, is missing here. Notice that this assumption was used only to ensure that the sum $\sum_{k=1}^{n} w_{nk}\lambda_{nk}$ lying between the bounds of essential spectrum. But the above choice of $w_{nk}$ guarantees that the sum $\sum_{k=1}^{n} w_{nk}\lambda_{nk}$ converges to some number between the bounds of essential spectrum. And that number will create a gap in the essential spectrum, as observed in the proof of Theorem (2.2.1).

**Remark 2.2.6.** The above observations show that we may be able to
predict the existence of spectral gaps, relaxing the assumptions of Theorem (2.2.1). But the freedom for choosing the weights \( w_{nk} \) to be arbitrary, is lost here.

It is not clear whether the converse of Theorem (2.2.1) is true for an arbitrary self-adjoint operator. The converse is proved below in the case of operators in the Arveson’s class.

**Theorem 2.2.2.** Let \( A \) be a bounded self-adjoint operator in the Arveson’s class. And suppose that there exists a gap in the essential spectrum. Then there exists a set of numbers \( \{w_{nk} : k = 1, 2, \ldots, n\} \) such that \( 0 \leq w_{nk} \leq 1 \) and \( \sum_{k=1}^{n} w_{nk} = 1 \) and a \( \delta > 0 \) such that

\[
\# \left\{ \lambda_{nj} : \left| \sum_{k=1}^{n} w_{nk} \lambda_{nk} - \lambda_{nj} \right| < \delta \right\} < K,
\]

for some \( K > 0 \).

**Proof.** Let \( (a, b) \) be a gap in the essential spectrum. Then there exists sequences of eigenvalues of truncations \( \lambda_{nl}, \lambda_{nm} \) such that

\[
\lim_{n_l \to \infty} \lambda_{nl} = a \text{ and } \lim_{n_m \to \infty} \lambda_{nm} = b.
\]

Fix a \( t \in (0, 1) \) and define the sequence \( w_{nk} \) as

\[
w_{nk} = \begin{cases} t, & \text{if } k = l, \\ 1 - t, & \text{if } k = m, \\ 0, & \text{otherwise,} \end{cases}
\]
Since the number $c_t = ta + (1 - t)b$ is in the interval $(a, b)$, it is not an essential point. Also since $A$ is in the Arveson’s class, all such points are transient points by Theorem (2.1.3). Hence there exists a $\delta_1 > 0$ such that $\sup_n (c_t - \delta_1, c_t + \delta_1) < K_1$ for some $K_1 > 0$. Also

$$\sum_{k=1}^{n} w_{nk} \lambda_{nk} = t\lambda_{nj} + (1 - t)\lambda_{n} \rightarrow ta + (1 - t)b = c_t \text{ as } n \rightarrow \infty.$$ 

Therefore there exists an $N$ such that

$$\left| c_t - \sum_{k=1}^{n} w_{nk} \lambda_{nk} \right| < \frac{\delta_1}{2} \text{ for all } n > N.$$ 

Now if for some $n > N$, $\left| \sum_{k=1}^{n} w_{nk} \lambda_{nk} - \lambda_{nj} \right| < \frac{\delta_1}{2}$, then $|c_t - \lambda_{nj}| < \delta_1$.

Therefore,

$$\# \left\{ \lambda_{nj} : \left| \sum_{k=1}^{n} w_{nk} \lambda_{nk} - \lambda_{nj} \right| < \frac{\delta_1}{2} \right\} < N_n (c_t - \delta_1, c_t + \delta_1) < K_1, \forall n > N.$$ 

Now choosing $K = \sup \{K_1, N\}$ and $\delta = \frac{\delta_1}{2}$, the proof is completed. \qed

Remark 2.2.7. In the above proof, the sequence $\{w_{nk}\}$ and the bound $K$ will depend on the particular choice of $t \in (0, 1)$.

### 2.3 Gap prediction methods

The concepts of second order relative spectra and quadratic projection method, which are almost synonyms of the other, were used in the spec-
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tral pollution problems and in determining the eigenvalues in the gaps by E.B. Davies, Levitin, Shagorodsky, etc. (see [34],[35], [51],[52]). Analogous to them, a new method is proposed in this section, to use in the spectral gap prediction problems. In short, the spectral gap prediction problems are reduced into the determination of nonzero values of a particular function. This particular function can be approximated by a sequence of functions uniformly. And this sequence of functions comes directly from the eigenvalues of truncations of the operator under concern.

The idea is to open the gap by translating and squaring the operator and identify each numbers in the interval $(\nu, \mu)$ as the lower bound of essential spectrum of a positive definite operator. And there the truncation methods, in particular, Theorem (2.1.7) are applied to compute this lower bound. The idea of squaring the operator to get information about its spectrum was used before. First, we shall briefly mention the work done by E.B.Davies in [34] and [35], which is of great interest, where he considered functions which are related to the distance from the spectrum.

