Chapter 5

Polarization of Radiation

5.1 Introduction

Quite often, measurements on the electromagnetic radiation are carried out mainly on the basis of the intensity of radiation as a function of wavelength. However, an important and frequently overlooked aspect of electromagnetic radiation is its state of polarization, which is related to the orientation of the electric field of the wave.

The states of polarization of any given beam of electromagnetic radiation can be conveniently characterized in terms of four quantities that can independently be measured with appropriate instruments. These observables are the four Stokes parameters \((I, Q, U, V)\) which were formulated by George Stokes in 1852 and introduced into astrophysics by Chandrasekar in 1946. The advantage of this description is its capacity to treat partially polarized beams as well.
In this chapter, we will present a brief account of the classical description of radiation and introduce the polarization states of the radiation and their representation on the Poincaré sphere. Then we will show how the polarization properties of a transverse plane wave radiation can be characterized in terms of a Stokes vector. We use the radiation density matrix formalism. We also relate the Stokes parameters to irreducible tensors of polarized radiation field. Finally, the radiation field is described in terms of its electric and magnetic multipole states, in a second quantized formalism.

5.2 Radiation in the Coulomb gauge

The vector potential \( \mathbf{A}(\mathbf{r}, t) \) representing the electromagnetic radiation satisfies the free field Maxwell equation

\[
(\nabla^2 - \frac{\partial^2}{\partial t^2}) \mathbf{A}(\mathbf{r}, t) = 0 ,
\]

in the Coulomb gauge

\[
\nabla \cdot \mathbf{A} = 0 .
\]

The associated electric and magnetic fields \( \mathbf{E} \) and \( \mathbf{H} \) are then given by the well known expressions

\[
\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} , \quad \mathbf{H} = \nabla \times \mathbf{A} .
\]

A monochromatic plane wave solution of Eq. (5.1), characterized by the wave vector \( \mathbf{k} \) with polar coordinates \( (k, \theta_k, \phi_k) \) and polarization \( \hat{e} \), is of the form

\[
\mathbf{A}(\mathbf{r}, t) = \hat{e} e^{i\mathbf{k} \cdot \mathbf{r}} e^{-i\omega t} ,
\]
where \( \omega = k = |k| \). The polar angles \((\theta_k, \phi_k)\) of \(k\) may be defined (see Fig. 5.1) with respect to a conveniently chosen right-handed Cartesian coordinate system, say, the atmospheric frame.

### 5.3 Polarization states of radiation

If \(\hat{\epsilon}_x, \hat{\epsilon}_y\) denote a pair of mutually orthogonal unit vectors in a plane perpendicular to \(k\) such that \((\hat{\epsilon}_x, \hat{\epsilon}_y, \hat{k})\) constitute a right-handed frame, the polarization \(\hat{\epsilon}\) in Eq. (5.4) may, in general, be represented (Ramachandran et al. 1995) by

\[
\hat{\epsilon}(\alpha, \beta) = \cos \beta (\hat{\epsilon}_x \cos \alpha + \hat{\epsilon}_y \sin \alpha) + \sin \beta (-\hat{\epsilon}_x \sin \alpha + \hat{\epsilon}_y \cos \alpha) ,
\]

in terms of two angles \(0 \leq \alpha < \pi\), which represents orientation and \(-\pi/4 \leq \beta \leq \pi/4\) which represents the ellipticity of the polarization. By setting \(\beta = 0\) in Eq. (5.5), all possible linear polarizations are obtained, whereas \(\alpha = 0, \beta = \pm \pi/4\) correspond to right circular and left circular states as defined by Shore and Menzel (1968), i.e.,

\[
\hat{\epsilon}_{\pm 1} = \frac{1}{\sqrt{2}} \left( \hat{\epsilon}_x \pm i \hat{\epsilon}_y \right) = \hat{\epsilon}(0, \pm \pi/4) .
\]

The above two states of circular polarization correspond to helicity \(\mu = \pm 1\) eigenstates of a photon (with spin \(J = 1\)), and they are invariant under Lorentz transformations.

