CHAPTER 1

INTRODUCTION

The purpose of this chapter is to introduce basic concept, preliminary notions and some fundamental results which we need in the development of the present thesis. Thus we have given a brief resume of the history of Riemannian geometry and some of the basic concepts in geometry of Riemannian manifold, almost contact structure, Lorentzian para–Sasakian manifold, Lorentzian $\alpha$–Sasakian manifold, almost complex manifold, their allied structures, curvatures and the geometry of these manifolds. Much though all these literatures are readily available in review articles ([19], [23], [24], ) and some in standard books e.g., N. J. Hicks [12] and R. S. Mishra ([20], [21]) S. Kobayashi [17] etc., nevertheless, we have collected them here to fix up our terminology and for ready references.

In 1854 at Gottingen (Germany), G. Riemann (1826–1866) discussed the problem how to define an $n$–dimensional space, how to equip such space with metric, definition of curvature and the relationship of such geometry to the world we live in. He introduced the concept of differential geometry of more than 3–dimension. Soon after, the publication of his work in 1868, several mathematicians like Beltrami (1869), Lipschitz (1869), Christoffel (1869) and other enriched the ideas of Riemann by introducing Baltrani parameters, covariant differentiation, Gauss equation and Christoffel symbols. J. A. Schouten (1930) and D. Van Dantzing (1931) were the first to transfer
the results in differential geometry of Riemannian spaces with complex structure. This opened an era of complex manifolds.

Ehresmann (1947) defined an almost complex manifold as an even dimensional differentiable manifold $M^n (n = 2m)$ of differentiability class $C^{r+1}$ such that there exist a tensor field $f$ of type $(1, 1)$ and differentiability class $C^r$, satisfying the equation $f^2 + I_n = 0$, where $I_n$ being unit tensor field.

Sasaki (1960) defined and studied almost contact structure and its integrability conditions. In 1970, Yano and Okumara, Vanzura (1972) defined and studied an almost $r$–contact structure manifold.

The hyperbolic contact manifold was given by Sasaki (1960) and A. Al–Aqueel, A. Hamoli, M. D. Upadhayay (1987) have studied some properties of such manifold. Other type of structures on differentiable manifolds were defined and studied by Helgason (1962), Yano (1963), Duggal (1964), Yano and Ishihara (1965), Upadhayay and Dube (1973), Kon (1973), Anur and Hegde (1974), Sinha and Singh (1975). Hussain and Sharafuddin (1977), Upadhayay and Agrawal (1981) and several others.

Mishra (1981) has considered differentiable manifolds when it is (i) even dimensional, (ii) odd dimensional, (iii) even or odd dimensional. He obtained different structures on a manifold using algebraic methods.

1.1 Differentiable Manifold

Let $M^n$ be a set. An $n$–coordinate pair $(\phi, U)$ on $M^n$ is a pair consisting of a subset $U$ of $M^n$ and a one to one map $\phi$ of $U$ onto an open set in $R^n$, where $R^n$ is the product space of ordered $n$–tuples of real numbers. An $n$–coordinate pair $(\phi, U)$ is called $C^r$ $(C^\infty)$ related to another $n$–coordinate pair $(\theta, V)$ if the mappings $\theta \circ \phi^{-1}$ and $\phi \circ \theta^{-1}$ are $C^r$ $(C^\infty)$ maps. A $C^r$ $(C^\infty)$ $n$–sub atlas on $M^n$ is the collection of $C^r$ $(C^\infty)$ related $n$–coordinate pairs.
such that each pair is \( C^r(C^\infty) \) related to every other member of the collection and union of the set \( U_h \) is \( M^n \), i.e., \( \cup U_h = M^n \). A maximal collection of \( C^r(C^\infty) \) \( n \)-sub atlas is called \( C^r(C^\infty) \) \( n \)-atlas on \( M^n \). The manifold \( M^n \) is the set \( M^n \) together with \( C^r(C^\infty) \) \( n \)-atlas on \( M^n \). An \( n \)-dimensional \( C^r \) manifold for which \( r \neq 0 \) is called a differentiable manifold or smooth manifold of class \( C^r \). If \( r = 0 \), \( M^n \) is called a topological manifold.

Each \( n \)-coordinate pair \( (\phi, U) \) on \( M^n \) induces a set of \( n \) real valued functions on \( M^n \) defined by

\[
X^i = u_i \circ \phi,
\]

where \( i = 1,2,3, \ldots, n \) and \( u_1, u_2, u_3, \ldots, u_n \) are slot functions defined on \( R^n \). The functions \( X^1, X^2, \ldots, X^n \) are called coordinate system or coordinate functions and \( U \) is called domain of the coordinate system.

Let \( M^n \) be a \( C^\infty \) manifold. An open set in \( M^n \) is a subset \( V \) such that \( \phi(U \cap V) \) is open in Euclidian space \( R^n \) for every \( n \)-coordinate pair \( (\phi, U) \) in \( M^n \). It can be easily shown that the set \( M^n \) is a topological space with the above definition of open sets. If \( m \) is a point in \( M^n \), a neighborhood of \( m \) is any open set containing \( m \).

