Chapter 11

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11.1 INTRODUCTION

The phenomenon of scattering illustrates the fact that quantum mechanics does not predict the outcome of an individual measurement, but rather the statistical distribution of all possible outcomes. Scattering is important in quantum mechanics and is intimately related to bound state problems, as much of the information about the interaction between particles is derivable from scattering experiments. Essentially, in non-relativistic quantum mechanics, scattering theory is concerned with the solution of the Schrodinger wave equation for different potentials with appropriate boundary conditions. Common approach to scattering problems are the method of partial waves and the Born approximation method. While the former deals with low-energy scattering by short-range central potentials, the latter is suitable for high energies and potentials, which are not necessarily central. Since relativistic quantum mechanics essentially concerns high energies, born approximation would be appropriate to describe relativistic scattering phenomena.

11.2 GREEN'S FUNCTION APPROACH TO SCATTERING

In order to exploit the connection between differential cross-section and scattering amplitude, one often examines the stationary scattering states. As in the non-relativistic case, a simple approach to solve the scattered wave equation in the presence of a potential is to use the Green's function method.

We have seen that the Klein-Gordon equation with a scalar potential, $S(x)$, reads

$$\left\{ p^2 c^2 + (mc^2 + S)^2 \right\} \psi = E^2 \psi.$$  \hspace{1cm} (11.2.1)
The operator form of the above equation is

\[ \{ c^2 \hbar^2 \nabla^2 + (E^2 - m^2 c^4) \} \psi = \{ 2mc^2 S + S^2 \} \psi \]  (11.2.2)

Eqn.(11.2.2) may well be written as

\[ (\nabla^2 + k^2) \psi_k (\vec{r}) = V_{eff} \psi (\vec{r}), \]  (11.2.3)

where \( k^2 = \frac{E^2 - m^2 c^4}{c^2 \hbar^2} > 0 \) for scattering  (11.2.4)

and \( V_{eff}^S = \frac{2mc^2 S + S^2}{c^2 \hbar^2} \).  (11.2.5)

Eqn.(11.2.3) may be compared with the similar equation in the non-relativistic case [Merzbacher, 1970] and rewritten in terms of the 'source function', \( S_0 (\vec{r}) \) as

\[ (\nabla^2 + k^2) \psi_k (\vec{r}) = S_0 (\vec{r}) \]  (11.2.6)

with solutions

\[ \psi_k (\vec{r}) = \frac{1}{\nabla^2 + k^2} S_0 (\vec{r}) \]  (11.2.7)

where \( S_0 (\vec{r}) = V_{eff}^S (\vec{r}) \psi (\vec{r}) \).

It is convenient to replace the differential equation by an integral equation. Conventionally, the Green's function is some kind of "inverse" of the differential operator and is defined to be the solution of the partial differential equation with the source \( S_0 (\vec{r}) \) replaced by a point source described by the three dimensional \( \delta \)-function [Morse and Feshbach,1953] as

\[ (\nabla^2 + k^2) G_k (\vec{r}, \vec{r}') = -\delta^3 (\vec{r} - \vec{r}') \]  (11.2.8)

Knowing the Green's function, the solution of the original equation may be constructed as

\[ \psi_k (\vec{r}) = \psi_0 (\vec{r}) - \int G_k (\vec{r}, \vec{r}') d^3 r' S_0 (\vec{r}') \]  (11.2.9)

where \( \vec{r} \) refers to the field point and \( \vec{r}' \), the source point. Here \( \psi_0 (\vec{r}) \) is the free particle solution of the KG equation satisfying

\[ (\nabla^2 + k^2) \psi_0 (\vec{r}) = 0. \]  (11.2.10)
The homogeneous solution corresponds to the case where there is no scattering and the solution may be written as

$$\psi_0 (\vec{r}) = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k} \cdot \vec{r}}. \quad (11.2.11)$$

The solution of Eqn.(11.2.9) may be readily written as

$$\psi_{k} (\vec{r}) = \psi_0 (\vec{r}) - \int d^3 \vec{r}' G_{k} (\vec{r}, \vec{r}') \mathcal{V}_{ff} (\vec{r}') \psi_k (\vec{r}'). \quad (11.2.12)$$

