CHAPTER 3

SOME RESULTS ON 3–DIMENSIONAL \((LCS)_n\) MANIFOLDS

3.1 Introduction

In (2003), A. A. Shaikh [11] introduced the notion of Lorentzian concircular structure manifolds (briefly \((LCS)_n\) manifolds) with an example, which generalizes the notion of Lorentzian para–Sasakian manifolds introduced by Matsumoto. The notion of local symmetry of a Riemannian manifold has been studied by many authors in several ways to a different extent. As a weaker version of local symmetry, in (1977), Takahashi [15] introduced the notion of locally \(\varphi\)–symmetric Sasakian manifold and obtained their several interesting results.

The importance of concircular curvature tensor is very well known in the differential geometry of certain \(F\)–structure such as complex, almost complex, Kahler, almost Kahler, contact and almost contact structure etc.

A transformation of an \(n\)–dimensional Riemannian manifold \(M^n\), which transform every geodesic circle of \(M^n\) in to a geodesic circle, is called a concircular transformation. A concircular transformation is always a conformal transformation. Here geodesic circle means a curve in \(M^n\) whose first curvature is constant and second curvature is identically zero. Thus the geometry of concircular transformations, that is, the concircular geometry is a generalization of inversive geometry in the sense that the change of metric is
more general than that induced by a circle preserving diffeomorphism. An interesting invariant of a concircular transformation is the Concircular curvature tensor.

The Concircular curvature tensor $C$ of type $(1,3)$ on a Riemannian manifold is defined as

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)} [g(Y,Z)X - g(X,Z)Y].$$

An $(LCS)_n$ manifold is said to be an $\eta$–Einstien manifold if its Ricci tensor $S$ is of the form

$$S(X,Y) = a g(X,Y) + b\eta(X)\eta(Y),$$

where $a, b$ are associated functions on the manifold. An $(LCS)_n$ manifold is said to be space form if the manifold is a space of constant curvature.

The object of the present chapter is to study the 3–dimensional $(LCS)_n$ manifold. After preliminaries in section 3.3, some basic results on 3–dimensional $(LCS)_n$ manifold are given. In section 3.4, we prove that a 3–dimensional $(LCS)_n$ manifold satisfying the condition $R(X,Y) \cdot S = 0$, where $R(X,Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for the tangent vectors $X, Y$ is a manifold of constant curvature. In section 3.5, we prove that a 3–dimensional $\psi$–recurrent $(LCS)_n$ manifold is of constant curvature. Finally we prove that a 3–dimensional locally $\psi$–concircularly symmetric $(LCS)_n$ manifold is symmetric if and only if the scalar curvature tensor $r$ is constant.

### 3.2 Preliminaries

An $n$–dimensional Lorentzian manifold $M^n$ is a smooth connected para compact Hausdorff manifold with a Lorentzian metric $g$ of type $(0,2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to R$ is a non–degenerate
inner product of signature \((-, +, \ldots, +)\), where \(T_p M\) denotes the tangent space of \(M\) at \(p\) and \(R\) is the real number space. A non–zero vector \(v \in T_p M\) is said to be timelike (resp. non–space like, null, space like) if it satisfies \(g_p(v, v) < 0\) (resp. < 0, \(= 0, > 0\)) ([11], [7]).

In a Lorentzian manifold \((M^n, g)\), a vector field \(P\) is defined as
\[
g(X, P) = A(X).
\]

For any \(X \in \chi(M)\) is said to be a concircular vector field if
\[
(\nabla_X A)(Y) = \alpha\{g(X, Y) + \omega(X)A(Y)\},
\]
where \(\alpha\) is non–zero scalar and \(\omega\) is a closed 1–form.

Let \(M^n\) be a Lorentzian manifold admitting a unit timelike concircular vector field \(\xi\), called the generator of manifold, then we have
\[
g(\xi, \xi) = -1. \quad (3.2.1)
\]

Since \(\xi\) is a unit concircular vector field, it follows that there exists a non zero 1–form \(\eta\), such that for
\[
g(X, \xi) = \eta(X). \quad (3.2.2)
\]

The equation of the following form holds
\[
(\nabla_X \eta)(Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\} \quad (\alpha \neq 0) \quad (3.2.3)
\]
for all vector fields \(X, Y\). Where \(\nabla\) denotes the operator of covariant differentiation with respect to the Lorentzian metric \(g\) and \(\alpha\) is a non–zero scalar function satisfies
\[
\nabla_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X), \quad (3.2.4)
\]
where \(\rho\) being a certain scalar function given by \(\rho = -(\xi\alpha)\).

