CHAPTER 2

$m$–PROJECTIVE CURVATURE TENSOR ON A LORENTZIAN PARA–SASAKIAN MANIFOLD

2.1 Introduction

The notation of Lorentzian para contact manifold was introduced by K. Matsumoto. The properties of Lorentzian para contact manifolds and their different classes, viz LP–Sasakian and LSP–Sasakian manifolds have been studied by several authors. In [18], M.Tarafdar and A. Bhattacharya proved that a LP–Sasakian manifold with conformally flat curvature tensor is locally isomorphic with a unit sphere $S^n(1)$. Further, they obtained that a LP–Sasakian manifold with $R(X,Y) \cdot C = 0$ is locally isomorphic with a unit sphere $S^n(1)$, where $C$ is the conformal curvature tensor of type $(1,3)$ and $R(X,Y)$ denotes the derivation of tensor of tensor algebra at each point of the tangent space. J.P. Singh [13] proved that an $m$–projectively flat para–Sasakian manifold is an Einstein manifold $R(\xi,X) \cdot W = 0$ holds and then it is locally isomorphic with a unit sphere $H^n(1)$. Also, an $n$–dimensional $\eta$–Einstien Para Sasakian manifold satisfying $W(\xi,X) \cdot R = 0$ if and only if either manifold is locally isomorphic to the hyperbolic space $H^n(–1)$ or the scalar curvature tensor $r$ of the manifold is $–n(n−1)$. S.K. Chaubey [3], studied the properties of $m$–projective curvature tensor in LP–Sasakian, Einstien LP–Sasakian and $\eta$–
Einstien LP–Sasakian manifold. LP–Sasakian manifolds have also studied by Matsumoto and Mihai [7], Takashashi [14], De, Matsumoto and Shaikh [4], Prasad [11], Venkatesha and Bagewadi [19].

LP–Sasakian manifolds equipped with $m$–projective curvature tensor. Section 2.1 is introductory. Section 2.2 deals with brief account of Lorentzian para–Sasakian manifolds. In section 2.3, we proved that an $m$–projectively flat LP–Sasakian manifold is an Einstien manifold and an LP–Sasakian manifold satisfying $(C^1_{\tau}W)(Y,Z) = 0$ is of constant curvature is $m$–projectively flat. In section 2.4, we proved that an Einstien LP–Sasakian manifold is $m$–projectively conservative if and only if the scalar curvature is constant. In section 2.5, we proved that an $n$–dimensional $\varphi$–$m$–projectively flat LP–Sasakian manifold is an $\eta$–Einstien manifold. In last, we proved that an $n$–dimensional quasi $m$–projectively flat LP–Sasakian manifold $M^n$ is locally isomeric to the unit sphere if and only if $M^n$ is locally isomeric to the unit sphere $S^n(1)$ if and only if $M^n$ is $m$–projectively flat.

2.2 Preliminaries

An $n$–dimensional differentiable manifold $M^n$ is a Lorentzian para–Sasakian (LP–Sasakian) manifold, if it admits a $(1,1)$–tensor field $\phi$, vector field $\xi$, 1–form $\eta$ and Lorentzian metric $g$ which satisfy:

(2.2.1) $\phi^2X = X + \eta(X)\xi$,
(2.2.2) $\eta(\xi) = -1$,
(2.2.3) $g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y)$,
(2.2.4) $g(X, \xi) = \eta(X)$,
(2.2.5) $(D_X\phi)(Y) = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi$. 
for arbitrary vector fields $X$ and $Y$. Where $D_X$ denote covariant differentiation with respect to $X$, (Matsumoto [6] and Matsumoto and Mihai [7]).

In an LP–Sasakian manifold $M^n$ with structure $(\phi, \xi, \eta, g)$, it is easily seen that

$$\phi \xi = 0, \quad \eta(\phi X) = 0, \quad \text{rank} \phi = (n - 1).$$

Let us put

$$F(X, Y) = g(\phi X, Y),$$

then, the tensor field $F$ is symmetric $(0, 2)$ tensor field.

$$F(X, Y) = F(Y, X),$$

and

$$F(X, Y) = (D_X \eta)(Y)$$

and

$$(D_X \eta)(Y) - (D_Y \eta)(X) = 0.$$

An LP–Sasakian manifold $M^n$ is said to be Einstien manifold if its Ricci tensor $S$ is of the form

$$S(X, Y) = Kg(X, Y),$$

where $K = (n - 1)$.

