Chapter 8

Relativistic Version of WKB Method and Virial Theorem
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8.1 INTRODUCTION

The WKB method, named after its proponents Wentzel, Kramers and Brillouin, is a semi-classical approximation method extensively used in quantum mechanical problems for slowly varying potentials. The WKB technique involves power series expansion of the wavefunction in terms of the Planck's constant. The wave function is smoothly matched on either side of the classical turning points. This leads to the quantisation condition from which the energy eigenvalues may be approximately determined for ground and excited states. The WKB method has been extensively used in non-relativistic quantum mechanics for various potentials.

Several authors have discussed the WKB method using the Schrödinger wave equation. In the early work of Langer [1937], the WKB method has been employed to obtain the solutions of the radial wave equation for attractive and repulsive Coulomb potential. While Friedrich [1996] has addressed the bound states in Woods-Saxon potential, power-law potential, quadratic potential using WKB approximation, Trost [1997] has compared the WKB and exact ground state energy for spiked harmonic oscillator potential. The phase loss in WKB waves due to reflection by a potential has been considered by Friedrich [1996]. In literature, the WKB treatment to relativistic wave equations has not been mentioned.

In relativistic quantum mechanics, however, the way of incorporating the potential is not unambiguously defined. For many problems of physical interest, exact solutions are not available even in one dimension and hence it is highly important to investigate whether the
methods of approximation known in the non-relativistic case are applicable to relativistic situations. Obviously, it is seen that extension of the well-known non-relativistic approximation might be possible if the relativistic wave equations are reduced to the Schrodinger-like form.

In this chapter, we discuss the relativistic version of the WKB method and also the relativistic generalisation of the Virial Theorem.

8.2 Review of the WKB Method

In the WKB method, one uses an appropriate formal development of the solutions of the Schrodinger equation in powers of $\hbar$. The quasi classical wavefunction is smoothly matched on the different sides of the classical turning points. This leads to the well-known quantisation condition,

$$\oint p \, dq = \left( n + \frac{1}{2} \right) \hbar, \quad n = 0, 1, 2, \ldots$$

(8.2.1)

This equation determines the possible discrete values of energy, $E$. The integral $\oint p \, dq$ determines the area of the phase space enclosed by a closed curve of energy $E$. Kagali et.al.,[1997] have shown that the same area of the phase space, might be obtained by integrating in the $p$-variable, leading to the condition,

$$\oint q \, dp = \left( n + \frac{1}{2} \right) \hbar \quad n = 0, 1, 2, \ldots$$

(8.2.2)

In this chapter, we see the possible extension of this novel and innovative approach, called the 'Phase space integration technique' to include bound states in relativistic quantum mechanics.
8.3 Extension of WKB Method To Bound States in Relativistic Quantum Mechanics

Only a few eigenvalue problems in relativistic quantum mechanics can be solved exactly and approximate methods are therefore of great practical importance. The well-known WKB approximation might be extended to obtain the relativistic bound state energies.

As in non-relativistic quantum mechanics, we choose a simple well-shaped effective potential with two classical turning points \( a \) and \( b \). It is known that WKB approximation could be used in regions 1, 2 and 3 away from the turning points and connection formulas would serve near \( x = a \) and \( x = b \).

![Diagram](image)

Fig. F 8.1

One major drawback in relativistic quantum mechanics is that for high energies, the wavefunction, \( \psi \) has a very short wavelength in the classically accessible region and hence is a rapidly oscillating function.

Owing to the fact that the Klein-Gordon equation could be transformed to the Schrödinger form, approximately the same results follow with \( E_{\text{eff}} \) and \( V_{\text{eff}} \) replacing \( E \) and \( V \) respectively. This treatment, however, cannot be generalised and is only valid when the effective potential is a slowly varying function of \( x \). Since \( V_{\text{eff}} \) is energy independent in scalar prescription, extension of WKB to pure scalar potentials would be feasible.
8.31 The Method

The one-dimensional Klein-Gordon equation, which can be cast into the Schrodinger form

\[ \frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} (E_{\text{eff}} - V_{\text{eff}}) \psi = 0 \]  

(8.3.1)

may be written as

\[ \frac{d^2\psi}{dx^2} + \frac{p_R^2}{\hbar^2} \psi = 0 \]  