This integral equation for $\psi_{k} (\vec{r})$ which resembles the non-relativistic equation, with $\mathcal{V}_{ff} (\vec{r}')$ replacing $V (\vec{r}')$ may rightly be called the 'Relativistic Scattering Integral Equation'. As in the non-relativistic case, it may be shown that the Green's function solution has the explicit form [Sakurai, 2001]

$$G_{k} (\vec{r}, \vec{r}') = \frac{1}{4\pi} \left[ \frac{e^{ik |\vec{r} - \vec{r}'|}}{|\vec{r} - \vec{r}'|} \right]. \quad (11.2.13)$$

As a novel extension of relativistic scattering theory to include interactions in the vector coupling prescription, we begin with the Klein-Gordon equation

$$\left( p^2 c^2 + m_2 c^4 \right) \psi = (E - V)^2 \psi. \quad (11.2.14)$$

Taking the positive root of the above equation, which is true for the outgoing wave and transforming it to the operator equation, we may write the Relativistic Schrödinger Equation as

$$\left\{ -\sqrt{-\hbar^2 c^2 \nabla^2 + m_2 c^4 + E} \right\} \psi = V \psi. \quad (11.2.15)$$

As before, the Green's function approach is best suited to address this problem. We define the new Green's function, $G_{E} (\vec{r}, \vec{r}')$ as satisfying the equation

$$\left\{ -\sqrt{-\hbar^2 c^2 \nabla^2 + m_2 c^4 + E} \right\} G_{E} (\vec{r}, \vec{r}') = -\delta^3 (\vec{r} - \vec{r}') \quad (11.2.16)$$

with $E = \sqrt{p_0^2 c^2 + m_2 c^2}$ and $p_0 = \hbar k_0$. 

The Fourier transform of the Green's function is given by

\[ G_E (\vec{r}, \vec{r}') = \frac{1}{(2\pi)^{3/2}} \int e^{i \vec{K} \cdot (\vec{r} - \vec{r}')} \tilde{G}_E (K') d^3 K' \]  

(11.2.17)

We thus see that the Green's function in the coordinate space is expanded in terms of the Green's function in the momentum space.

The Fourier transform of the $\delta$-function is

\[ \delta = \frac{1}{(2\pi)^3} \int e^{i \vec{K} \cdot (\vec{r} - \vec{r}')} d^3 K. \]  

(11.2.18)

Using Eqns. (11.2.17) and (11.2.18) in Eqn. (11.2.16), we obtain

\[ \frac{1}{(2\pi)^{3/2}} \int \tilde{G}_E (\vec{K}) \left\{ -\sqrt{\hbar^2 c^2 K^2 + m^2 c^4} + E \right\} d^3 K e^{i \vec{R} \cdot (\vec{r} - \vec{r}')} = -\frac{1}{(2\pi)^3} \int e^{i \vec{K} \cdot |\vec{r} - \vec{r}'|} d^3 K \]  

(11.2.19)

It follows from the above equation that

\[ \tilde{G}_E (\vec{K}) = - (2\pi)^{3/2} \left\{ \frac{1}{E - \sqrt{\hbar^2 c^2 K^2 + m^2 c^4}} \right\} \]  

(11.2.20)

so that

\[ G_E (\vec{r}, \vec{r}') = \frac{-1}{(2\pi)^{3/2}} \int \frac{e^{i \vec{K} \cdot (\vec{r} - \vec{r}')} d^3 K}{\sqrt{E - \sqrt{\hbar^2 c^2 K^2 + m^2 c^4}}} \]  

(11.2.21)

or equivalently,

\[ G_E (\vec{r}, \vec{r}') = \frac{-1}{(2\pi)^{3/2}} \int \frac{e^{i \vec{K} \cdot (\vec{r} - \vec{r}')} d^3 K}{\sqrt{\sqrt{\hbar^2 c^2 k_0^2 + m^2 c^4} - \sqrt{\hbar^2 c^2 K^2 + m^2 c^4}}} \]  

(11.2.22)

This integral can be evaluated in spherical polar co-ordinates ($K, \theta_K, \phi_K$) corresponding to $\vec{K}$.