### 1.2 Vectors and Vector Fields

Let \( M^n \) be a \( C^\infty \) manifold and let \( C^\infty(m) \) denote the set of all real valued functions that are \( C^\infty \) on some neighbourhood of \( m \) in \( M^n \). A tangent vector at \( M^n \) is a real valued function \( X_m \) on \( C^\infty(m) \) which satisfies the following conditions:

\[
\begin{align*}
(1.2.1) \quad & (i) \quad X_m(f + g) = X_m f + X_m g, \\
(1.2.2) \quad & (ii) \quad X_m(af) = aX_m f, \\
(1.2.3) \quad & (iii) \quad X_m(fg) = f(m)X_m g + g(m)X_m f,
\end{align*}
\]
where \( f \) and \( g \) are \( C^\infty \)-functions in a neighbourhood of \( m \) and ‘\( a \)’ is a real number.

The tangent space of \( M^n \) at \( m \in M^n \) is denoted by \( T_m(M^n) \) and is the set of all tangent vectors at \( m \in M^n \). It is a vector space over the field of real numbers under the operations of vector addition and scalar multiplication given below

\[
\begin{align*}
(1.2.4) \quad (i) \quad (X_m + Y_m)f &= X_m f + Y_m f, \\
(ii) \quad (aX_m)f &= aX_m f
\end{align*}
\]

for \( X_m, Y_m \in T_m(M^n) \), where \( f \) is \( C^\infty \) function and ‘\( a \)’ is a real number.

A vector field \( X \) on an open set \( U \) in \( M^n \) is mapping that assigns to each point \( m \) in \( U \) a vector \( X_m \in T_m(M^n) \). The vector field \( X \) is \( C^\infty \) on \( U \), if \( f \) is open and for each real valued \( C^\infty \) function \( f \) on \( U \), the function \( Xf \) is \( C^\infty \) on \( U \cap V \). For a function \( f \) which is \( C^\infty \) on \( V \) and \( X \), a \( C^\infty \) field on \( U \), we define a vector \( fX \) on \( U \cap V \) as

\[
(1.2.5) \quad (fX)_m = f(m)X_m.
\]

It is easy to show that \( fX \) is again a \( C^\infty \) vector field. The set of all \( C^\infty \) vector fields on \( M^n \), therefore, forms a module over the ring of real valued \( C^\infty \) functions on \( M^n \). Let us denote this by \( \chi(M^n) \).

If \( X, Y \in \chi(M^n) \), their Lie bracket \( [X, Y] \) is defined as

\[
(1.2.6) \quad [X, Y]_m f = X_m(Yf) - Y_m(Xf).
\]

If \( f \) and \( g \) are \( C^\infty \) functions, then

\[
(1.2.7) \quad [X, Y](f + g) = [X, Y]f + [X, Y]g
\]

and

\[
(1.2.8) \quad [X, Y](af) = a[X, Y]f,
\]

where ‘\( a \)’ is a real number.
Further,

\[(1.2.9)\quad [X,Y](fg) = f[X,Y]g + g[X,Y]f.\]

Thus, \([X,Y]\) is a vector field.

\([X,Y]\) is skew symmetric in \(X, Y\). That is

\[(1.2.10)\quad [X,Y] = -[Y,X].\]

Also,

\[(1.2.11)\quad [X,X] = 0.\]

Again,

\[(1.2.12)\quad [X + Y, Z] = [X, Z] + [Y, Z]\]

and

\[(1.2.13)\quad [fX, gY] = f(Xg)Y - g( Yf )X + f g[X,Y]\]

for all \(X, Y, Z \in \chi(M^n)\).

The Lie bracket satisfies the Jacobi identity

\[(1.2.14)\quad [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.\]

### 1.3 Tensors and Forms

Let \(M^n\) be an \(n\)–dimensional \(C^\infty\)–manifold and let \(m\) be a point of \(M^n\).

Let \(T_m(M^n)\) be the tangent space of \(M^n\) at \(m\). For an integer \(p > 0\), a \(p\)–co tensor at a point \(m\) is a real valued \(p\)–linear function on \(T_m(M^n) \times T_m(M^n) \times \ldots \ldots \times T_m(M^n)\) \((p\)–times). Thus in particular, \(\alpha\) is a 2–cotensor at \(m\) if

\[(1.3.1)\]

(i) \(\alpha(X,Y)\) is real,

(ii) \(\alpha(X + Y, Z) = \alpha(X,Z) + \alpha(Y,Z),\)

(iii) \(\alpha(X, Y + Z) = \alpha(X,Y) + \alpha(X,Z),\)
(iv) \( \alpha(aX,Y) = \alpha(X,aY) = \alpha(a(X,Y)) \)

for all \( X,Y,Z \in T_m(M^n) \) and \('a'\) is a real number.

Let \( T_m(M^n)^* \) be the dual space of \( T_m(M^n) \). Thus \( T_m(M^n)^* \) is the set of all linear functions on \( T_m(M^n) \) into the set of real numbers. An element of \( T_m(M^n)^* \) is called a covariant vector or \( 1\)–form at the point \( m \). For \( p > 0 \), a \( p\)–contra variant tensor or a tensor of type \((p,0)\) at \( m \) is a real valued \( p\)–linear function on \( T_m(M^n)^* \times T_m(M^n)^* \times \ldots \times T_m(M^n)^* \) \((p\)–times). In particular, \( T \) is a \( 2\)–contra tensor at \( m \) if

(1.3.2) (i) \( T(\alpha,\beta) \) is real,

(ii) \( T(\alpha + \beta,\gamma) = T(\alpha,\gamma) + T(\beta,\gamma) \),

(iii) \( T(\alpha,\beta + \gamma) = T(\alpha,\beta) + T(\alpha,\gamma) \),

(iv) \( T(a\alpha,\beta) = T(\alpha,a\beta) = aT(\alpha,\beta) \),

where \( \alpha,\beta,\gamma \in T_m(M^n)^* \) and \('a'\) is real number. This process can be generalized for the tensor of type \((r,s)\).