(8.3.2)

where

\[ p_R = \sqrt{2m (E_{\text{eff}} - V_{\text{eff}})}. \]  

(8.3.3)

For a potential, \( S(x) \), treated as a Lorentz scalar, it is trivial to note that

\[ p_R^2 = \frac{E^2 - (mc^2 + S)^2}{c^2}. \]  

(8.3.4)

As in the non-relativistic case, when the effective potential varies smoothly, the wavefunction may be written as

\[ \psi(x) = \phi(x) \exp \left[ \pm \frac{i}{\hbar} \int_{x_1}^{x_2} p_R(x) \, dx \right] \]  

(8.3.5)

where \( \phi(x) \) is a slowly varying function. This is the basis for WKB method[Shankar,1994].

Carrying out the details of the calculation as in the non-relativistic case, one would obtain the quantisation condition as

\[ 2 \int_{x_1}^{x_2} p_R \, dx = \left( n + \frac{1}{2} \right) \hbar, \quad \text{where} \quad n = 0, 1, 2, \ldots \]  

(8.3.6)

It is interesting to note that the turning points \( a \) and \( b \) may be obtained using

\[ V_{\text{eff}}(a) = V_{\text{eff}}(b) = E_{\text{eff}}. \]  

(8.3.7)

Analogously, for potentials in the vector coupling scheme, Eqn.(8.3.6) is satisfied with \( p_R \) given by

\[ p_R^2 = \frac{(E - V)^2 - m^2c^4}{\lambda^2}. \]  

(8.3.8)
However the WKB approximation is admissible in vector prescription only for those favourable potentials which allow binding.

The Phase Space Integration method, an innovative technique introduced by Kagali [1997], serves as an alternate approach to the usual WKB method. While the WKB approximation works in the co-ordinate space, the new method deals with the momentum space. The WKB, together with the Phase space integration method may be extended to the relativistic domain, which enhances the validity of these methods. It is seen that for bound states the modified method gives the same energy eigenvalues as the standard WKB procedure.

8.4 APPLICATION OF WKB METHOD

As a classic application of the WKB method, we consider simple one dimensional potentials like the screened Coulomb potential, discussed in Chapter 2 and the linear potential addressed in Chapter 3 to explore the bound state energy eigenvalues, non-relativistically and relativistically. Also it is seen that the eigenenergies obtained by the Phase Space Integration method are in excellent agreement with those obtained by the standard WKB method.

8.4.1 Linear Potential

We have seen that a linear finite range potential may be written as

\[ V(x) = -\frac{V_0}{a} (a - |x|) \]  

(8.4.1)

with \( V_0 \) denoting the depth and \( a \), the range of the potential.

The WKB quantisation condition reads

\[ 2 \int_{\xi_1}^{\xi_2} p dx = \left( n + \frac{1}{2} \right) \hbar \]  

(8.4.2)

with

\[ p^2 = 2m \left[ E + V_0 \left(1 - \frac{|x|}{a}\right) \right]. \]  

(8.4.3)
It is straightforward to check that, for the ground state \((n = 0)\), Eqn.(8.4.2) reduces to
\[
2 \int_{x_1}^{x_2} \sqrt{\tilde{E} + \tilde{V}_0 \left(1 - \frac{|x|}{a}\right)} \, dx = a\pi
\] (8.4.4)

where \(\tilde{E} = \frac{E}{\hbar^2/2ma^2}\) and \(\tilde{V}_0 = \frac{V_0}{\hbar^2/2ma^2}\).

The turning points \(x_1\) and \(x_2\) are obtained by setting \(E - V(x) = 0\). This leads to
\[x_1 = -a \left(1 + \frac{E}{V_0}\right)\quad \text{and} \quad x_2 = +a \left(1 + \frac{E}{V_0}\right).
\]

Eqn.(8.4.4) may be written as
\[
\int_0^{a(1+\frac{E}{V_0})} \sqrt{\tilde{E} + \tilde{V}_0 \left(1 - \frac{x}{a}\right)} \, dx = \frac{a\pi}{4}.
\] (8.4.5)

Carrying out the integration, we obtain
\[
\tilde{E} = \left(\frac{3\pi\tilde{V}_0}{8}\right)^{2/3} - \tilde{V}_0.
\] (8.4.6)

The above equation may be readily solved to obtain the ground state energy for various well-depths.