If the $K_z$ axis is chosen to be along $(\vec{r} - \vec{r}')$, we may write

\[ \vec{K} \cdot (\vec{r} - \vec{r}') = |\vec{K}| |\vec{r} - \vec{r}'| \cos \theta_K. \]
Eqn.(11.2.22) becomes

\[ G_E (\vec{r}, \vec{r}') = \frac{-1}{(2\pi)^3} \int_0^{2\pi} d\phi_K \int_0^\pi d\theta_K \int_0^\infty dK \frac{e^{iK|\vec{r}-\vec{r}'|} \cos \theta_K}{\sqrt{K^2 - m^2} - \sqrt{K^2 + m^2}} \]  

wherein we have adapted natural units and set \( c = \hbar = 1 \). On carrying out the integration, the above equation reduces to

\[ G_E (\vec{r}, \vec{r}') = \frac{1}{(2\pi)^3} \frac{1}{i |\vec{r}-\vec{r}'|} \int_0^\infty K dK \frac{\{e^{iK|\vec{r}-\vec{r}'|} - e^{-iK|\vec{r}-\vec{r}'|}\}}{\sqrt{K^2 + m^2} - \sqrt{k_0^2 + m^2}} \]  

The integral is readily evaluated by extending the lower limit to \(-\infty\)

\[ G_E (\vec{r}, \vec{r}') = \frac{1}{(2\pi)^3} \frac{1}{i |\vec{r}-\vec{r}'|} \int_0^\infty K dK \frac{e^{iK|\vec{r}-\vec{r}'|}}{\sqrt{K^2 + m^2} - \sqrt{k_0^2 + m^2}}. \]  

Rationalising the denominator, it is straightforward to check that the above equation becomes

\[ G_E (\vec{r}, \vec{r}') = \frac{1}{(2\pi)^3} \frac{1}{i |\vec{r}-\vec{r}'|} \int_{-\infty}^\infty K \frac{\left(\sqrt{K^2 + m^2} + \sqrt{k_0^2 + m^2}\right)}{K^2 - k_0^2} e^{ik|\vec{r}-\vec{r}'|} dK. \]  

Eqn.(11.2.26) further simplifies to

\[ G_E (\vec{r}, \vec{r}') = \frac{1}{(2\pi)^3} \frac{1}{i |\vec{r}-\vec{r}'|} \int_{-\infty}^\infty K \frac{\left(\sqrt{K^2 + m^2} + \sqrt{k_0^2 + m^2}\right)}{(K + k_0)(K - k_0)} e^{ik|\vec{r}-\vec{r}'|} dK. \]  

It is clearly seen that this integrand has simple poles on the real axis in the complex K-plane at \( K = \pm k_0 \). We now invoke the '\( i\epsilon \)' prescription for convergence, and write the above equation as

\[ G_E (\vec{r}, \vec{r}') = \int_{-\infty}^\infty K \frac{\left(\sqrt{K^2 + m^2} + \sqrt{k_0^2 + m^2}\right)}{K^2 - k_0^2 + i\epsilon} e^{ik|\vec{r}-\vec{r}'|} dK. \]  

This integrand has poles at

\[ K = \pm k_0 \sqrt{1 \pm \frac{i\epsilon}{k_0^2}} \approx \pm k_0 \pm i\epsilon' \quad \text{where} \quad \epsilon' = \frac{\epsilon}{2k_0}. \]
Using the method of residues, the above integral on evaluation gives
\[
G_+^V (\vec{r}, \vec{r}') = \frac{1}{4\pi} \frac{e^{+ik_0|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \sqrt{k_0^2 + m^2} \tag{11.2.29}
\]

\( G_+^V (\vec{r}, \vec{r}') \) is the new Green's function for the outgoing wave for any potential in the vector coupling scheme.

Alternately, Eqn.(11.2.17) may be rewritten as
\[
(\nabla^2 + k^2) \psi_k (\vec{r}) = S_0^V (\vec{r})
\]
with \( k^2 = \frac{E^2-m^2c^4}{c^2\hbar^2} > 0 \), \( V_{eff}^V = \frac{2EV-V^2}{c^2\hbar^2} \) and \( S_0^V (\vec{r}) = V_{eff}^V \psi_k (\vec{r}) \psi (\vec{r}) \) is the 'Source function for vector interaction'.