The set of \( p\)–co tensor at \( m \) is denoted by \( T_m^{0,p}(M^n) \). It is a vector space over the field of real numbers. Also the natural vector space formed by \( p\)–contra tensors is denoted by \( T_m^{p,0}(M^n) \).

Finally, for \((p,q) > 0\) \( p\)–co and \( q\)–contra tensor at \( m \) is a real valued \((p + q)\)–linear function on \( T_m(M^n)^* \times T_m(M^n)^* \times \ldots \times T_m(M^n)^* \) \((p\)–times) \((T_m(M^n)^* \times T_m(M^n)^* \times \ldots \times T_m(M^n)^*)\) \((q\)–times) and the vector space of such tensors is denoted by \( T_m^{q,p}(M^n) \). If \( p > 0 \) and \( q > 0 \), elements of \( T_m^{q,p}(M^n) \) are called mixed tensors. A tensor of type \((0,1)\) is an element of \( T_m(M^n)^* \) and is also called \( 1\)–form at \( m \). Similarly a vector at \( m \) is a \( 1\)–contra tensor at \( m \).

A tensor is said to be symmetric if and only if its value remains the same for all possible permutations of its arguments. A tensor is skew–
symmetric or alternating if and only if its value after any permutation of its argument is the product of its value before the permutation and the sign of the permutation. For \( p > 0 \), a \( p \)-form at \( m \) is an alternating \( p \)-co tensor at \( m \) and the set of \( p \)-forms at \( m \) is denoted by \( F^p(T_m(M^n)) \).

### 1.4 Tensor Fields and Tensor Product

A tensor field of type \((r, s)\) is \(r\)-contra variant and \(s\)-covariant tensor field on an open set \( A \) and is defined to be the mapping that assigns to each point \( m \) in \( A \), a tensor of type \((r, s)\) at that point. The set of all tensor fields of type \((r, s)\) on \( A \) forms a vector space under usual vector addition and scalar multiplication and this vector space is denoted by \( T^{r,s}_m(M^n) \).

A covariant vector field on a set \( A \) is called of class \( C^\infty \) if

1. \( A \) is open,
2. \( \bar{\omega}(X) \) is a \( C^\infty \) function on \( A \) for all \( C^\infty \) contravariant vector fields on \( A \).

A \( 0 \)-form on an open set \( A \) is an element of \( F^0(A) \) and \( 1 \)-form on \( A \) is a \( C^\infty \) covariant vector field on \( A \). A \( p \)-form on \( A \) is a skew–symmetric \( p \)-covariant tensor field and the set of all \( p \)-forms on \( A \) is denoted by \( F^p(A) \).

If \( \alpha \in T^{0,p}_m(A) \) and \( \beta \in T^{0,q}_m(A) \), then the tensor product of covariant tensor \( \alpha \) and \( \beta \) denoted by \( \alpha \otimes \beta \) is defined to be an element in \( T^{0,p+q}_m(A) \) given by

\[
(\alpha \otimes \beta)(X_1, X_2, \ldots, X_{p+q}) = \alpha(X_1, X_2, \ldots, X_p)\beta(X_{p+1}, X_{p+2}, \ldots, X_{p+q})
\]

for all vector field \( X_1, X_2, \ldots, X_{p+q} \) on \( A \).

The operation of tensor product satisfies the following relations:
where \( b \) is a real number and \( \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \) are co–tensors.

However,
\[
\alpha \otimes \beta \neq \beta \otimes \alpha,
\]
in general, but
\[
(\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma).
\]

Thus, tensor product \( \alpha \otimes \beta \) is bilinear and associative but not symmetric in general. The tensor product of contra variant tensor or mixed tensor can also be defined analogously.

1.5 **Riemannian Manifold and Connections**

A Riemannian manifold is a \( C^\infty \) manifold \( M^n \) on which one has singled out a \( C^\infty \) real valued, bilinear, symmetric and positive definite metric \( g \) for each pair of tangent vectors at all points. A metric tensor \( g \) is called bilinear if it satisfies:

\[
(1.5.1) \quad (i) \quad g(X + Y, Z) = g(X, Z) + g(Y, Z),
\]

\[
(1.5.2) \quad (ii) \quad g(X, Y + Z) = g(X, Y) + g(X, Z),
\]

\[
(1.5.3) \quad (iii) \quad g(aX, Y) = g(X, aY) = ag(X, Y)
\]

for all \( X, Y, Z \) tangent to \( M^n \) and \( 'a' \) in \( R \).

The metric \( 'g' \) is called symmetric if it satisfies
\[
(1.5.4) \quad g(X, Y) = g(Y, X).
\]
If $g(X,Y) > 0$ for all $X \neq 0$ and $g(X,X) = 0 \implies X = 0$, the metric ‘$g$’ is called positive definite.