Phase Space Integration Method

As an alternate approach, the non-relativistic energy relation,
\[
E = \frac{p^2}{2m} + V(x),
\] (8.4.7)

with the potential in Eqn.(8.4.1) becomes
\[
E = \frac{p^2}{2m} - \frac{V_0}{a} (a - x).
\] (8.4.8)

This leads to
\[
\frac{V_0}{a} = \frac{a}{V_0} \left( (E + V_0) - \frac{p^2}{2m} \right).
\] (8.4.9)

The quantisation condition in the phase-space is
\[
2 \int_{p_1}^{p_2} x dp = (n + \frac{1}{2}) \hbar.
\] (8.4.10)
The turning points obtained by setting \( E - V(x) = 0 \) reduce to

\[
p = \pm \sqrt{2m(E + V_0)}.
\]  

(8.4.11)

In view of Eqns. (8.4.9) and (8.4.11), Eqn. (8.4.10) becomes

\[
\int_0^{\sqrt{2m(E + V_0)}} \left\{ \frac{a}{V_0} \left( (E + V_0) - \frac{p^2}{2m} \right) \right\} dp = \left( n + \frac{1}{2} \right) \frac{\hbar}{4}.
\]  

(8.4.12)

On integration and simplification, we obtain, for the ground state energy,

\[
\frac{8a}{3V_0} (2m)^{1/2} (E + V_0)^{3/2} = \frac{\hbar}{2}.
\]  

(8.4.13)

With \( \tilde{E} \) and \( \tilde{V}_0 \) as defined earlier, the above equation reduces to

\[
\tilde{E} = \left( \frac{3\pi V_0}{8} \right)^{2/3} - V_0,
\]  

(8.4.14)

which is identical with Eqn. (8.4.6). We thus see that the phase space integration technique is an elegant method for obtaining the bound state energies and is complementary to the standard WKB approximation.

**8.42 Screened Coulomb Potential**

As an interesting example, we consider the screened Coulomb potential introduced in Chapter 2.

\[
V(x) = -\frac{g}{2a} e^{-|x|/a}
\]  

(8.4.15)

to explore the bound state energies.

**a. Non-relativistic Case**

The WKB quantisation condition Eqn. (8.4.2) for the above potential becomes

\[
2 \int_{z_1}^{z_2} \left[ 2m \left( E + \frac{g}{2a} e^{-|x|/a} \right) \right] \frac{1}{2} dx = \left( n + \frac{1}{2} \right) \hbar.
\]  

(8.4.16)
Multiplying by \(\left(\frac{a}{\hbar}\right)\) and defining, \(\bar{a} = \frac{a}{\hbar/mc}\), \(\bar{\gamma} = \frac{\gamma}{mc}\) and \(\bar{E} = \frac{E}{mc^2}\) the above equation may be written as

\[
\int_{z_1}^{z_2} \left[ 2\bar{E}\bar{a}^2 + \bar{a}\bar{\gamma}e^{-|z|/\bar{a}} \right]^\frac{1}{4} dx = \left( n + \frac{1}{2} \right) a\pi.
\] (8.4.17)

The turning points are \(x = \pm a \ln \left(\frac{2\bar{f}\bar{a}}{\bar{\gamma}}\right)\) with \(\bar{f} = -\bar{E}\).

Eqn.(8.4.17) then becomes

\[
2 \int_0^{a \ln \left(\frac{2\bar{f}\bar{a}}{\bar{\gamma}}\right)} \left[ -2\bar{f} \bar{a}^2 + \bar{a} \bar{\gamma} e^{-|z|/\bar{a}} \right]^\frac{1}{4} dx = \left( n + \frac{1}{2} \right) a\pi.
\] (8.4.18)

The above integration is evaluated using Mathematica and the energy eigenvalues for ground and excited states are computed for various values of \(\bar{a}\) and \(\bar{\gamma}\). The WKB values are compared with the exact values in Table T 8.4.1. We see that the agreement is good even for small values of \(n\) and essentially exact for higher states.