An LP–Sasakian manifold $M^n$ is said to be $\eta$–Einstien manifold if its Ricci tensor $S$ is of the form

$$S(X, Y) = \alpha g(X, Y) + \beta \eta(X)\eta(Y)$$

for any vector fields $X$ and $Y$, where $\alpha, \beta$ are the functions on $M^n$.

Let $M^n$ be an $n$–dimensional LP–Sasakian manifold with structure $(\varphi, \xi, \eta, g)$, then we have (Matsumoto and Mihai [7], Mihai, Shaikh and De [4]).
for any vector fields \( X, Y, Z \), where \( R(X, Y)Z \) is the Riemannian curvature tensor of type \((1, 3)\), \( S \) is a Ricci tensor of type \((0, 2)\), \( Q \) is Ricci tensor of type \((1, 1)\) and \( r \) is the scalar curvature.

\[
g(QX, Y) = S(X, Y)
\]

for all vector fields \( X, Y \).

\( m \)-projectively curvature tensor \( W \) on a Riemannian manifold \((M^n, g)\) \((n > 3)\) of type \((1, 3)\) is defined as follows (G.P. Pokhariyal and R.S. Mishra [10]).

\[
W(X, Y)Z = R(X, Y)Z - \frac{1}{2(n-1)} [ S(Y, Z)X - S(X, Z)Y \\
+ g(Y, Z)QX - g(X, Z)QY ].
\]

So that

\[
'W(X, Y, Z, U) \equiv g(W(X, Y)Z, U) = 'W(Z, U, X, Y).
\]

On an \( n \)-dimensional LP–Sasakian manifold, the concircular curvature tensor \( C \) is defined as

\[
C(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)} [ g(Y, Z)X - g(X, Z)Y ].
\]
Now in the view of \( S(X,Y) = \frac{r}{n} g(X,Y) \), (2.2.20) becomes

\[
W(X,Y)Z = C(X,Y)Z.
\]

Thus, in an Einstein LP–Sasakian manifold, \( m \)--projectively curvature tensor \( W \) and the concircular curvature tensor \( C \) coincides.

### 2.3 \( m \)--projectively Flat LP–Sasakian Manifold

In this section we assume that \( W(X,Y)Z = 0 \), then from (2.2.20) we get

\[
R(X,Y)Z = \frac{1}{2 (n-1)} [S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY].
\]

Contracting (2.3.1) with respect to \( X \), we get

\[
S(Y,Z) = \frac{r}{n} g(Y,Z).
\]

Hence we can state the following theorem:

**Theorem 2.3.1.** Let \( M^n \) be an \( n \)--dimensional \( m \)--projectively flat LP–Sasakian manifold, then \( M^n \) be an \( \eta \)--Einstein manifold.

Contracting (2.2.20) with respect to \( X \), we get

\[
(C^1_1 W)(Y,Z) = S(Y,Z) - \frac{1}{2 (n-1)} [n S(Y,Z) - S(X,Z)Y + r g(Y,Z) - g(X,Z)]
\]

(2.3.3)

\[
(C^1_1 W)(Y,Z) = \frac{n}{2 (n-1)} [S(Y,Z) - \frac{r}{n} g(Y,Z)],
\]

(2.3.4)

where \( (C^1_1 W)(Y,Z) \) is the contraction of \( W(X,Y)Z \) with respect to \( X \).