We may also express

\[
\hat{\epsilon}(\alpha, \beta) = \sum_{\mu=\pm 1} c_\mu(\alpha, \beta) \hat{\epsilon}_\mu ,
\]

in terms of the circular polarization states Eq. (5.6), where

\[
c_{\pm 1}(\alpha, \beta) = \frac{1}{\sqrt{2}} (\cos \beta \pm \sin \beta) e^{\mp i \alpha} .
\]
Clearly, $\hat{e}(\alpha, \beta)$ satisfies the transversality condition $k \cdot \hat{e}(\alpha, \beta) = 0$ which follows from Eq. (5.2). The domain $0 < \beta < \pi/4$ corresponds to right elliptical states whereas $-\pi/4 < \beta < 0$ refers to left elliptical states of polarization. The linear polarization states are represented on the equatorial line of the Poincâtre sphere with north and south poles corresponding to right and left circular states of polarization respectively, if we define the polar angles $(\theta_P, \phi_P)$ on the Poincâtre sphere through $\theta_P = \pi/2 - 2\beta$ and $\phi_P = 2\alpha$ (see Fig. 5.2). It is well known that every pair of diametrically opposite points $(\theta_P, \phi_P)$ and $(\pi - \theta_P, \pi + \phi_P)$ on the Poincâtre sphere represent mutually orthogonal states of polarization. Noting that $\hat{e}$ is a vector in a 2-dimensional complex linear vector space, we have

$$\langle \hat{e}(\alpha, \beta)|\hat{e}(\pi/2 + \alpha, -\beta)\rangle = 0 ,$$

(5.9)

using the Dirac notation for the inner product. If $I(\alpha, \beta)$ denotes the intensity of radiation with polarization $\hat{e}(\alpha, \beta)$, the Stokes parameters $I, Q, U, V$ as defined by Chandrasekhar (1950) are given by

$$I = I(\alpha, \beta) + I(\pi/2 + \alpha, -\beta) = S_0 ,$$

$$Q = I(0, 0) - I(\pi/2, 0) = S_1 ,$$

$$U = I(\pi/4, 0) - I(3\pi/4, 0) = S_2 ,$$

$$V = I(0, \pi/4) - I(0, -\pi/4) = S_3 .$$

(5.10)

It may be noted that $Q, U, V$ are essentially $I(\alpha, \beta) - I(\pi/2 + \alpha, -\beta)$ for $(\alpha, \beta) = (0, 0), (\pi/4, 0)$ and $(0, \pi/4)$ respectively. In fact $I(\alpha, \beta) - I(\pi/2 + \alpha, -\beta)$ for any $(\alpha, \beta)$ may be evaluated in terms of $Q, U, V$. If the radiation is unpolarized, $I(\alpha, \beta) =
\[ I(n/2 + \alpha, -\beta) \text{ for every } (\alpha, \beta) \text{ and hence } Q = U = V = 0. \] The Stokes vector is denoted by

\[
\mathbf{S}(k) = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix}, \tag{5.11}
\]

for polarized radiation with wave vector \( k \).

### 5.4 Density matrix formalism for radiation

The state of polarization of radiation may also be described using the density matrix formalism (McMaster 1961). Since there are two states of polarization for radiation, the density matrix \( \rho^7 \) is a \( (2 \times 2) \) hermitian matrix. Using the important properties of Pauli matrices, any two-dimensional hermitian matrix can be expressed as a linear combination of the unit matrix \( 1 \) and the Pauli matrices \( \sigma \). Therefore, the radiation density matrix assumes the form (Blum 1996)

\[
\rho^7 = \frac{1}{2} [1 + \sigma \cdot \mathbf{S}], \tag{5.12}
\]

where the rows and columns of the Pauli matrices are labelled by the left and right circular polarization states \( |\mu = \pm 1\rangle \) of the radiation.