Let $M^n$ be a $C^\infty$ manifold, a connection $\nabla$ on $M^n$ is an operator that assigns to each pair of vector fields $X, Y$ on $A$, a $C^\infty$ vector field $\nabla_X Y$ on $A$, such that

$$\nabla: (X, Y) \to \nabla_X Y.$$ 

Let $Z$ be a $C^\infty$ vector field on $A$. Then for each real valued $C^\infty$ function on $A$, $\nabla$ satisfies the following properties:

(1.5.5) (i) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z,$

(1.5.6) (ii) $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z,$

(1.5.7) (iii) $\nabla_{fX} Y = f \nabla_X Y,$

(1.5.8) (iv) $\nabla_X (fY) = (Xf)Y + f \nabla_X Y.$

The connection $\nabla$ is said to be the Riemannian if

(1.5.8) $\nabla_X (g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

and

(1.5.10) $\nabla_X Y - \nabla_Y X = [X,Y],$

where ‘$g$’ is metric tensor on the manifold $M^n$ and $X, Y, Z$ are $C^\infty$ vector fields with common domain.

The Riemannian connection $\nabla$ is said to be metric connection if

(1.5.11) $\nabla_X g(Y,Z) = 0.$

The Riemannian connection $\nabla$ is said to be non–metric connection if

(1.5.12) $\nabla_X g(Y,Z) \neq 0.$
The torsion tensor of a connection $\nabla$ is a vector valued tensor $T$ that assigns to each pair of $C^\infty$ vector fields $X$ and $Y$ with common domain $A$, a $C^\infty$ vector field $T(X,Y)$ with the domain $A$ and is given by

\begin{equation}
T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],
\end{equation}

where $[X,Y]$ is the Lie bracket and is defined as

\begin{equation}
[X,Y] = XY - YX.
\end{equation}

The above equation shows that $T(X,Y)$ is skew symmetric. If the torsion tensor further satisfies

\begin{equation}
T(X,Y) = u(Y)X - u(X)Y,
\end{equation}

where $u$ is 1–form on $M^n$, the connection $\nabla$ is said to be semi–symmetric connection.

A semi–symmetric connection is said to be metric if it satisfies equation (1.5.11).

If torsion tensor $T$ satisfies

\begin{equation}
T(X,Y) = u(Y)FX - u(X)FY,
\end{equation}

then connection $\nabla$ is said to be quarter symmetric.

A quarter symmetric connections is said to be metric if equation (1.5.11) is also satisfied.

A linear connection $\tilde{\nabla}$ on $M^n$ given by

\begin{equation}
\tilde{\nabla}_X Y = \nabla_X Y - u(X)Y + g(X,Y)P,
\end{equation}

where $\tilde{\nabla}$ is the Riemannian connection and $P$ is a vector field such that

\begin{equation}
u(X) = g(P,X).
\end{equation}

Moreover, $\tilde{\nabla}$ is called quarter symmetric semi–metric connection if
(1.5.20) \[ (\nabla_X g)(Y, Z) = 2u(X)(FY, Z) - u(Y)g(FX, Z) - u(Z)g(FX, Y). \]

1.6 Semi–Riemannian Manifold

A Semi–Riemannian manifold is a \( C^\infty \) manifold \( M^n \) together with a bilinear, symmetric and non–singular metric \( g \) for each pair of tangent vector at all point. A metric tensor \( g \) is called bilinear if it satisfies the following:

\begin{align*}
(1.6.1) \quad & (i) \quad g(X + Y, Z) = g(X, Z) + g(Y, Z), \\
(1.6.2) \quad & g(X, Y + Z) = g(X, Y) + g(X, Z)
\end{align*}

and

\begin{equation}
(1.6.3) \quad g(aX, Y) = g(X, aY) = ag(X, Y),
\end{equation}

for all \( X, Y, Z \) tangent to \( M^n \) and ‘\( a \)’ in \( R \).

The metric \( g \) is called symmetric if it satisfies

\begin{equation}
(1.6.4) \quad (ii) \quad g(X, Y) = g(Y, X).
\end{equation}

The metric \( g \) is non singular if

\begin{equation}
(1.6.5) \quad (iii) \quad \text{If } g(X, Y) = 0 \text{ for all } X \text{ implies } Y = 0.
\end{equation}

1.7 The Lie Derivative

Let \( M^n \) be \( n–\)dimensional \( C^\infty \) manifold and let \( X \) be a \( C^\infty \) vector field on an open subset \( A \) of \( M^n \). An operator \( \mathcal{L}_X \), called the Lie derivative with respect to a vector field \( X \) which maps \( T^{r,s}(A) \) into itself, is defined as follows:

\begin{equation}
(1.7.1) \quad (i) \quad \text{If } f \text{ is a function on } A, \text{ i.e., } f \in F^0(A), \text{ then } \mathcal{L}_X f = Xf,
\end{equation}
(iii) If $Y$ is a vector field on $A$, i.e., $Y \in T^{1,0}(A)$, then
\[ \mathcal{L}_Yf = [X,Y], \]

(iii) For a 1–form $\omega$ on $A$ i.e., $\omega \in T^{0,1}(A)$, then
\[ (\mathcal{L}_X \omega)(Y) = X_\omega(Y) - \omega([X,Y]), \]

(iv) If $\theta \in T^{r,s}(A)$, $\omega_1, \omega_2, \omega_3, \ldots, \omega_r$ in $T^{1,0}(A)$
and $Y_1, Y_2, Y_3, \ldots, Y_s$ in $T^{0,1}(A)$, then
\[ (\mathcal{L}_X \theta)((\omega_1, \omega_2, \omega_3, \ldots, \omega_r, Y_1, Y_2, Y_3, \ldots, Y_s)) \]
\[ = \mathcal{L}_X\{\theta(\omega_1, \omega_2, \omega_3, \ldots, \omega_r, Y_1, Y_2, Y_3, \ldots, Y_s)\} \]
\[ -\theta\{\mathcal{L}_X \omega_1, \omega_2, \omega_3, \ldots, \omega_r, Y_1, Y_2, Y_3, \ldots, Y_s\} \]
\[ -\theta\{\omega_1, \mathcal{L}_X \omega_2, \omega_3, \ldots, \omega_r, Y_1, Y_2, Y_3, \ldots, Y_s\} \]
\[ -\theta\{\omega_1, \omega_2, \omega_3, \ldots, \omega_r, Y_1, Y_2, Y_3, \ldots, \mathcal{L}_X Y_s\}. \]