If

\[
(C^1_1 W)(Y,Z) = 0,
\]

we get
(2.3.5) \[ S(Y, Z) = \frac{r}{n}g(Y, Z). \]

Using (2.3.5) and (2.3.1), we get
(2.3.6) \[ 'R(X, Y, Z, W) = \frac{r}{n(n-1)} [g(Y, Z)g(X, W) \]
\[- g(X, Z)g(Y, W)]. \]

Hence we can state the following theorem:

**Theorem 2.3.2.** An $m$–projectively flat LP–Sasakian manifold satisfying \((C_1^1 W)(Y, Z) = 0\), is a manifold of constant curvature.

Using (2.3.5) and (2.3.6) in (2.2.18), we get

\[ W(X, Y)Z = 0, \]

i.e., the manifold \(M^n\) is $m$–projectively flat. Hence we can state the following theorem:

**Theorem 2.3.3.** An LP–Sasakian manifold \((M^n, g) (n > 3)\) satisfying \((C_1^1 W)(Y, Z) = 0\) of constant curvature, is $m$–projectively flat.

### 2.4 Einstein LP–Sasakian Manifold Satisfying \((div W)(X, Y)Z = 0\).

A manifold \((M^n, g) (n > 3)\) is called $m$–projectively conservative if (Hicks N.J. [5]),

(2.4.1) \[ div(W) = 0, \]

where ‘\(div\)’ denotes divergence.

Now differentiating (2.2.20) conveniently, we get
(2.4.2) \[ (D_u W)(X, Y)Z = (D_u R)(X, Y)Z - \frac{1}{2(n-1)} [(D_u S)(Y, Z)X \]
\[- (D_u S)(X, Z)Y + g(Y, Z)(D_w Q)X - g(X, Z)(D_w Q)Y] \]
which gives on contraction.

\[(2.4.3) \quad \text{div}(W)(X,Y) = \text{div}(R)(X,Y)Z - \frac{1}{2(n-1)} [(D_XS)(Y,Z) - (D_YS)(X,Z) + g(Y,Z)\text{div}(Q)X - g(X,Z)\text{div}(Q)Y].\]

But

\[\text{div}(Q) = \frac{1}{2} dr.\]

Using in (2.4.3), we get

\[(2.4.4) \quad \text{div}(W)(X,Y) = \text{div}(R)(X,Y)Z - \frac{1}{2(n-1)} [(D_XS)(Y,Z) - (D_YS)(X,Z) + \frac{1}{2} g(Y,Z)dr(X) - \frac{1}{2} g(X,Z)dr(Y)].\]

Now, from (Eisenhart L.P. (1926)), we have

\[(2.4.5) \quad \text{div}(R)(X,Y)Z = (D_XS)(Y,Z) - (D_YS)(X,Z).\]

Using (2.4.5) in (2.4.4), we get

\[(2.4.6) \quad \text{div}(W)(X,Y)Z = \frac{(2n-3)}{2(n-1)} (D_XS)(Y,Z) - (D_YS)(X,Z)
- \frac{1}{4(n-1)} [ g(Y,Z)dr(X) - g(X,Z)dr(Y)].\]

If LP–Sasakian manifold is an Einstei manifold, then from (2.1.12) and (2.4.5), we obtain

\[(2.4.7) \quad \text{div}(R)(X,Y)Z = 0.\]

From (2.4.6) and (2.4.7), we get

\[(2.4.8) \quad \text{div}(W)(X,Y)Z = - \frac{1}{4(n-1)} [ g(Y,Z)dr(X) - g(X,Z)dr(Y)].\]

From (2.4.1) and (2.4.8), we get

\[ [g(Y,Z)dr(X) - g(X,Z)dr(Y)] = 0\]
which shows that $r$ is constant. Again if $r$ is constant then from (2.4.8), we get
\[ \text{div}(W)(X, Y)Z = 0. \]

Hence we can state the following theorem:

**Theorem 2.4.1.** An Einstein LP–Sasakian manifold $(M^n, g)$ ($n > 3$) is $m$–projectively conservative if and only if the scalar curvature is constant.

### 2.5 $\varphi$–$m$–Projectively Flat LP–Sasakian Manifold

A differentiable manifold $(M^n, g)$, $n > 3$, satisfying the condition
\[ \varphi^2 \mathcal{W}(\varphi X, \varphi Y) \varphi Z = 0, \]

is called $\varphi$–$m$–projectively flat LP–Sasakian manifold.