Thus the wavefunction in the presence of the scatterer may be written as
\[
\psi_k (\vec{r}) = \psi_0 (\vec{r}) - \int G_E (\vec{r}, \vec{r}') d^3r' S_0 (\vec{r}') \tag{11.2.30}
\]

Since \( G_E (\vec{r}, \vec{r}') \) is known from Eqn.(11.2.15), one may easily evaluate the above integral for known energies and specific potentials.

### 11.3 RELATIVISTIC BORN APPROXIMATION

In situations where the kinetic energy of the colliding particles is large in comparison with the interaction energy, the scattering potential may be regarded as a perturbation. The scattered wavefunction at points far from the interaction region may then be expressed as plane wave solutions, leading to the celebrated Born Approximation [Schiff,1955].

For scalar interaction, Eqn.(11.2.12) reads
\[
\psi_k (\vec{r}) = \psi_0 (\vec{r}) - \int d^3r' G_k^S (\vec{r}, \vec{r}') V_{eff}^S (\vec{r}') \psi_k (\vec{r}') \tag{11.3.1}
\]

When the field point is away from the source point, we may write
\[
|\vec{r} - \vec{r}'| = r \sqrt{1 - 2 \frac{\vec{r} \cdot \vec{r}'}{r^2} + \left(\frac{r'}{r}\right)^2}, \quad \text{which leads to}
\]
\[
|\vec{r} - \vec{r}'| \approx r - \hat{r} \cdot \vec{r}' \quad \text{where} \quad \hat{r} \quad \text{is a unit vector along} \quad \vec{r}.
\]
The Green's function \( G^S_k (\vec{r}, \vec{r}') \) of Eqn.(11.2.13) takes the form

\[
G^S_k (\vec{r}, \vec{r}') \approx \frac{1}{4\pi} \frac{e^{i k r}}{r} e^{-i \vec{k}' \cdot \vec{r}'}
\]

(11.3.2)

where \( \vec{k}' = k \hat{r} \).

Using Eqn.(11.3.2) in Eqn.(11.3.1) we obtain the asymptotic expression

\[
\psi_k (\vec{r}) \sim e^{i \vec{k} \cdot \vec{r}} - \frac{1}{4\pi} \frac{e^{i k r}}{r} \int e^{-i (\vec{k}' - \vec{k}) \cdot \vec{r}'} V^S_{ef} (\vec{r}') d^3 \vec{r}'.
\]

(11.3.3)

The above equation may be written in the standard form as

\[
\psi_k (\vec{r}) \sim e^{i \vec{k} \cdot \vec{r}} + \frac{1}{4\pi} \frac{e^{i k r}}{r} f_k (\theta, \phi)
\]

(11.3.4)

where

\[
f_k (\theta, \phi) = - \frac{1}{4\pi} \int e^{-i \vec{q} \cdot \vec{r}'} V^S_{ef} (\vec{r}') d^3 \vec{r}'
\]

(11.3.5)

with \( \vec{q} = \vec{k}' - \vec{k} \). \( f_k (\theta, \phi) \) is the First order relativistic scattering amplitude. Apparently, in relativistic quantum mechanics, owing to the dependence of the scattering amplitude on the effective potential, an interaction may result in different amplitudes depending upon the type of coupling. Thus \( f_k (\theta, \phi) \) aside having angular dependence, is not uniquely defined for a specific potential. This novel feature is contrary to what is observed in non-relativistic quantum mechanics.

Knowing the relativistic scattering amplitude, the differential scattering cross-section may be found using

\[
\frac{d\sigma}{d\Omega} = |f_k (\theta, \phi)|^2.
\]

(11.3.6)

Born approximation may be applied to obtain the expression for the scattering amplitude when the potential is treated as a Lorentz vector. Following the same prescription as before, we obtain

\[
f_k (\theta, \phi) = - \frac{1}{4\pi} \int e^{-i \vec{q} \cdot \vec{r}'} V^V_{ef} (\vec{r}') d^3 \vec{r}'
\]

(11.3.7)

where

\[
V^V_{ef} = \frac{2EV - V^2}{c^2 \hbar^2}.
\]

(11.3.8)
However, starting from the new Green's function defined by Eqn.(11.2.29), the scattering amplitude may be shown to be

\[ f_k (\theta, \phi) = -\frac{1}{4\pi} \int e^{-i\vec{q}\cdot\vec{r}'}/\sqrt{k_0^2 + m^2} V(\vec{r}) d^3r'. \]  

(11.3.9)

It is interesting to note that while the scattering amplitude in Eqn.(11.3.7) involves the effective vector potential, the one in Eqn.(11.3.9) contains the actual interaction.