The Lie derivative preserves the forms and satisfies the following properties:

(1.7.2) (i) $\mathcal{L}_X(\theta_1 + \theta_2) = \mathcal{L}_X \theta_1 + \mathcal{L}_X \theta_2$,

(1.7.3) (ii) $\mathcal{L}_X(\theta_1 \otimes \theta_2) = (\mathcal{L}_X \theta_1) \otimes \theta_2 + \theta_1(\mathcal{L}_X \theta_2)$
and

(1.7.4) (iii) $\mathcal{L}_X(\alpha \wedge \beta) = (\mathcal{L}_X \alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_X \beta)$,

where $\theta_1$ and $\theta_2$ are $C^\infty$ tensor fields of the same type with common domain. Also $\alpha$ and $\beta$ are 1–forms with common domain in the manifold.
1.8 Exterior Derivative

For \( p \geq 0 \), we define the exterior derivative: \( F^p(A) \to F^{p+1}(A) \), where \( F^p(A) \) is the set of \( C^\infty \) \( p \)-form on an open set \( A \) in \( M^n \). If \( f \in F^0(A) \), i.e., \( f \) is a \( C^\infty \) function on \( A \), then we define a 1–form \( df \) by
\[
(1.8.1) \quad df(X) = 0
\]
for all \( C^\infty \) vectors fields on \( A \).

The operator \( d \) satisfies the following properties:

(i) \( d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2 \),

(ii) \( d(\alpha_1 \wedge \alpha_2) = d\alpha_1 \wedge \alpha_2 + (-1)^p (\alpha_1 \wedge d\alpha_2) \),

(1.8.2) (iii) \( d^2 f = 0 \),

where \( \omega_1, \omega_2, \alpha_1 \in F^p(A) \) and \( \alpha_2 \) is an arbitrary form on \( A \).

1.9 The General Covariant Derivative

Let \( D \) be the connection on a \( C^\infty \) manifold \( M^n \) and \( X \) be a \( C^\infty \) vector field on an open set \( A \). An operator, which map \( T^{r,s}(A) \) into itself, called general covariant derivative with respect to \( X \) is defined by
\[
(1.9.1) \quad (i) \quad \nabla_X f = Xf \quad \text{for } f \in F^0(A),
\]
\[
(1.9.2) \quad (ii) \quad (\nabla_X \alpha)Y = X(\alpha(Y)) - \alpha(\nabla_X Y),
\]
where \( \alpha \in T^{0,1}(A) \) and \( Y \) is a vector field on \( A \).

If \( \theta \in T^{r,s}(A) \), \( \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r \) in \( T^{1,0}(A) \) and \( Y_1, Y_2, Y_3, \ldots, Y_s \) in \( T^{0,1}(A) \), then \( \nabla_X \theta \) is defined by
(1.9.3) \[(\nabla_X \theta)((\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r, Y_1, Y_2, Y_3, \ldots, Y_s))\]
\[= \nabla_X \{\theta(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r, Y_1, Y_2, Y_3, \ldots, Y_s)\}\]
\[-\theta \{\nabla_X \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r, Y_1, Y_2, Y_3, \ldots, Y_s\}\]
\[-\theta \{\alpha_1, \nabla_X \alpha_2, \alpha_3, \ldots, \alpha_r, Y_1, Y_2, Y_3, \ldots, Y_s\} \ldots\]
\[\ldots \ldots -\theta \{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r, Y_1, Y_2, Y_3, \ldots, \nabla_X Y_s\}\].

We call \(\nabla_X\) is the general covariant derivative. Also we have the following properties:
(1.9.4) \[(i)\] \(\nabla_X (\theta_1 + \theta_2) = \nabla_X \theta_1 + \nabla_X \theta_2\),
(1.9.5) \[(ii)\] \(\nabla_X (\theta_1 \otimes \theta_2) = (\nabla_X \theta_1) \otimes \theta_2 + \theta_1 \otimes (\nabla_X \theta_2),\)
(1.9.6) \[(iii)\] \(\nabla_X (\alpha \wedge \beta) = (\nabla_X \alpha) \wedge \beta + \alpha \wedge (\nabla_X \beta),\)

where \(\theta_1, \theta_2\), and \(\theta_3\) are \(C^\infty\) tensor fields of the same type with common domain \(A\) and \(\alpha, \beta\) are 1–forms on \(A\).

1.10 Submanifolds

A \(C^\infty\) manifold \(M^n\) is called a submanifold of a \(C^\infty\) manifold \(M^m\) if for each point \(p \in M^m\) there is a co–ordinate neighborhood \(\overline{U}\) of \(M^n\) with the co–ordinate function \(\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_n\), such that the set \(U = \{p \in \overline{U} : \bar{X}_{m+1}(p) = \bar{X}_{m+2}(p) = \ldots = \bar{X}_n(p) = 0\}\) is a co–ordinate neighborhood of \(p\) on \(M^n\). These co–ordinate systems are called special or adapted co–ordinate systems.