Suppose that $(M^n, g)$, $n > 3$ is a $\varphi$–$m$–projectively flat LP–Sasakian manifold. It is easy to see that
\[ \varphi^2 \mathcal{W}(\varphi X, \varphi Y) \varphi Z = 0, \]
holds if and only if
\[ g(\mathcal{W}(\varphi X, \varphi Y) \varphi Z, \varphi W) = 0 \]
for any vector fields $X, Y, Z, W$.

By the use of (2.2.18), $\varphi$–$m$–projectively flat means
\[ 'R(\varphi X, \varphi Y, \varphi Z, \varphi W) = \frac{1}{2(n-1)} [S(\varphi Y, \varphi Z)g(\varphi X, \varphi W) \]
\[ -S(\varphi X, \varphi Z)g(\varphi Y, \varphi W) + g(\varphi Y, \varphi Z)S(\varphi X, \varphi W) \]
\[ -g(\varphi X, \varphi Z)S(\varphi Y, \varphi W)], \]

where $'R(X, Y, Z, W) = g(R(X, Y)Z, W)$. 

Let \( \{e_1, e_2, \ldots, e_{n-1}, \xi\} \) be a local orthonormal basis of vector fields in \( M^n \) by using the fact that \( \{\varphi e_1, \varphi e_2, \ldots, \varphi e_{n-1}, \varphi \xi\} \) is also orthonormal basis, if we put \( X = W = e_i \) in (2.5.2) and sum up with respect to \( i \), we have

\[
\sum_{i=1}^{n-1} R(\varphi e_i, \varphi Y, \varphi Z, \varphi e_i) = \frac{1}{2(n-1)} \sum_{i=1}^{n-1} [S(\varphi Y, \varphi Z) \\
g(\varphi e_i, \varphi e_i) - S(\varphi e_i, \varphi Z)g(\varphi Y, \varphi e_i) + g(\varphi Y, \varphi Z) \\
S(\varphi e_i, \varphi e_i) - g(\varphi e_i, \varphi Z)S(\varphi Y, \varphi e_i)].
\]

On an LP–Sasakian manifold, we have

\[
\sum_{i=1}^{n-1} \Gamma(\varphi e_i, \varphi Y, \varphi Z, \varphi e_i) = S(\varphi Y, \varphi Z) + g(\varphi Y, \varphi Z),
\]

\[
\sum_{i=1}^{n-1} S(\varphi e_i, \varphi e_i) = r + (n - 1),
\]

\[
\sum_{i=1}^{n-1} g(\varphi e_i, \varphi Z)S(\varphi Y, \varphi e_i) = S(\varphi Y, \varphi Z),
\]

\[
\sum_{i=1}^{n-1} g(\varphi e_i, \varphi e_i) = (n + 1),
\]

\[
\sum_{i=1}^{n-1} g(\varphi e_i, \varphi Z)g(\varphi Y, \varphi e_i) = g(\varphi Y, \varphi Z).
\]

So, by the virtue of (2.5.4)–(2.5.4), the equation (2.5.3) takes the form

\[
S(\varphi Y, \varphi Z) = \left[\frac{r}{n-1} - 1\right]g(\varphi Y, \varphi Z).
\]

By making the use of (2.2.3) and (2.2.17) in (2.5.9), we get

\[
S(Y, Z) = \left[\frac{r}{n-1} - 1\right]g(Y, Z) + \left[\frac{r}{n-1} - n\right]\eta(Y)\eta(Z).
\]

Hence we can state the following theorem:

**Theorem 2.5.1.** Let \( M^n \) be an \( n \)–dimensional \( n > 3, \varphi \)–\( m \)–projectively flat LP–Sasakian manifold and then \( M^n \) be an \( \eta \)–Einstien manifold with constants

\[
\alpha = \left[\frac{r}{n-1} - 1\right] \text{ and } \beta = \left[\frac{r}{n-1} - n\right].
\]

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2.6 Quasi $m$–Projectively Flat LP–Sasakian Manifold

An LP–Sasakian manifold $M^n$ is said to be quasi $m$–projectively flat, if

$$g(W(X,Y)Z,\varphi U) = 0$$

for any vector fields $X, Y, Z, U$.