Using the above Eq. (5.12), the Stokes parameters are also obtained through the expressions

\[
S_0 = Tr \rho^7 = (\rho_{+1,+1} + \rho_{-1,-1}) = I,
\]

\[
S_1 = Tr(\rho^7 \sigma_x) = (\rho_{+1,-1} + \rho_{-1,+1}) = Q,
\]
\[ S_2 = Tr(\rho^7 \sigma_y) = i(\rho_{+1,-1} - \rho_{-1,+1}) = U, \]
\[ S_3 = Tr(\rho^7 \sigma_z) = (\rho_{+1,+1} - \rho_{-1,-1}) = V. \]  

Thus, we have
\[
\rho^7 = \frac{1}{2} \begin{bmatrix}
I + V & 0 & Q - iU \\
0 & 0 & 0 \\
Q + iU & 0 & I - V
\end{bmatrix},
\]
explicitly.

5.5 The relation between Stokes parameters and polarized radiation field tensors

Noting that the spin of the photon is \( J = 1 \), we can also express the Stokes parameters in terms of the irreducible tensor parameters \( \mathcal{T}_Q \) of the polarized radiation field, noting that the elements in the second row and second column of Eq. (4.19) will be zero due to the absence of longitudinal state \(|1, 0\rangle\), when the axis of quantization is chosen parallel to the direction \( k \) of propagation. Thus, the radiation density matrix, \( \rho^7 \) assumes the form
\[
\rho^7 = \frac{1}{2} \begin{bmatrix}
1 + \sqrt{3/2} \mathcal{T}_0^1 + \sqrt{1/2} \mathcal{T}_0^2 & 0 & \sqrt{3} \mathcal{T}_2^2 \\
0 & 0 & 0 \\
\sqrt{3} \mathcal{T}_2^2 & 0 & 1 - \sqrt{3/2} \mathcal{T}_0^1 + \sqrt{1/2} \mathcal{T}_0^2
\end{bmatrix}
\]

By comparing Eqs. (5.15) and (5.14), the Stokes parameters are given by
\[ Q = \frac{1}{\sqrt{3}}(\mathcal{T}_2^2 + \mathcal{T}_2^2), \]
\[ U = \frac{1}{\sqrt{3}}(\mathcal{T}_2^2 - \mathcal{T}_2^2), \]
\[ V = \sqrt{2/3} \tau_0^1. \] (5.16)

### 5.6 Multipole states of the radiation field

Expressing the operator \( \nabla^2 \) in Eq. (5.1) in terms of polar coordinates \((r, \theta, \phi)\) and defining the vector spherical harmonics \( T_{LM}(\hat{r}) \) following Rose (1957), monochromatic 'magnetic' and 'electric' \(2L\)-pole solutions of the Maxwell equation are obtained as

\[ M_{LM}(r) = f_L(kr) T_{LM}(\theta, \phi), \] (5.17)

\[ E_{LM}(r) = \left[ -\sqrt{\frac{L}{2L+1}} f_{L+1}(kr) T_{L+1M}(\theta, \phi) \right. \]
\[ \left. + \sqrt{\frac{L+1}{2L+1}} f_{L-1}(kr) T_{L-1M}(\theta, \phi) \right], \] (5.18)

where \( f_L(kr) \) denote solutions of the radial equation with \( k = \omega \). They are orthogonal to each other and are states with opposite parity. Whereas Rose (1957) refers to \( \hat{e}(0, \pm \pi/4) \) states as 'left' and 'right' circular polarization states, we follow here the nomenclature of Shore and Menzel (1968) and Stenflo (1994).