Let \(Z\) be a \(C^\infty\) vector field on a special co–ordinate neighborhood \(U\) of a point in \(M^m\) and let \(U\) be the associated co–ordinate neighborhood of \(p\) in \(M^n\). A \(C^\infty\) vector field \(\tilde{Z}\) on \(\overline{U}\) such that
\[(1.10.1) \quad \{\tilde{Z}(f)\}_p = \{Z(f)/U\}_p,\]
where $f$ is a $C^\infty$ function of an open set $\bar{U}$ and $f/U$ denotes the restriction of $f$ on $U$. Thus (1.9.1) is called the $C^\infty$ extension of $Z$.

If $\bar{X}, \bar{Y}$ be $C^\infty$ extension on $\bar{U}$ of $C^\infty$ vector fields $X, Y$ on $U$ respectively, then

$$[X,Y]_p f = \bar{X}_p\{\bar{Y}(f)\} - \bar{Y}_p\{\bar{X}(f)\}$$

$$= \bar{X}_p\{Y(f)/U\} - \bar{Y}_p\{X(f)/U\}$$

$$= X_p\{Y(f)/U\} - Y_p\{X(f)/U\}$$

$$= [X,Y]_p f/U.$$

Thus, $[\bar{X}, \bar{Y}]$ is $C^\infty$ extension of $[X,Y]$.

## 1.11 Almost Complex Manifold

Let $M^{2n}$ be a $2n$–dimensional (even dimensional) differentiable manifold of class $C^\infty$. Let there exists on $M^{2n}$ a $C^\infty$ tensor field $f$ of type (1, 1) satisfying

$$f^2 = -I,$$

where $I$ denotes the unit tensor field, then we say that (1, 1) tensor field $f$ gives an almost complex structure to the manifold. We call such a manifold $M^{2n}$ an almost complex manifold.

In an almost complex manifold with an almost complex structure $f$, the Nijenhuis tensor $N(X,Y)$ is given by


$N(X,Y)$ is skew–symmetric in $X$ and $Y$ and satisfies the following:

(i) $N(fX,Y) = N(X,fY) = -fN(X,Y)$,

(ii) $N(fX,fY) = -N(X,Y)$. 

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An almost Hermitian manifold is an almost complex manifold with a Riemannian metric \( g \) satisfying
\[
g(fX, fY) = g(X, Y). \tag{1.11.5}
\]

By the virtue of (1.10.5), it can be easily shown that the operator \( f \) applied to contravariant vector preserves their lengths and also the angles. If we put
\[
F(X, Y) \overset{\text{def}}{=} g(fX, Y), \tag{1.11.6}
\]
then by virtue of (1.10.1), (1.10.5) and (1.10.6), we can show that
\[
F(X, Y) = -F(Y, X)
\]
which shows that \( F \) is a skew–symmetric 2–form.

Also,
\[
g(fX, X) = F(X, X) = -F(X, X)
\]
or
\[
(fX, X) = 0.
\]

Thus, an arbitrary vector field \( X \) and its transform \( fX \) are \( g \)–orthogonal.

A necessary and sufficient condition that the almost complex structure \( f \) be induced by a complex structure is that \( N(X, Y) = 0; \ N(X, Y) \) being the Nijenhuis tensor of the structure tensor \( f \).

An almost Hermitian manifold is called Hermitian manifold if and only if the Nijenhuis tensor vanishes identically.

A 2–cotensor \( \alpha \) in an almost complex manifold is pure if
\[
\alpha(fX, fY) = -\alpha(X, Y), \tag{1.11.7}
\]
while \( \alpha \) is called hybrid, if
\[
\alpha(fX, fY) = \alpha(X, Y). \tag{1.11.8}
\]
### 1.12 Almost Contact Metric Manifold

Let $\bar{M}^n$ be an almost contact metric manifold with an almost contact metric structure $(\phi, \xi, \eta, g)$, i.e., $\phi$ is a $(1,1)$ tensor field, $\xi$ is a vector field, $\eta$ is a $1$–form and $g$ is a Riemannian metric on $\bar{M}^n$ such that:

\begin{align}
(1.12.1) \quad \phi^2 &= -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0, \\
(1.12.2) \quad g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y) \\
(1.12.3) \quad g(X, \phi Y) &= -g(\phi X, Y), \quad g(X, \xi) = \eta(X)
\end{align}

for all $X, Y \in T\bar{M}^n$.

Let $\bar{M}^n$ be an almost contact metric manifold with an almost contact metric structure $(\phi, \xi, \eta, g)$. Let $\psi$ denote the fundamental $2$–form in $\bar{M}^n$ given by $\psi(X, Y) \equiv g(X, \phi Y)$. The almost contact metric manifold is called a contact metric manifold if $\psi = d\eta$. If the structure vector field $\xi$ is killing with respect to the Riemannian metric $g$, the contact metric manifold is said to be a $K$–contact manifold. It is known that a contact metric manifold is $K$–contact if and only if

$$\bar{\nabla}_X \xi = -\phi X$$

for each $X \in T\bar{M}^n$. Where $\bar{\nabla}$ is the Levi–Civita connection on $\bar{M}^n$. The almost contact structure $(\phi, \xi, \eta)$ becomes normal if

\begin{equation}
(1.12.4) \quad N^{(1)} \equiv [\phi, \phi] + 2d\eta \otimes \xi = 0,
\end{equation}

where $[\phi, \phi]$ is the Nijenhuis tensor of $\phi$ and $d$ denotes the exterior derivative operator.