2.3.1 Analytical approach

In his paper published in 1998 [34], E.B.Davies considered the function $F$ defined by

$$F(t) = \inf \left\{ \frac{\|A(x) - tx\|}{\|x\|} : 0 \neq x \in \mathbb{L} \right\} \quad (2.10)$$

where $\mathbb{L}$ is a subspace of $\mathbb{H}$. Then he observed the following (Lemma 1 and its corollary in [34]).

- $F$ is Lipschitz continuous and satisfies $|F(s) - F(t)| \leq |s - t|$, for
all $s, t \in \mathbb{R}$.

- $F(t) \geq d(t, \sigma(A)) = \text{dist}(t, \sigma(A))$

- If $0 \leq F(t) \leq \delta$, then $\sigma(A) \cap [t - \delta, t + \delta] \neq \emptyset$.

From these observations, he obtained some bounds for the eigenvalues in the spectral gap of $A$, and found it useful in some concrete situations. For the efficient computation of the function $F$, he considered family of operators $N(s)$ on the given finite dimensional subspace $L$, defined by

$$N(s) = A_L^* A_L - 2sPA_L + s^2 I_L$$

where $P$ is the projection onto $L$ and the notation $A_L$ means $A$ restricted to $L$. The eigenvalues of these finite dimensional operators form sequence of real analytic functions. He used these sequence to approximate the function $F$ and thereby obtain information about the spectral properties of $A$. The main result is stated below (special case of Theorem 9 in [34], under the assumption that $A$ is bounded).

**Theorem 2.3.1.** Suppose $\{L_n\}_{n=1}^{\infty}$ is an increasing sequence of closed subspaces of $\mathbb{H}$. If $F_n$ the functions associated with $L_n$ according to (2.10), then $F_n$ decreases monotonically and converge locally uniformly to $d(., \sigma(A))$. In particular, $s \in \sigma(A)$ if and only if

$$\lim_{n \to \infty} F_n(s) = 0.$$ 

In his paper on spectral pollution [35] in 2004, he tried to link the above method with various techniques that were known in the past due
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to Lehmann [49], Behnke and Goerisch [11], Zimmerman and Mertin [72]. There he tried to resolve the problem of spurious eigenvalues in a spectral gap. He considered the function

\[ G(t) = \inf \{ \| A(x) - tx \| : x \in \mathbb{H}, \| x \| = 1 \} \]

and wanted to evaluate \( G \) numerically and to locate spectrum of \( A \), using the fact that \( G(t) = d(t, \sigma(A)) \). He introduced the approximating sequence of functions as

\[ G_n(t) = \inf \{ \| A(x) - tx \| : x \in L_n, \| x \| = 1 \} \]

where \( L_n \) is an increasing sequence of subspaces whose union is dense in \( \mathbb{H} \), and used them to obtain some results as listed below (page no. 422-425 of [35]).

- Given \( \epsilon > 0 \), there exists an \( N_\epsilon \) such that \( n \geq N_\epsilon \) implies
  \[ G(t) \leq G_n(t) \leq G(t) + \epsilon \text{ for all } t \in \mathbb{R} \]

- \( \sigma(A) \cap [t - G_n(t), t + G_n(t)] \neq \emptyset \) for every \( t \in \mathbb{R} \).

Using these and with some assumptions on \( G_n \), he obtained some bounds for the eigenvalues between the bounds of essential spectrum. He also produced some numerical evidence for the implementation of these techniques in bounding the eigenvalues of some particular operators.

Levitin and Shargorodsky considered the problem of spectral pollution in [52]. They suggested the usage of second order relative spectra, to deal
the problem. For the sake of completion, the definition is given below.

**Definition 2.3.1.** [52] Let $\mathbb{L}$ be a finite-dimensional subspace of $\mathbb{H}$. A complex number $z$ is said to belong to the second order spectrum $\sigma_2(A, \mathbb{L})$ of $A$ relative to $\mathbb{L}$ if there exists a nonzero $u$ in $\mathbb{L}$ such that

$$\langle (A - zI)u, (A - \bar{z}I)v \rangle = 0, \text{ for every } v \in \mathbb{L}$$

They proved the following. Consider a disc in the complex plane with diameter is an interval on the real line which intersect with the spectrum of $A$. Every such discs will have nonempty intersection with the second order relative spectrum (Lemma 5.2 of [52]). They also provided some numerical results in case of some Multiplication and Differential operators, which indicated the effectiveness of second order relative spectra in avoiding the spectral pollution. In [51], Boulton and Levitin used the quadratic projection method to avoid spectral pollution in the case of some particular Schrodinger operators. Before introducing the new method, we list down a couple of theorems from [54] which considered operators with disconnected essential spectrum and useful in our context.