Thus, the plane wave solutions Eq. (5.4) corresponding to 'right' and 'left' circular polarization states are obtained by setting \( \hat{e} \) as \( \hat{e}_\mu \) with \( \mu = \pm 1 \) in Eq. (5.4) respectively. Following Rose (1957), we may also express the plane wave solution with \( \hat{e} = \hat{e}_\mu \) in terms of the multipole solutions of Eqs. (5.17) and (5.18) as

\[ A_{k,\mu}(r,t) = (2\pi)^{1/2} \sum_{L=1}^{\infty} (i)^L (2L+1)^{1/2} \sum_{M=-L}^{L} D_{LM}^\mu(\phi_k, \theta_k, 0) \]
\[ \left[ M_{LM}(\hat{r}) + i \mu E_{LM}(\hat{r}) \right] e^{-i\omega t}. \] (5.19)
Starting with the solutions given by Eqs. (5.17) and (5.18) for the vector potential \( A \), we obtain the corresponding 'magnetic' or 'electric' \( 2^L \)-pole solutions of the electric and magnetic fields \( E \) and \( H \) using Eq. (5.3). We denote the electric and magnetic fields so derived, by \( E_{LM}^{(m)}, H_{LM}^{(m)} \) and \( E_{LM}^{(e)}, H_{LM}^{(e)} \) where the superfix \((m)\) corresponds to Eq. (5.17) and the superfix \((e)\) corresponds to Eq. (5.18). It follows that

\[
H_{LM}^{(m)} = E_{LM}^{(e)}, \quad H_{LM}^{(e)} = -E_{LM}^{(m)}.
\]  

(5.20)

We may also express Eq. (5.4) with \( \hat{\varepsilon} = \hat{\varepsilon}(\alpha, \beta) \) as

\[
A_{k,\hat{\varepsilon}(\alpha,\beta)}(r, t) = \sum_{\mu} c_{\mu}(\alpha, \beta) A_{k,\mu}(r, t),
\]

(5.21)

using Eq. (5.7) for any arbitrary state of polarization \( \hat{\varepsilon}(\alpha, \beta) \).

### 5.7 Quantum field

We know that Maxwell's equations are the basic equations describing all classical electromagnetic phenomena. In quantum field theory, the Maxwell equations are quantized which leads to the notion of the vacuum also as a state of the field and this facilitates spontaneous emission. Spontaneous emission as well as induced emission and absorption processes are accommodated in quantum electrodynamics (QED) by treating the vector potential \( A(r, t) \) as a quantum field

\[
A(r, t) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \frac{d^3k}{\sqrt{2\omega_k}} \sum_{\mu=\pm1} \left[ a_{k,\mu} A_{k,\mu}(r, t) + a_{k,\mu}^\dagger A_{k,\mu}(r, t)^* \right],
\]

(5.22)

where \( \mu = \pm 1 \) denote the two circular polarization states orthogonal to \( k \), while \( a_{k,\mu}^\dagger \) and \( a_{k,\mu} \) are
the creation and annihilation operators respectively. They satisfy the commutation relations

\[ [a_{k,\mu}, a_{k',\mu'}^\dagger] = \delta(k - k') \delta_{\mu,\mu'} \] (5.23)

Any plane wave, \( A_{k,\mu}(r, t) \) is essentially an eigenstate of linear momentum, characterized by an appropriate wave number \( k \), whereas atomic processes involve absorption and emission of radiation which are eigenstates of total angular momentum \( L \) and parity (represented by the ‘electric’ and ‘magnetic’ solutions). When an atomic electron makes a transition from a bound state to another bound state, angular momentum and parity are conserved, which is in contrast to processes like Compton effect on a free electron, where linear momentum is conserved. Therefore, the multipolar nature of the radiation field has to be exactly taken into account, while describing the processes like emission or absorption by a bound electron. Consequently, it is advantageous to use Eq. (5.19) and express \( A_{k,\mu}(r, t) \) and \( A_{k,\mu}(r, t)^* \) in Eq. (5.22) in terms of the multipole solutions given by Eqs. (5.17) and (5.18).
Figure 5.1: The diagram representing the polarization states of radiation.
Figure 5.2: The polarization states of radiation on the Poincaré sphere.