The Mathematica Program using the module RootsInRange, to extract the energy eigenvalues is given in Appendix A 8.4.1

b. Relativistic Case

To test the validity of the Relativistic WKB method, we begin with the quantisation condition,

\[
2 \int_{z_1}^{z_2} p_R dx = \left( n + \frac{1}{2} \right) \hbar.
\] (8.4.19)

For the potential \(V(x)\), defined in Eqn.(8.4.15), in the vector coupling prescription, the above condition reduces to

\[
2 \int_{z_1}^{z_2} \left[ \frac{1}{c^2} \left\{ \left( E + \frac{\theta}{2\bar{a}} \exp \left( -\frac{|z|}{\bar{a}} \right) \right)^2 - m^2c^4 \right\} \right]^\frac{1}{4} dx = \left( n + \frac{1}{2} \right) \hbar.
\] (8.4.20)
On multiplying by $\frac{a}{h}$ and defining $\tilde{a}, \tilde{g}$ and $\tilde{E}$ as before, we see that the above equation may be written as

$$\int_{\tilde{z}_1}^{\tilde{z}_2} \left[ \tilde{E}^2 \tilde{a}^2 + \frac{\tilde{g}^2}{4} e^{-2\tilde{z}/\tilde{a}} + \tilde{E} \tilde{g} \tilde{a} e^{-i\tilde{z}/\tilde{a}} - \tilde{a}^2 \right]^\frac{1}{4} dx = \left( n + \frac{1}{2} \right) a \pi \quad (8.4.21)$$

Using the criteria, $E_{\text{eff}} - V_{\text{eff}} = 0$ or equivalently $E - V = mc^2$, the turning points are found to be $x = \pm a \ln \left\{ \frac{2\tilde{a} \tilde{f}}{\tilde{g}} \right\}$ with $\tilde{f} = 1 - \tilde{E}$.

Eqn.(8.4.21) would then become

$$2 \int_{0}^{a \ln \left\{ \frac{2\tilde{a} \tilde{f}}{\tilde{g}} \right\}} \left[ \left\{ (1 - \tilde{f}) \tilde{a} + \frac{\tilde{g}}{2} e^{-x/\tilde{a}} \right\}^2 - \tilde{a}^2 \right]^\frac{1}{4} dx = \left( n + \frac{1}{2} \right) a \pi \quad (8.4.22)$$

This integral is evaluated using Mathematica. The eigenenergies are obtained using RootsInRange by specifying $\tilde{a}$ and $\tilde{g}$. The complete Mathematica program is listed in Appendix A 8.4.2.

The ground state energy for typical values of $\tilde{a}$ and $\tilde{g}$ obtained by the WKB method is compared with the exact values in the accompanying Table T 8.4.2. The excellent agreement in the values upholds the relativistic generalisation of the WKB method.

**Table T. 8.4.2**

<table>
<thead>
<tr>
<th>$\tilde{a}$</th>
<th>$\tilde{g}$</th>
<th>$\tilde{f}_{\text{WKB}}$</th>
<th>$\tilde{E}_{\text{WKB}} = 1 - \tilde{f}$</th>
<th>$\tilde{E}_{\text{exact}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.165661</td>
<td>0.833439</td>
<td>0.830395</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0.446018</td>
<td>0.553982</td>
<td>0.553443</td>
</tr>
</tbody>
</table>
8.5 Relativistic Virial Theorem

The virial theorem which formulates a general relation between the mean values of the kinetic energy, $\langle \hat{T} \rangle$ and the potential, $\langle V \rangle$, is valid both in classical mechanics and quantum mechanics. Virial Theorem may be expressed in various forms and finds application in many areas of physics like the central force problem, thermodynamics, stellar astrophysics, etc.,.

In the present section, we initially focus on the relativistic generalisation of the classical virial theorem and then address the quantum mechanical approach of Virial Theorem using the relativistic wave equations.

8.5.1 Generalisation of Virial Theorem - Classical Approach

In Classical Mechanics, virial theorem essentially deals with the time average of various mechanical quantities and is explicitly written as [Goldstein, 1969]

$$2\langle T \rangle = \langle \vec{r} \cdot \nabla V(r) \rangle \quad (8.5.1)$$

where $T$ denotes the kinetic energy and $V$, the potential energy. However in quantum mechanics, starting with the non-relativistic Hamiltonian and taking the expectation value of $\frac{d}{dt}(\vec{r}, \vec{p})$, virial theorem in exactly the same the same form is recovered.

