Chapter 3

σ(∗)-RINGS

For a ring endomorphism $\sigma$ of a ring $R$, $\sigma$ is said to be a rigid endomorphism if $a\sigma(a) = 0$ implies $a = 0$ for $a \in R$, see [52]. A ring $R$ is called a $\sigma$-rigid ring if there exists a $\sigma$-rigid endomorphism of $R$.

In [53], Kwak extend the $\sigma$-rigid property of a ring $R$ to the prime radical $P(R)$ of $R$. We study the characterisation of a $\sigma(\ast)$-ring and their related properties. Note that any rigid endomorphism is a monomorphism and $\sigma$-rigid rings are reduced rings (i.e. rings without non-zero nilpotent elements), but there exist an endomorphism of a commutative reduced ring which is not rigid, see ([49], Example 9). The next definition appears in [53].

**Definition 3.1.** Let $\sigma$ be an endomorphism of a ring $R$. We say that $R$ is a $\sigma(\ast)$-ring if $a\sigma(a) \in P(R)$ implies that $a \in P(R)$ for $a \in R$.

**Example 3.1.** (Example (2) of Kwak [53]) Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where $F$ is a field and $\sigma : R \to R$ is defined by $\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. Then $P(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ and it can be seen that $\sigma$ is an endomorphism of $R$ and that $R$ is a $\sigma(\ast)$-ring.
We note that the above ring is not \( \sigma \)-rigid. For let \( 0 \neq a \in F \). Then
\[
\begin{pmatrix}
0 & a \\
0 & 0
\end{pmatrix}
\sigma
\begin{pmatrix}
0 & a \\
0 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}, \text{ but } \begin{pmatrix}
0 & a \\
0 & 0
\end{pmatrix} \neq \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}
\]

**Example 3.2.** (Example (1.3) of Bhat [18]) Let \( R = \mathbb{C} \), the field of complex numbers. Then \( \sigma : R \to R \) be the map defined by \( \sigma(a + \iota b) = a - \iota b \), for all \( a, b \in R \) is an automorphism of \( R \) and \( R \) is a \( \sigma \)-rigid ring.

### 3.1 Completely prime ideals of skew polynomial rings over \( \sigma(\ast) \)-rings

We begin this section with the following proposition:

**Proposition 3.1.** (Proposition (2) of [19]) Let \( R \) be a ring and \( \sigma \) an endomorphism of \( R \). If \( R \) is a \( \sigma(\ast) \)-ring, then \( P(R) \) is completely semiprime.

**Proof.** Let \( a \in R \) be such that \( a^2 \in P(R) \). Then
\[
a\sigma(a)\sigma(a) = \sigma(a)\sigma(a)\sigma^2(a) \in \sigma(P(R)) = P(R).
\]
So \( a\sigma(a) \in P(R) \) and it follows that \( a \in P(R) \).

**Proposition 3.2.** (Proposition (3) of [19]) Let \( R \) be a \( \sigma(\ast) \) ring and \( U \in MinSpec(R) \) such that \( \sigma(U) = U \). Then \( U(S(R)) = U[x; \sigma] \) is a completely prime ideal of \( S(R) = R[x; \sigma] \).

**Proof.** The Proposition (3.1) implies that \( P(R) \) is completely semiprime ideal of \( R \) and \( U \) is completely prime by ([69], Proposition 1.11). Note that \( \sigma \) can be extended to an automorphism \( \overline{\sigma} \) of \( R/U \). It is well known that \( S/U(S(R)) \simeq (R/U)[x; \overline{\sigma}] \) and consequently \( U(S(R)) \) is a completely prime ideal of \( S(R) \).
Theorem 3.3. (Theorem (2.4) of Bhat and Kumari [15]) Let $R$ be a Noetherian ring and $\sigma$ an automorphism of $R$. Then $R$ is a $\sigma(\ast)$-ring if and only if for each minimal prime $U$ of $R$, $\sigma(U) = U$ and $U$ is completely prime ideal of $R$.

Proof. Let $R$ be a Noetherian ring such that $\sigma(U) = U$, $U$ is a completely prime ideal of $R$ and $a \in R$ such that $a\sigma(a) \in P(R) = \cap_{i=1}^{n}U_i$, where $U_i$ are the minimal primes of $R$. Since for each $i, \sigma(a) \in U_i$, then $a \in U_i$. Hence, $a \in P(R)$ and it follows that $R$ is a $\sigma(\ast)$-ring.

Conversely, suppose that $R$ is a $\sigma(\ast)$-ring. Then by ([14], Proposition (2.1)) we have that $P(R)$ is a completely semiprime ideal of $R$ and $\sigma(U) = U$, for all $U \in \text{MinSpec}(R)$.