**Lemma 2.3.1.** [54] Let $A$ be a bounded self-adjoint operator with the essential spectrum, $\sigma_e(A) = [a, b] \cup \{c\}$ where $a < b < c$. Assume that $b$ is not an accumulation point of the discrete spectra of $A$. Then $a,b,c$ can be computed by truncation method.

Next theorem will give information about one endpoint of the spectral gap, provided the other end point is known.

**Theorem 2.3.2.** [54] Let $A$ be a bounded self-adjoint operator and
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\[ \sigma_e(A) = [a,b] \cup [c,d], \text{ where } a < b < c < d. \] Assume that b is known and not an accumulation point of the discrete spectra of A. Then c can be computed by truncation method.

### 2.3.2 The new method

To predict the existence of a gap in the essential spectrum, we need to know whether a number \( \lambda \) in \((\nu, \mu)\) belongs to the spectrum or not. If it is not a spectral value, then there exists an open interval between \((\nu, \mu)\) as a part of the complement of the spectrum, since the complement is an open set. We observe that the spectral gap prediction is possible by computing values of the following function.

**Definition 2.3.2.** Define the nonnegative valued function \( f \) on the real line \( \mathbb{R} \) as follows.

\[
f(\lambda) = \nu_\lambda = \inf \sigma_e((A - \lambda I)^2).
\]

The primary observation is that we can predict the existence of a gap inside the essential spectrum by evaluating the function and checking whether it attains a nonzero value. The nonzero values of this function give the indication of spectral gaps.

**Theorem 2.3.3.** The number \( \lambda \) in the interval \((\nu, \mu)\) is in the gap if and only if \( f(\lambda) > 0 \). Also one end point of the gap will be \( \lambda \pm \sqrt{f(\lambda)} \).

**Proof.** Using the spectral mapping theorem, we observe that \( f(\lambda) \) is the square of the distance of \( \lambda \) to the essential spectrum of \( A \). The details
are given below.

\[ \inf \sigma_e((A - \lambda I)^2) = d(0, \sigma_e(A - \lambda I)^2) = d(0, \sigma_e(A - \lambda I))^2 = d(\lambda, \sigma_e(A))^2 \]

Hence \( \lambda \) is in the essential spectrum of \( A \) if and only if \( f(\lambda) = 0 \), since essential spectrum is a closed set. Therefore the number \( \lambda \) in the interval \((\nu, \mu)\) is in the gap if and only if \( f(\lambda) > 0 \). Now if \( \lambda \) is in the gap, then one of the end points will be at a distance \( \sqrt{f(\lambda)} \) from \( \lambda \). Hence that end point will be \( \lambda \pm \sqrt{f(\lambda)} \).

\[ \begin{array}{c}
\text{Figure 2.1: Graph of } f(\lambda) \\
\end{array} \]

The advantage of considering \( f(\lambda) \) is that, it is the lower bound of the essential spectrum of the operator \((A - \lambda I)^2\), which we can compute by using the finite dimensional truncations with the help of Theorem (2.1.7).
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So the computation of $f(\lambda)$, for each $\lambda$, is possible. This enables us to predict the gap using truncations. Also here we are able to compute one end point of a gap. The other end point is possible to compute as discussed in Theorem (2.3.2).

Coming back to the Arveson’s class, we observe that the essential points and hence the essential spectrum is fully determined by the zeros of the function in the definition (2.3.2)

**Corollary 2.** If $A$ is a bounded self-adjoint operator in the Arveson’s class, then $\lambda$ is an essential point if and only if $f(\lambda) = 0$.

**Proof.** This follows easily from Theorems (2.1.3) and (2.3.3). \qed

When one wishes to apply the above results to determine the gaps in the essential spectrum of a particular operator, one has to face the following problems. To check for each $\lambda$ in $(\nu, \mu)$, is a difficult task in the computational point of view. Also taking truncations of the square of the operator may lead to difficulty. Note that $(P_n A P_n)^2$ and $P_n A^2 P_n$ are entirely different. So we may have to do more computations to handle the problem.

Another problem is the rate of convergence and estimation of the remainder term. For each $\lambda$ in $(\nu, \mu)$ the value of the function $f(\lambda)$ has to be computed. This computation involves truncation of the operator $(A - \lambda I)^2$ and the limiting process of sequence of eigenvalues of each truncation. The rate of convergence of these approximations and the remainder estimate are the questions of interest.
Below, the function $f(\cdot)$ is approximated by a double sequence of functions, which arise from the eigenvalues of truncations of operators.