As an extension of virial theorem to special theory of relativity, we begin with the relativistic expressions for total energy ($E$) and momentum ($\vec{p}$) as

$$E = \frac{mc^2}{\sqrt{1-\beta^2}} \quad \text{and} \quad \vec{p} = \frac{m\vec{\beta}c}{\sqrt{1-\beta^2}} \quad (8.5.2)$$

with $\beta = \frac{v}{c}$.

Starting with the celebrated equation of motion $\frac{d\vec{p}}{dt} = \vec{F}$ or equivalently

$$\frac{d}{dt} \left( \frac{m\vec{\beta}c}{\sqrt{1-\beta^2}} \right) = \vec{F}, \quad (8.5.3)$$
we take the scalar product with \( \vec{r} \) and integrate w.r.t time, to obtain

\[
\int dt \, \vec{r} \cdot \frac{d}{dt} \left( \frac{m\beta c}{\sqrt{1 - \beta^2}} \right) = \int dt \, (\vec{r} \cdot \vec{F}).
\]  

(8.5.4)

Taking the time average of the above equation over an interval \( T \), we obtain

\[
\frac{1}{T} \int_0^T dt \, \vec{r} \cdot \frac{d}{dt} \left( \frac{m\beta c}{\sqrt{1 - \beta^2}} \right) = \frac{1}{T} \left[ \frac{c}{\sqrt{1 - \beta^2}} \right]^T - \left( \frac{mc^2\beta^2}{\sqrt{1 - \beta^2}} \right).
\]  

(8.5.5)

If the system is periodic and or bounded, the first term on the right hand side vanishes and we write

\[-\left( \frac{mc^2\beta^2}{\sqrt{1 - \beta^2}} \right) = (\vec{r} \cdot \vec{F}).\]

(8.5.6)

Writing \(-\beta^2 = (1 - \beta^2) - 1\), it is straightforward to check that the above equation reduces to

\[\langle L_0 + E \rangle = -\langle \vec{F} \cdot \vec{r} \rangle,\]

(8.5.7)

noting \( L_0 = -mc^2\sqrt{1 - \beta^2} \), the form of the Lagrangian in the absence of external forces. Clearly, Eqn.(8.5.7) is the relativistic analogue of the classical virial theorem, for motions bounded in space and such that the velocities involved do not approach \( c \), the speed of light, indefinitely.

The non-relativistic limit of Eqn.(8.5.7) is accomplished by substituting for \( L_0 \) and \( E \), which leads to

\[\langle -mc^2\sqrt{1 - \beta^2} + \frac{mc^2}{\sqrt{1 - \beta^2}} \rangle = -\langle \vec{F} \cdot \vec{r} \rangle.\]

(8.5.8)

The above equation may be written as

\[\langle -mc^2\sqrt{1 - \beta^2} \left[ 1 - (1 - \beta^2)^{-1} \right] \rangle = -\langle \vec{F} \cdot \vec{r} \rangle.\]

(8.5.9)

Taylor expansion upto the first term reduces Eqn.(8.5.9) to

\[\langle mv^2\sqrt{1 - \beta^2} \rangle = -\langle \vec{F} \cdot \vec{r} \rangle.\]

(8.5.10)
In the low-speed Newtonian limit, \( v \ll c, 1 - \beta^2 \approx 1 \), we regain \( \langle 2T \rangle = -\langle \vec{F} \cdot \vec{r} \rangle \).

where \( T = \frac{1}{2} m v^2 \) is the non-relativistic kinetic energy. We notice that although neither \( L_0 \) nor \( E \) corresponds exactly to the kinetic energy in classical mechanics, their sum, \( (L_0 + E) \) indeed plays the same role as twice the kinetic energy in the classical virial theorem. This suggests that the relativistic version of the classical virial theorem has an appropriate non-relativistic limit.