A normal contact metric manifold is called a Sasakian manifold. Each Sasakian manifold is a $K$–contact metric manifold.

An almost contact metric manifold is Sasakian if and only if
(1.12.5) \[(\bar{\nabla}_X \phi)Y = g(X, Y)\xi - \eta(Y)X\]

for any \(X, Y \in T\bar{M}^n\).

An almost contact metric manifold is known to be a Kenmotsu manifold if

(1.12.6) \[(\bar{\nabla}_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)X\]

and nearly Kenmotsu manifold if

(1.12.7) \[(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = -\eta(Y)\phi X - \eta(X)\phi Y.\]

### 1.13 Lorentzian Para–Sasakian Manifold

An \(n\)–dimensional differentiable manifold \(M^n\) is a Lorentzian para–Sasakian (LP–Sasakian) manifold, if it admits a \((1,1)\)–tensor field \(\phi\), vector field \(\xi\), 1–form \(\eta\) and Lorentzian metric \(g\) which satisfy

(1.13.1) \[\phi^2 X = X + \eta(X)\xi,\]

(1.13.2) \[\eta(\xi) = -1,\]

(1.13.3) \[g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),\]

(1.13.4) \[g(X, \xi) = \eta(X),\]

(1.13.5) \[(D_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi\]

and

(1.13.6) \[D_X \xi = \phi X\]

for arbitrary vector fields \(X\) and \(Y\), where \(D_X\) denote covariant differentiation with respect to \(X\).

In an LP–Sasakian manifold \(M^n\) with structure \((\phi, \xi, \eta, g)\), it is easily seen that

(1.13.7) \[\phi \xi = 0, \quad \eta(\phi X) = 0, \quad \text{rank } \phi = (n - 1).\]
1.14  **Lorentzian Concircular Structure Manifold (LCS)\(_n\)**

An Lorentzian manifold \(M^n\) together with the unit timelike concircular vector field \(\xi\), its associated 1–form \(\eta\) and \((1,1)\) tensor field \(\psi\) is said to be Lorentzian concircular structure manifold (briefly \((LCS)_n\)–manifold).

Especially, if we take \(\alpha = 1\), then we can obtain the LP–Sasakian structure of Matsumoto ([18]). In a \((LCS)_n\)–manifold, the following relations holds:

\[
\begin{align*}
(1.14.1) & \quad \eta(\xi) = -1, \psi \xi = 0, \eta(\psi X) = 0, \\
(1.14.2) & \quad g(\psi X, \psi Y) = g(X, Y) + \eta(X) \eta(Y), \\
(1.14.3) & \quad \eta(R(X,Y)Z) = (\alpha^2 - \rho)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \\
(1.14.4) & \quad S(X, \xi) = (n - 1)(\alpha^2 - \rho)\eta(X), \\
(1.14.5) & \quad R(X,Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \\
(1.14.6) & \quad R(\xi, X)Y = (\alpha^2 - \rho)[g(X,Y)\xi - \eta(Y)X], \\
(1.14.7) & \quad R(\xi, X)\xi = (\alpha^2 - \rho)[\eta(X)\xi + X], \\
(1.14.8) & \quad (\nabla_X \psi)(Y) = \alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \\
(1.14.9) & \quad (X\rho) = d\rho(X) = \beta\eta(X).
\end{align*}
\]

1.15  **Lorentzian \(\alpha\)–Sasakian Manifold**

A differentiable manifold of dimension \(n\) is called Lorentzian \(\alpha\)–Sasakian manifold if it admits a \((1,1)\)–tensor field \(\varphi\), a contravariant vector field \(\xi\), a covariant vector field \(\eta\) and Lorentzian metric \(g\) which satisfy the following:

\[
(1.15.1) \quad \eta(\xi) = -1, \quad \varphi^2 = I + \eta \otimes \xi,
\]
for all $X, Y \in TM$.

Lorentzian $\alpha$–Sasakian manifold $M$ also satisfies the condition

\begin{align}
(1.15.4) \quad \nabla_X \xi &= -\alpha \varphi(X) \nabla_Z = 0,
\end{align}

and

\begin{align}
(1.15.5) \quad (\nabla_X \eta)Y &= -\alpha g(\varphi X, Y),
\end{align}

where $\nabla$ denotes the operator of covariant differentiation with respect to the Lorentzian metric $g$.

A differentiable manifold $(M^n, g), n > 2$, satisfying the condition

\begin{align}
\varphi^2 \bar{P}(\varphi X, \varphi Y)\varphi Z = 0,
\end{align}

is called $\varphi$–pseudo projectively flat Lorentzian $\alpha$–Sasakian manifold.

A differentiable manifold $(M^n, g), n > 2$, satisfying the condition

\begin{align}
\varphi^2 \bar{C}(\varphi X, \varphi Y)\varphi Z = 0,
\end{align}

is called $\varphi$–quasi conformally flat.

A differentiable manifold $(M^n, g), n > 2$, satisfying the condition

\begin{align}
\varphi^2 \bar{V}(\varphi X, \varphi Y)\varphi Z = 0,
\end{align}

is called $\varphi$–quasi concircularly flat Lorentzian $\alpha$–Sasakian manifold.