Applications:

To test the validity of our method, we proceed to calculate the relativistic scattering amplitude for specific potentials like the modified Yukawa potential, commonly encountered in nuclear reactions and the screened Coulomb potential, often dealt with in atomic collisions.

1. Modified Yukawa Potential

As a classic application of the relativistic Born approximation, we consider the elastic scattering of a relativistic particle by a scalar modified Yukawa potential of the form

\[ S(r) = \left( \frac{g}{2a} \right) \frac{e^{-r/a}}{r} \]  

(11.3.10)

where \( g \) is the coupling constant and \( a \) the range of the potential. Using Eqn.(11.2.5), valid for scalar interaction, Eqn.(11.3.5) becomes

\[ f_k (\theta, \phi) = -\frac{1}{4\pi c^2 \hbar^2} \int e^{-i\vec{q}\cdot\vec{r}'}(2mc^2S + S^2) d^3r'. \]  

(11.3.11)

which further simplifies to

\[ f_k (\theta, \phi) = -\frac{m}{2\pi \hbar^2} \int e^{-i\vec{q}\cdot\vec{r}'} S(\vec{r}) d^3r' - \frac{1}{4\pi c^2 \hbar^2} \int e^{-i\vec{q}\cdot\vec{r}'} S^2 d^3r'. \]  

(11.3.12)

While the first term on the righthand side gives the Fourier Transform of \( S \), the second is the Fourier Transform of \( S^2 \).

With the insertion of the above potential, Eqn.(11.3.12) simplifies to

\[ f(\theta) = -\frac{1}{q} \frac{2m}{\hbar^2} \left( \frac{g}{2a} \right) \int_0^\infty e^{-\xi} \sin qr \, dr - \frac{1}{q c^2 \hbar^2} \left( \frac{g}{2a} \right)^2 \int_0^\infty e^{-2\xi} \frac{\sin qr}{r} \, dr. \]  

(11.3.13)
The standard integrals reduce to the form [Gradshteyn, 1965]

\[
\int_{0}^{\infty} e^{-z} \sin qr \, dr = \frac{\sin (aq)}{q^2 + \left( \frac{1}{a^2} \right)} ,
\]

\[
\int_{0}^{\infty} e^{-z} \sin qr \, dr = \tan^{-1} \left( \frac{aq}{2} \right) .
\]

The scattering amplitude may now be written as

\[
f(\theta) = - \frac{mg}{\hbar^2} \left( \frac{a}{a^2q^2 + 1} \right) - \frac{g^2}{4a^2c^2\hbar^2 q} \tan^{-1} \left( \frac{aq}{2} \right) . \tag{11.3.14}
\]

It is straightforward to check that

\[a^2q^2 = 2k^2 (1 - \cos \theta),\]

\[a^2q^2 = 2 \left( \frac{E^2 - m^2c^4}{c^2\hbar^2} \right) (1 - \cos \theta).\]

Expressing \( \bar{a} = \frac{a}{\hbar/mc}, \quad \bar{E} = \frac{E}{mc^2} \) and \( \bar{g} = \frac{g}{\hbar^2/m} \), we see that

\[a^2q^2 = 2a^2 \left( \bar{E}^2 - 1 \right) (1 - \cos \theta) .\]

It follows from Eqn.(11.3.14) that the complete scattering amplitude may be written as

\[
f(\theta) = -a\bar{g} \left\{ \frac{1}{2a^2(\bar{E}^2 - 1)(1 - \cos \theta) + 1} \right\} - \frac{1}{4} \left( \frac{\bar{g}}{\bar{a}} \right)^2 \frac{1}{q} \tan^{-1} \left( \frac{aq}{2} \right) . \tag{11.3.15}
\]

Apparently \( f(\theta) \) has the dimensions of length. Knowing the strength and range of the potential and also the energy and direction of the incoming particle, the relativistic scattering amplitude may be computed from Eqn.(11.3.15). While the first term corresponds to the non-relativistic scattering amplitude, the second gives the first order relativistic correction to \( f(\theta) \). For very weak potentials (small \( g \) and large \( a \)) under which Born approximation is valid, the relativistic correction term is extremely small and \( f(\theta) \) approaches
the non-relativistic limit, as expected. The differential scattering cross-section is obtained using

\[ \frac{d\sigma}{d\Omega} = |f(\theta)|^2. \]