For spherically symmetric potentials, i.e., \( V(r) \sim r^n \), virial theorem yields

\[
2\langle T \rangle = n\langle V \rangle. \tag{8.5.11}
\]

### 8.52 Relativistic Analogue of Virial Theorem - Quantum Mechanical Approach

To elucidate the correspondence between classical and quantum mechanics, we begin with the Hamiltonian for the relativistic Schrödinger equation,

\[
H = \sqrt{p^2 c^2 + m^2 c^4} + V(r). \tag{8.5.12}
\]

The commutator \([\vec{r}, \vec{p}, H]\) may be obtained for the various components. For the \( x \)-component, we write,

\[
[x p_x, H] = \left[ x p_x, \sqrt{p^2 c^2 + m^2 c^4} \right] + [x p_x, V(x)] \tag{8.5.13}
\]

Noting that \( \hat{x} \phi \rightarrow i \hbar \frac{d}{d p_x} \), in the momentum representation it is trivial to check that the above commutator simplifies to

\[
[x p_x, H] = \frac{i \hbar p_x^2 c^2}{\sqrt{p^2 c^2 + m^2 c^4}} - i \hbar \frac{\partial V}{\partial x}. \tag{8.5.14}
\]

Similar expressions follow for the \( y \) and \( z \) components, in view of which we may write

\[
[\vec{r}, \vec{p}, H] = \frac{i \hbar}{\sqrt{p^2 c^2 + m^2 c^4}} \left( p_x^2 + p_y^2 + p_z^2 \right) c^2 - i \hbar \left( x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} + z \frac{\partial V}{\partial z} \right) \tag{8.5.15}
\]

or equivalently,

\[
\frac{1}{i \hbar} [\vec{r}, \vec{p}, H] = \frac{p^2 c^2}{\sqrt{p^2 c^2 + m^2 c^4}} - (\vec{r} \cdot \vec{\nabla} V). \tag{8.5.16}
\]
Introducing a complete set of states and taking the expectation values we immediately notice that for stationary states, Eqn.(8.5.16) becomes

$$\left< \sqrt{p^2 c^2 + m^2 c^4} \right> - \left< \frac{m^2 c^4}{\sqrt{p^2 c^2 + m^2 c^4}} \right> - \left< \vec{r} \cdot \vec{\nabla} V \right> = 0 \tag{8.5.17}$$

wherein we have used the fact that $$\left< [\vec{r} \cdot \vec{p}, H] \right> = 0$$. Eqn.(8.5.17) may well be written as

$$\langle H \rangle - \langle mc^2 \sqrt{1 - \beta^2} \rangle - \langle \vec{r} \cdot \vec{\nabla} V \rangle = 0 \tag{8.5.18}$$

or equivalently,

$$\langle L_0 + H \rangle = -\langle \vec{r} \cdot \vec{F} \rangle \tag{8.5.19}$$

with $$L_0 = mc^2 \sqrt{1 - \beta^2}$$, as before. Eqn.(8.5.19) is precisely in the same form as Eqn.(8.5.7).

Starting with the Dirac Hamiltonian, $$\hat{H}_D = c \hat{\alpha} \cdot \vec{p} + \hat{\beta} mc^2 + V(r)$$, we write

$$[\vec{r} \cdot \vec{p}, H_0] = [\vec{r} \cdot \vec{p}, c \hat{\alpha} \cdot \vec{p}] + [\vec{r} \cdot \vec{p}, \hat{\beta} mc^2] + [\vec{r} \cdot \vec{p}, V] \tag{8.5.20}$$

For the $$x$$-component, we obtain

$$[xp_x, H] = [xp_x, c\alpha_1 p_x] + [xp_x, \beta mc^2] + [xp_x, V] \tag{8.5.21}$$

Since the following commutator relations are obeyed,

$$[\hat{\alpha}, x] = 0, \quad [\hat{\alpha}, \hat{p}] = 0,$$

$$[\hat{\beta}, x] = 0, \quad [\hat{\beta}, \hat{p}] = 0$$

we note that

$$[xp_x, H] = i\hbar c \alpha_1 p_x - i\hbar x \frac{\partial V}{\partial x}$$, in view of $$[xp_x, \beta mc^2] \equiv 0$$.

With similar relations for the other components, we arrive at

$$[\vec{r} \cdot \vec{p}, H] = i\hbar c \vec{\alpha} \cdot \vec{p} - i\hbar (\vec{r} \cdot \vec{\nabla} V) \tag{8.5.22}$$
Taking the expectation value, it follows for stationary states

\[ \langle c \vec{a} \cdot \vec{p} \rangle + \langle \vec{r} \cdot \vec{F} \rangle = 0. \]  (8.5.23)

which is equivalent to

\[ \langle c \vec{a} \cdot \vec{p} + \beta mc^2 \rangle - \langle \beta mc^2 \rangle = -\langle \vec{r} \cdot \vec{F} \rangle. \]  (8.5.24)

This implies that

\[ \langle \hat{H} \rangle - \langle \beta mc^2 \rangle = -\langle \vec{r} \cdot \vec{F} \rangle. \]  (8.5.25)

The left hand side of the above equation, defines the kinetic energy of the Dirac particle, thus leading to the virial theorem. This justifies the agreement between quantum and classical equations of motion and establishes an equivalence between Quantum Relativistic Virial Theorem and the Classical Relativistic Virial Theorem.