### 1.16 Riemannian Curvature Tensor

Let $(M^n, g), n > 3$, be a connected semi Riemannian manifold of class $C^\infty$ and $\nabla$ be its Levi–Civita connection. Riemannian curvature tensor $R$ is defined by

\[ R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z. \]
1.17 **Pseudo–Projective Curvature Tensor**

Pseudo–projective curvature tensor $\tilde{P}$ on a Riemannian manifold $(M^n, g), n > 2$ of type $(1, 3)$ is defined as follows

\[
\tilde{P}(X, Y)Z = a R(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y]
- \frac{r}{n} \left[ \frac{a}{n-1} + b \right] [g(Y, Z)X - g(X, Z)Y].
\]

1.18 **Quasi–Conformal Curvature Tensor**

Quasi–conformal curvature tensor $\tilde{C}$ on a Riemannian manifold $(M^n, g), n > 2$ of type $(1, 3)$ is defined as follows

\[
\tilde{C}(X, Y)Z = a R(X, Y) + b[S(Y, Z)X
- S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY]
+ \frac{r}{(n-1)(n-2)} [g(Y, Z)X - g(X, Z)Y] = C(X, Y)Z.
\]

1.19 **Quasi–Concircular Curvature Tensor**

Quasi–concircular curvature tensor $\tilde{V}$ on a Riemannian manifold $(M^n, g), n > 2$ of type $(1, 3)$ is defined as follows

\[
\tilde{V}(X, Y)Z = a R(X, Y)Z + \frac{r}{n} \left[ \frac{a}{n-1} + 2b \right] [g(Y, Z)X - g(X, Z)Y].
\]
1.20 \textbf{$m$–Projective Curvature Tensor}

\textit{m–projective curvature tensor $W$ on a Riemannian manifold $(M^n, g), n > 3$ of type $(1, 3)$ is defined as follows}

\begin{equation*}
W(X, Y)Z = R(X, Y)Z - \frac{1}{2^{(n-1)}} [S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY].
\end{equation*}

1.21 \textbf{Almost Pseudo Ricci–Symmetric Manifold $A(PR S)_n$}

A Riemannian manifold $(M^n, g)$ is called an almost pseudo Ricci–symmetric manifold if its Ricci tensor $S$ of type $(0, 2)$ is not identically zero and satisfies a relation

\begin{equation*}
(D_X S)(Y, Z) = \{A(X) + B(X)\} S(Y, Z) + A(Y) S(X, Z) + A(Z) S(Y, X),
\end{equation*}

where $D$ denotes the operator of covariant differentiation with respect to Riemannian metric $g$ and $A, B$ are nowhere vanishing 1–forms such that

\[ g(X, \rho) = A(X) \]

and

\[ g(X, \mu) = B(X) \]

for all $X$ and $\rho$ and $\mu$ are the basic vector fields of the manifold.

The 1–forms $A$ and $B$ are called the associated 1–form and $n$–dimensional manifold of this kind is denoted by $A(PR S)_n$.
1.22 **Generalized Pseudo Symmetric Manifold** $G(PS)_n$

A non–flat differentiable manifold $(M^n, g)$ ($n > 3$) is called generalized pseudo symmetric $(PS)_n$, if there exists a vector field $P$ and 1–form $A, B, C, D$ on $M$ such that


1.23 **Generalized Pseudo Ricci–Symmetric Manifold** $G(PRS)_n$

A non–flat differentiable manifold $(M^n, g)$ ($n > 2$) is called generalized pseudo Ricci–symmetric $G(PRS)_n$, if it satisfies the condition

$$(D_X S)(Y, Z) = 2A(X)S(Y, Z) + B(Y)S(X, Z) + C(Z)S(X, Y),$$

where $A, B, C$ are the non zero 1–forms and $D$ denotes the operator of covariant differentiation with respect to $g$.

1.24 **Generalized Ricci–Recurrent Manifold** $G(R)_n$

A differentiable manifold $(M^n, g)$ ($n > 2$), whose Ricci tensors of type $(0, 2)$ satisfies the condition

$$(D_X S)(Y, Z) = A(X)S(Y, Z) + B(X)g(Y, Z),$$

where $A$ and $B$ are two non–zero 1–forms, $P$ and $Q$ are two vector fields such that

$$g(X, P) = A(X),$$
\[ g(X, Q) = B(X). \]

Such a manifold is called generalized Ricci–recurrent manifold and an \( n \)–dimensional manifold of this kind were denoted by \( G(R)_n \).

### 1.25 Semi Pseudo Symmetric Manifold \((SPS)_n\)

A type of non–flat differentiable manifold \((M^n, g) (n > 3)\), whose curvature tensor \( R \) satisfies the condition

\[
+ A(Z)R(Y, X)W + A(W)R(Y, Z)X \\
+ A(W)R(Y, Z)X,
\]

where \( A \) is a non zero 1–form and

\[ g(X, P) = A(X). \]

Such a manifold is called a semi pseudo symmetric and an \( n \)–dimensional manifold of this kind is denoted by \((SPS)_n\).

### 1.26 Semi Pseudo Ricci–Symmetric Manifold \((SPRS)_n\)

A type of non–flat Riemannian manifold \((M^n, g) (n > 3)\), whose Ricci–tensor of type \((0, 2)\) satisfies the condition

\[
(D_X S)(Y, Z) = A(Y)S(X, Z) + A(Z)S(X, Y),
\]

where \( A \) is a non zero 1–form. Such a manifold were called by them Semi pseudo Ricci–symmetric and an \( n \)–dimensional manifold of this kind is denoted by \((SPRS)_n\).
REFERENCES