If we put
\[
\psi X = \frac{1}{\alpha}\nabla_X \xi, \quad (3.2.5)
\]
then from (3.2.3) and (3.2.5), we get
(3.2.6) \[ \psi X = X + \eta(X)\xi. \]

From which it follows that \( \psi \) is symmetric \((1,1)\) tensor and called the structure tensor of the manifold. Thus the Lorentzian manifold \( M^n \) together with the unit timelike concircular vector field \( \xi \), its associated \( 1 \)–form \( \eta \) and \((1,1)\) tensor field \( \psi \) is said to be Lorentzian concircular structure manifold (briefly \((LCS)_n\)–manifold).

Especially, if we take \( \alpha = 1 \), then we can obtain the LP–Sasakian structure of Matsumoto ([7]). In a \((LCS)_n\)–manifold, the following relations holds ([11]):

(3.2.7) \[ \eta(\xi) = -1, \quad \psi \xi = 0, \quad \eta(\psi X) = 0, \]
\[ g(\psi X, \psi Y) = g(X, Y) + \eta(X)\eta(Y), \]

(3.2.8) \[ \eta(R(X,Y)Z) = (\alpha^2 - \rho)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \]

(3.2.9) \[ S(X, \xi) = (n - 1)(\alpha^2 - \rho)\eta(X), \]

(3.2.10) \[ R(X,Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \]

(3.2.11) \[ R(\xi,X)Y = (\alpha^2 - \rho)[g(X,Y)\xi - \eta(Y)X], \]

(3.2.12) \[ R(\xi,X)\xi = (\alpha^2 - \rho)[\eta(X)\xi + X], \]

(3.2.13) \[ (\nabla_X\psi)(Y) = \alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \]

(3.2.14) \[ (X\rho) = d\rho(X) = \beta \eta(X). \]

In a 3–dimensional Riemannian manifold, we have

(3.2.15) \[ R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y - \frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \]

where \( Q \) is a Ricci tensor, i.e., \( g(QX,Y) = S(X,Y) \) and \( r \) is a scalar curvature of the manifold.
3.3 Basic Results

**Theorem 3.3.1.** In a 3–dimensional $(LCS)_n$ manifold, the Ricci operator is given by

\[(3.3.1) \quad QX = \left[\frac{r}{2} - (\alpha^2 - \rho)\right]X + \left[\frac{r}{2} - 3(\alpha^2 - \rho)\right]\eta(X)\xi.\]

**Proof:** Taking $Z = \xi$ in (3.2.15) and using (3.2.6), (3.2.9) & (3.2.10), we get (3.3.1).

**Corollary 3.3.1.** In a 3–dimensional $(LCS)_n$ manifold, the Ricci tensor and curvature tensor are given respectively by

\[(3.3.2) \quad S(X, Y) = \left[\frac{r}{2} - (\alpha^2 - \rho)\right] g(X, Y)\]

\[+ \left[\frac{r}{2} - 3(\alpha^2 - \rho)\right]\eta(X)\eta(Y)\]

and

\[(3.3.3) \quad R(X, Y)Z = \left[\frac{r}{2} - 2(\alpha^2 - \rho)\right][g(Y, Z)X - g(X, Z)Y]\]

\[+ \left[\frac{r}{2} - 3(\alpha^2 - \rho)\right][g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y].\]

**Proof:** Equation (3.3.2) follows from (3.3.1). Using (3.3.1) and (3.3.2) in (3.2.15), we get (3.3.3).

**Remark 3.3.1.** Equation (3.3.2) shows that, a 3–dimensional $(LCS)_n$ manifold is an $\eta$–Einstein manifold.

**Lemma 3.3.1.** A 3–dimensional $(LCS)_n$ manifold, is a manifold of constant curvature if and only if the scalar curvature $r = 6(\alpha^2 - \rho)$.

**Proof:** From equation (3.3.3), the lemma follows.

**Lemma 3.3.2.** A 3–dimensional $(LCS)_n$–manifold is a space form if and only if the scalar curvature $r = 6(\alpha^2 - \rho)$. 