2. Screened Coulomb Potential

We now investigate the kinematics of a boson scattered by a screened Coulomb potential of the form

\[ V(r) = V_0 \frac{e^{-r/a}}{r} \]

where \( a \), as before, refers to the range of the potential. The screened Coulomb potential reduces to the pure Coulomb Potential, when the range is infinite. Going through the same mathematical steps as before, it is straightforward to check that the scattering amplitude reduces to

\[ f(\theta) = -\left( \frac{2m}{\hbar^2} \right) V_0 \left\{ \frac{1}{q^2 + \frac{1}{a^2}} \right\} - \frac{1}{qc^2\hbar^2} V_0^2 \tan^{-1} \left( \frac{aq}{2} \right). \]  

The differential scattering cross-section is

\[ \frac{d\sigma}{d\Omega} = \left( \frac{2mV_0}{\hbar^2} \right)^2 \frac{1}{(q^2 + \frac{1}{a^2})^2} + 2 \left( \frac{2m}{\hbar^2} \right) V_0^3 \left\{ \frac{1}{q^2 + \frac{1}{a^2}} \right\} \tan^{-1} \left( \frac{aq}{2} \right) + \left( \frac{V_0^2}{qc^2\hbar^2} \right)^2 \left[ \tan^{-1} \left( \frac{aq}{2} \right) \right]^2. \]

The screened Coulomb potential approaches the pure Coulomb potential in the limit \( a \to \infty \) and \( V_0 = Ze^2 \). The above equation in the NR limit becomes

\[ \frac{d\sigma}{d\Omega} \sim \frac{Z^2e^4}{16 \epsilon^2 \sin^4 (\theta/2)} + \frac{Z^2e^6}{2 \epsilon \sin^2 (\theta/2)} \left( \frac{\pi}{2} \right) \]  

where \( \epsilon = E - mc^2 \). We identify the first term to be the Rutherford scattering cross-section and the second term gives the first order relativistic correction.
11.4 RESULTS AND DISCUSSION

Elastic scattering is an important tool for obtaining information about the structural properties of matter. The relativistic approach to scattering illustrated in this chapter serves as a novel method to describe pion-nucleon scattering. In the simplest approximation, called the Impulse Approximation, [Greiner,1995] it is reasonable to assume that a pion interacts with only one of the nucleons in the nucleus and all the other nucleons just appear in the kinematic factors.

Transforming the Klein-Gordon equation to the Schrödinger-like form, with an effective energy, \( E_{\text{eff}} \) and effective potential, \( V_{\text{eff}} \), we have extended the formal Green's function approach to relativistic scattering of pions. Invoking the 'ie' prescription leads to the well-known Lippmann-Schwinger equation. Obviously, the poles of the new Green's function signal the eigenvalues of the system.

For sufficiently high energies and weak potentials, Born approximation serves as a good method to obtain the relativistic scattering amplitude and relativistic scattering cross-section. Though this method is not valid for any arbitrary potential, specific examples like the modified Yukawa potential and the screened Coulomb potential are examined. It is indeed interesting to note that scattering amplitude as well as the cross-section explicitly depend on the nature of the coupling scheme of the potential through the effective potential.

In our scattering formulation for mesons, we have also addressed the Green’s function technique to systems obeying the relativistic wave equations involving square-root operator. This novel approach, which involves the Fourier Transform of the Green’s function and the Calculus of Residues, yields an altogether different expression for the scattering amplitude, involving not the effective potential, but rather the interaction potential alone.

The validity of the alternate approaches to relativistic scattering phenomena may be tested by computing the relativistic correction to scattering cross-section for specific
potentials and comparing them with experimental data. Furthermore, the best agreement between the theoretical values and experimental results would shed light on the right type of coupling scheme suitable for the interaction.

While the Propagator theory to Collisions [Bjorken, 1964] is known in literature, for electromagnetic couplings, this novel method, apart from being elegant, is a straightforward extension of the non-relativistic approach to scattering, in wave mechanics.