From (2.2.18), we get

$$g(W(X,Y)Z,\varphi U) = g(R(X,Y)Z,\varphi U)$$

$$- \frac{1}{2(n-1)} [S(Y,Z)g(X,\varphi U) - S(X,Z)g(Y,\varphi U) + g(Y,Z)S(X,\varphi U) - g(X,Z)S(Y,\varphi U)].$$

Let $\{e_1, e_2, \ldots, e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in $M^n$. By using the fact that $\{\varphi e_1, \varphi e_2, \ldots, \varphi e_{n-1}, \xi\}$ is also orthonormal basis, if we put $X = \varphi e_i, U = e_i$ in (2.5.2) and sum up with respect to $i$, then we have

$$\sum_{i=1}^{n-1} g(W(\varphi e_i,Y)Z,\varphi e_i) = \sum_{i=1}^{n-1} g(R(\varphi e_i,Y)Z,\varphi e_i)$$

$$- \frac{1}{2(n-1)} \sum_{i=1}^{n-1} [S(Y,Z)g(\varphi e_i,\varphi e_i) - S(\varphi e_i,Z)g(Y,\varphi e_i)$$

$$+ g(Y,Z)S(\varphi e_i,\varphi e_i) - g(\varphi e_i,Z)S(Y,\varphi e_i)].$$

On an LP–Sasakian manifold by straightforward calculation, we get

$$\sum_{i=1}^{n-1} R(e_i,Y,Z,e_i) = \sum_{i=1}^{n-1} R(\varphi e_i,Y,Z,\varphi e_i)$$

$$= S(Y,Z) + g(\varphi Y,\varphi Z),$$

$$\sum_{i=1}^{n-1} S(\varphi e_i,Z)g(Y,\varphi e_i)$$

$$= S(Y,Z) - (n-1)\eta(Y)\eta(Z).$$
Using (2.5.4), (2.5.7), (2.6.4), (2.6.5) in (2.6.3), we get

\[ \sum_{i=1}^{n-1} g(W(\varphi e_i, Y)Z, \varphi e_i) = S(Y, Z) + g(\varphi Y, \varphi Z) - \frac{1}{2(n-1)}[(n - 1)S(Y, Z) + (r + n - 1)g(Y, Z) + 2(n - 1)\eta(Y)\eta(Z)]. \]

Using (2.2.3) in (2.6.6), we get

\[ \sum_{i=1}^{n-1} g(W(\varphi e_i, Y)Z, \varphi e_i) = \frac{1}{2} \left[ S(Y, Z) - \left( \frac{r}{n} - 1 \right) g(Y, Z) \right]. \]

If \( M^n \) is quasi \( m \)-projectively flat, then (2.6.7) reduces to

\[ S(Y, Z) = \left( \frac{r}{n} - 1 \right) g(Y, Z). \]

Putting \( Z = \xi \) in (2.6.8) and then using (2.2.6) & (2.2.16), we get

\[ r = n(n - 1). \]

Using (2.6.9) in (2.6.8), we get

\[ S(Y, Z) = (n - 1)g(Y, Z), \]

i.e., \( M^n \) is an Einstein manifold.

Now, using (2.6.10) in (2.2.18), we get

\[ W(X, Y)Z = R(X, Y)Z - [g(Y, Z)X - g(X, Z)Y]. \]

If LP–Sasakian manifold is \( m \)-projectively flat, then

\[ R(X, Y)Z = [g(Y, Z)X - g(X, Z)Y]. \]

Hence we can state the following theorem:

**Theorem 2.6.2.** In a quasi \( m \)-projectively flat LP–Sasakian manifold \( M^n \) is locally isometric to the unit sphere \( S^n(1) \) iff \( M^n \) is \( m \)-projectively flat.
REFERENCES