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Proof: From lemma (3.3.1), the lemma (3.3.2) follows.

3.4 3–dimensional \((LCS)_n\)–manifold satisfying \(R(X,Y) \cdot S = 0\).

Let us consider a 3–dimensional \((LCS)_n\)–manifold satisfying

\[(3.4.1) \quad R(X,Y) \cdot S = 0.\]

From (3.4.1), we have

\[(3.4.2) \quad S(R(X,Y)U,V) + S(U,R(X,Y)V) = 0.\]

Putting \(X = \xi\) in (3.4.2) and using (3.2.11), we get

\[(3.4.3) \quad (\alpha^2 - \rho)[g(Y,U)S(V,\xi) - \eta(U)S(Y,V)]
\quad + (\alpha^2 - \rho)[g(Y,V)S(U,\xi) - \eta(V)S(Y,U)] = 0.\]

Using (3.2.9) in (3.4.3), we get

\[(3.4.4) \quad (\alpha^2 - \rho)[2(\alpha^2 - \rho)g(Y,U)\eta(V) - \eta(U)S(Y,V)]
\quad + (\alpha^2 - \rho)[2(\alpha^2 - \rho)g(Y,V)\eta(U) - \eta(V)S(U,Y)] = 0.\]

Let \(\{e_i\}, i = 1,2,3\) be an orthonormal basis of tangent space at each point of the manifold. Then putting \(Y = U = e_i\) in (3.4.4) and taking summation over \(i\), for \(1 \leq i \leq 3\), we get

\[(3.4.5) \quad [r - 6(\alpha^2 - \rho)]\eta(V) = 0.\]

This gives \(r = 6(\alpha^2 - \rho)\) (since \(\eta(V) \neq 0\)) which implies by lemma 3.3.1 that the manifold is of constant curvature. Hence we can state the following theorem:

**Theorem 3.4.1.** A 3–dimensional \((LCS)_n\)–manifold satisfying \(R(X,Y) \cdot S = 0\), is manifold of constant curvature.
3.5 3–dimensional ψ–recurrent \((LCS)_n\) manifold

An \((LCS)_n\) manifold is said to be ψ–recurrent \((LCS)_n\) manifold if there exists a non zero 1–form \(\psi\) such that

\[(3.5.1) \quad \psi^2 ((D_W R)(X,Y)Z) = A(W)R(X,Y)Z \]

for arbitrary vector fields \(X, Y, Z, W\).

Taking covariant differentiation on both the sides of equation (3.3.3), we get

\[(3.5.2) \quad (D_W R)(X,Y)Z = \left[\frac{dr(W)}{2} - 2(2 \alpha \rho - \beta) \eta(W)\right] \]

\[\left[ g(Y,Z)X - g(X,Z)Y \right] + \left[\frac{dr(W)}{2} - 3(2 \alpha \rho - \beta) \eta(W)\right] \]

\[\left[ g(Y,Z) \eta(X) - g(X,Z) \eta(Y) \xi + \eta(\eta(Z)X - \eta(X)\eta(Z)Y) \right] \]

\[+ \left[\frac{r}{2} - 3(\alpha^2 - \rho)\right] \left[ g(Y,Z) \eta(X) - g(X,Z) \eta(Y) \right](D_w \xi) \]

\[+ \left[\frac{r}{2} - 3(\alpha^2 - \rho)\right] \left[ g(Y,Z)(D_w \eta)(X) - g(X,Z)(D_w \eta)(Y)\right]\xi \]

\[+ \left[\frac{r}{2} - 3(\alpha^2 - \rho)\right] \left[ (D_w \eta)(Y)\eta(Z)X + \eta(Y)(D_w \eta)(Z)X \right. \]

\[- (D_w \eta)(X)\eta(Z)Y - \eta(X)(D_w \eta)(Z)Y]. \]

Taking \(X, Y, Z, W\) orthogonal to \(\xi\) and using (3.2.7), we get

\[(3.5.3) \quad (D_W R)(X,Y)Z = \frac{dr(W)}{2} \left[ g(Y,Z)X - g(X,Z)Y \right] \]

\[+ \left[\frac{r}{2} - 3(\alpha^2 - \rho)\right] \left[ g(Y,Z)(D_w \eta)(X) - g(X,Z)(D_w \eta)(Y)\right]\xi. \]