**Theorem 2.3.4.** Let $f_{n,k}$ be the sequence of functions defined by $f_{n,k}(\lambda) = \lambda_{n+1-k} (P_n (A - \lambda I)^2 P_n)$. Then $f(\cdot)$ is the uniform limit of a subsequence of $\{f_{n,k}(\cdot)\}$ on all compact subsets of the real line.

**Proof.** By Theorem (2.1.7), we have for each $\lambda$,

$$f(\lambda) = \lim_{k \to \infty} \lim_{n \to \infty} f_{n,k}(\lambda), \quad \text{where } f_{n,k}(\lambda) = \lambda_{n+1-k} (P_n (A - \lambda I)^2 P_n).$$

Now the quantity $\Delta = |f_{n,k}(\lambda) - f_{n,k}(\lambda_0)|$ can be estimated as follows.

$$\Delta = |\lambda_{n+1-k} (P_n (A - \lambda I)^2 P_n) - \lambda_{n+1-k} (P_n (A - \lambda_0 I)^2 P_n)|$$

$$\leq \|P_n (A - \lambda I)^2 P_n - P_n (A - \lambda_0 I)^2 P_n\|$$

$$\leq \|(A - \lambda I)^2 - (A - \lambda_0 I)^2\| = \|((\lambda^2 - \lambda_0^2) I - 2(\lambda_0 - \lambda) A)\| \leq M |\lambda - \lambda_0|,$$

where $M = 2 (|\mu| + \|A\|)$. The first inequality follows from (2.8) and the second one from the fact that $\|P_n\| = 1$. Hence we have

$$|f_{n,k}(\lambda) - f_{n,k}(\lambda_0)| \leq M |\lambda - \lambda_0|. \quad (2.12)$$

Since the constant $M$ above is independent of $n, k$ or $\lambda$, $\{f_{n,k}(\cdot)\}$ forms an equicontinuous family of functions, also it is point wise bounded. Hence $\{f_{n,k}(\cdot)\}$ has a subsequence which converges uniformly on all compact subsets by Arzela-Ascoli theorem. Hence the proof is complete. \(\square\)

The following result makes the computation of $f(\lambda)$ much easier for a particular class of operators. When the operator is truncated first and
square the truncation rather than truncating the square of the operator, the difficulty of squaring a bounded operator is reduced. The computation needs only to square the finite matrices. Denote \( A - \lambda I \) by the symbol \( A_\lambda \).

**Theorem 2.3.5.** If \( \| P_n A - P_n \| \to 0 \) as \( n \to \infty \), then

\[
\lim_{k \to \infty} \lim_{n \to \infty} \lambda_{n+1-k} (P_n (A - \lambda I)^2 P_n) = \lim_{k \to \infty} \lim_{n \to \infty} \lambda_{n+1-k} (P_n (A - \lambda I) P_n)^2.
\]

**Proof.** Observe the following chain of equalities;

\[
\| P_n (A_\lambda)^2 P_n - (P_n (A_\lambda) P_n)^2 \| = \| P_n (A_\lambda) (A_\lambda) P_n - (P_n (A_\lambda) P_n) (P_n (A_\lambda) P_n) \|
\]

\[
= \| P_n (A_\lambda) (A_\lambda) P_n - (A_\lambda) P_n (A_\lambda) P_n + (A_\lambda) P_n P_n (A_\lambda) P_n - P_n (A_\lambda) P_n (A_\lambda) P_n \|
\]

using \( P_n^2 = P_n \) and adding and subtracting \( (A_\lambda) P_n (A_\lambda) P_n \). And notice that the latter is equal to

\[
\|[P_n (A_\lambda) - (A_\lambda) P_n] (A_\lambda) P_n - [P_n (A_\lambda) - (A_\lambda) P_n] P_n (A_\lambda) P_n\|
\]

\[
\|[P_n (A_\lambda) - (A_\lambda) P_n] [(A_\lambda) P_n - P_n (A_\lambda) P_n] \| \leq 2 \| A_\lambda \| \| P_n (A_\lambda) - (A_\lambda) P_n \| = 2 \| A_\lambda \| \| P_n A - AP_n \| \to 0,
\]

as the dimension \( n \) tends to infinity. The proof is completed by applying (2.12) to the matrices \( (P_n (A - \lambda I)^2 P_n) \) and \( (P_n (A - \lambda I) P_n)^2 \).

**Remark 2.3.1.** The function \( f(.) \) that is considered here is directly related to the distance from the essential spectrum, while Davies' function was related with the distance from the spectrum. Here the approximation results in [19], especially Theorem (2.1.7) are used to approximate the function. But it is still not known to us whether these results are useful in
a computational point of view. The methods due to Davies were applied in the case of some Schrödinger operators with a particular kind of potentials in [52] and [51]. We hope that a combined use of both methods may give a better understanding of the spectrum.