8.53 First Order Relativistic Correction

It is straightforward to check that Eqn.(8.5.7) may be written as

\[ \langle -mc^2 \sqrt{1 - \beta^2} + \frac{mc^2}{\sqrt{1 - \beta^2}} \rangle = -\langle \vec{F} \cdot \vec{r} \rangle \]  (8.5.26)

Taylor expansion of the above, upto the first three terms, yields,

\[ \langle m \nu^2 - \frac{1}{4} \frac{m \nu^4}{c^4} \rangle = -\langle \vec{F} \cdot \vec{r} \rangle. \]  (8.5.27)

The above equation may be transformed into

\[ \langle 2T - \frac{T}{2} \beta^2 \rangle = -\langle \vec{F} \cdot \vec{r} \rangle. \]  (8.5.28)

or equivalently

\[ \langle 2T \left(1 - \frac{1}{2} \left( \frac{T}{mc^2} \right) \right) \rangle = -\langle \vec{F} \cdot \vec{r} \rangle. \]  (8.5.29)

We remark that \( T \) is the non-relativistic kinetic energy and \( -\frac{1}{2} \left( \frac{T}{mc^2} \right) \) is the first order relativistic correction. Since \( T \ll mc^2 \), the correction is no doubt small. Higher order approximation might lead to finer corrections. Obviously, the Relativistic Virial Theorem necessarily leads to the formal classical virial theorem in the Newtonian limit.
8.6 **RESULTS AND DISCUSSION**

The WKB approximation is extensively used in quantum mechanical problems to obtain approximate expressions for the eigenenergies of a particle confined in a potential well. It is known that this method is particularly useful when one deals with slowly varying potentials. The phase space integration method, an innovative formulation introduced by Kagali serves as an alternate approach to the usual WKB method. While the former works in the co-ordinate space, the latter deals with the momentum representation. The WKB method, together with the phase space integration method may be extended to relativistic quantum mechanics, which enhances the validity of these methods.

Since the one-dimensional Klein-Gordon equation may be transformed, to the non-relativistic Schrodinger form, extension of these methods is possible in situations wherein the effective potential is a slowly-varying function of $x$ and favours binding. As specific examples, we have considered the screened Coulomb potential and the linear potential. Results of WKB calculations compare well with the exact values and the agreement is excellent for excited states. The linear finite ranged potential, we have considered is of great interest in studying the energy spectrum of a quark-antiquark bound system, called the quarkonium. More importantly, it is seen that the phase space integration method gives the same energy eigenvalues as those obtained by the standard WKB procedure.

The well-known virial theorem of classical mechanics which applies to a wide variety of systems ranging from an ideal gas to a cluster of galaxies is known to have a quantum analogue. Relativistic generalisation of the virial theorem is interesting in itself, as it would establish a correspondence between classical, non-relativistic and relativistic quantum mechanics. The classical and the quantum mechanical approaches are discussed in detail. Starting with the relativistic Hamiltonian, virial theorem in the standard form is accomplished and is seen to have an appropriate non-relativistic limit.

Thus the relativistic generalisation of the WKB method, phase-space integration method and virial theorem serve as novel approach to solve bound state problems in relativistic quantum mechanics.
**Table T 8.4.1**

Comparison between Exact and WKB Energies for Screened Soulomb Potential (NR Case)

\[ a \text{ in units of } \frac{\hbar}{mc}, g \text{ in units of } \hbar c \text{ and } V_0 \text{ and } E \text{ in } mc^2 \]

\[(a = 1\lambda)\]

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<th>(g)</th>
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<th>(n = 2)</th>
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<td>(E_{0}^{WKB})</td>
<td>(E_{1}^{exact})</td>
<td>(E_{1}^{WKB})</td>
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