Applying \(\psi^2\) on both the sides of (3.5.3) and using (3.2.7), we get

\[(3.5.4) \quad \psi^2(D_W R)(X,Y)Z = \frac{dr(W)}{2} \left[ g(Y,Z)X - g(X,Z)Y \right]. \]

By (3.5.1), equation (3.5.4) reduces to

\[(3.5.5) \quad A(W)R(X,Y)Z = \frac{dr(W)}{2} \left[ g(Y,Z)X - g(X,Z)Y \right]. \]
Putting \( W = \{e_i\} \), where \( \{e_i\} = 1, 2, 3 \), is an orthonormal basis of the tangent space of any point of the manifold and taking summation over \( i \), \( 1 \leq i \leq 3 \), we get

\[
R(X, Y)Z = \lambda [g(Y, Z)X - g(X, Z)Y],
\]

where \( \lambda = \frac{dr(e_i)}{2A(e_i)} \) is a scalar, since \( A \) is a non zero 1–form. Then by Schur’s theorem, \( \lambda \) will be a constant on the manifold. Therefore \( M^3 \) is of constant curvature \( \lambda \).

Hence we can state the following theorem:

**Theorem 3.5.1.** A 3–dimensional \( \psi \)–recurrent \( (LCS)_n \)–manifold is of constant curvature.

### 3.6 3–dimensional locally \( \psi \) concircularly symmetric \( (LCS)_n \) manifold

An \( (LCS)_n \) manifold is said to be locally \( \psi \) concircularly symmetric if the Concircular curvature tensor \( C \) satisfies

\[
\psi^2 \left( (D_W)C(X, Y)Z \right) = 0
\]

for arbitrary vector fields \( X, Y, Z, W \) orthonormal to \( \xi \).

Using (3.3.3) in (3.1.1) in a 3–dimensional \( (LCS)_n \) manifold, the Concircular curvature tensor \( C \) is given by

\[
C(X, Y)Z = \left[ \frac{r}{3} - 2(\alpha^2 - \rho) \right] [g(Y, Z)X - g(X, Z)Y] \\
+ \left[ \frac{r}{2} - 3(\alpha^2 - \rho) \right] [g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi] \\
+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y.
\]

Taking covariant differentiation on both sides of the equation (3.6.2), we get
(3.6.3) \( (D_W C)(X,Y)Z = \left[ \frac{dr(W)}{3} - 2(2 \alpha \rho - \beta)\eta(W) \right] [g(Y,Z)X - g(X,Z)Y] + \left[ \frac{dr(W)}{2} - 3(2 \alpha \rho - \beta)\eta(W) \right] [g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)] + \left[ \frac{r}{2} - 3(\alpha^2 - \rho) \right] [g(Y,Z)(D_W \eta)(X)\xi - g(X,Z)(D_W \eta)(Y)\xi - g(X,Z)\eta(Y)]

\[ (D_W \xi) + (D_W \eta)(Y)\eta(Z)X + \eta(Y)(D_W \eta)(Z)X - (D_W \eta)(X) \eta(Z)Y - \eta(X)(D_W \eta)(Z)Y. \]

Now, assume that \( X, Y \) and \( Z \) are horizontal vector fields. So the equation (3.6.3) becomes

(3.6.4) \( (D_W C)(X,Y)Z = \frac{dr(W)}{3} [g(Y,Z)X - g(X,Z)Y] + \left[ \frac{r}{2} - 3(\alpha^2 - \rho) \right] [g(Y,Z)(D_W \eta)(X)\xi - g(X,Z)(D_W \eta)(Y)\xi - g(X,Z)\eta(Y)]. \)

Applying \( \psi^2 \) on both the sides of (3.6.4) and using (3.2.7), we get

(3.6.5) \( \psi^2 (D_W C)(X,Y)Z = \frac{dr(W)}{3} [g(Y,Z)X - g(X,Z)Y + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi]. \)

Again, taking \( X, Y, Z, W \) orthogonal to \( \xi \), we get

(3.6.6) \( \psi^2 (D_W C)(X,Y)Z = \frac{dr(W)}{3} [g(Y,Z)X - g(X,Z)Y]. \)

Hence we can state the following theorem:

**Theorem 3.6.1.** A 3–dimensional \( (LCS)_n \) manifold is locally \( \psi \)–concircularly symmetric if and only if the scalar curvature \( r \) is constant.
REFERENCES


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