Now suppose that $U = U_1$ is not completely prime. Then there exists $a, b \in R \setminus U$ with $ab \in U$ and we obtain that $c$ be any element of $b(U_2 \cap U_3 \cap ... \cap U_n)a$. Then $c^2 \in \cap_{i=1}^{n}U_i = P(R)$ implies $b(U_2 \cap U_3 \cap ... \cap U_n)a \subseteq U$. Hence, $bR(U_2 \cap U_3 \cap ... \cap U_n)Ra \subseteq U$ and by the fact that $U$ is prime, we have that $a \in U, U_i \subseteq U$, for some $i \neq 1$ or $b \in U$. None of these can occur, so $U$ is completely prime. □

Proposition 3.4. (Proposition (2.1) of [24]) Let $R$ be a right Noetherian $\mathbb{Q}$-algebra, $\sigma$ an automorphism of $R$ such that $R$ is a $\sigma(\ast)$-ring and $\delta$ a $\sigma$-derivation of $R$. Then $\sigma(U) = U$ and $\delta(U) \subseteq U$, for all $U \in \text{MinSpec}(R)$.

Proof. We will first show that $P(R)$ is completely semiprime. In fact, let $a \in R$ such that $a^2 \in P(R)$. Then

$$a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a) \in \sigma(P(R)) = P(R).$$

Thus, $a\sigma(a) \in P(R)$ and we have that $a \in P(R)$.

We next show that $\sigma(U) = U$, for all $U \in \text{MinSpec}(R)$. In fact, let $U = U_1$
be a minimal prime ideal, $U_2, U_3, ..., U_n$ the other minimal primes of $R$ and suppose that $\sigma(U) \neq U$. Then $\sigma(U)$ is also a minimal prime ideal of $R$. Renumbering the minimal prime we have that $\sigma(U) = U_n$. Let $a \in \cap_{i=1}^{n-1} U_i$. Then $\sigma(a) \in U_n$, and so $a\sigma(a) \in \cap_{i=1}^{n} U_i = P(R)$. By assumption we have that $a \in P(R)$ and consequently $\cap_{i=1}^{n-1} U_i \subseteq U_n$ which implies that $U_i \subseteq U_n$, for some $i \neq n$, which is impossible. Hence $\sigma(U) = U$, for all $U \in \text{MinSpec}(R)$.

Let now $T = \{a \in U$ such that $\delta^k(a) \in U$, for all integers $k \geq 1\}$. First of all, we will show that $T$ is an ideal of $R$. Let $a, b \in T$. Then $\delta^k(a) \in U$ and $\delta^k(a) \in U$, for all integers $k \geq 1$. Now $\delta^k(a - b) = \delta^k(a) - \delta^k(b) \in U$, for all $k \geq 1$. Therefore, $a - b \in T$. Therefore, $T$ is a $\delta$-invariant ideal of $R$.

We will now show that $T \in \text{Spec}(R)$. Suppose $T \notin \text{Spec}(R)$. Let $a \notin T, b \notin T$ be such that $aRb \subseteq T$. Let $t, s$ be least such that $\delta^t(a) \notin U$ and $\delta^s(b) \notin U$. Now there exists $c \in R$ such that $\delta^t(a)c \sigma^t(\delta^s(b)) \notin U$. Let $d = \sigma^{-t}(c)$. Now $\delta^{t+s}(adb) \in U$ as $aRb \subseteq T$. This implies on simplification that $\delta^t(a)\sigma^t(d)\sigma^t(\delta^s(b)) + u \in U$, where $u$ is sum of terms involving $\delta^l(a)$ or $\delta^m(b)$, where $l < t$ and $m < s$. Therefore, by assumption $u \in U$ which implies that $\delta^t(a)\sigma^t(d)\sigma^t(\delta^s(b)) \in U$. This is a contradiction. Therefore, our supposition must be wrong. Hence $T \in \text{Spec}(R)$. Now $T \subseteq U$, so $T = U$ as $U \in \text{MinSpec}(R)$. Hence $\delta(U) \subseteq U$. \hfill $\square$

**Theorem 3.5.** (Theorem (2.6) of Bhat [15]) Let $R$ be a Noetherian ring and $\sigma$ an automorphism of $R$ such that $R$ is a $\sigma(\ast)$-ring. Then $R[x; \sigma]$ is also a $\sigma(\ast)$-ring.

**Proof.** First of all show that $\sigma(P) = P$, for all $P \in \text{MinSpec}(S(R))$. Let $P \in \text{MinSpec}(S(R))$. Then by ([10], Theorem (2.4)) there exists $U \in \text{MinSpec}(R)$ such that $P = U^0[x; \sigma]$. Now $R$ is a $\sigma(\ast)$-ring implies that
\[ \sigma(U) = U \] by Theorem (3.3) and therefore, \( U^0 = U \). So \( P = U[x; \sigma] \) and thus \( \sigma(P) = P \).

We now show that \( P \) is completely prime. Let

\[ f(x) = x^n a_n + x^{n-1} a_{n-1} + \ldots + a_0 \]

and

\[ g(x) = x^m b_m + x^{m-1} b_{m-1} + \ldots + b_0 \]

in \( R[x; \sigma] \) be such that

\[ f(x)g(x) \in P = U[x; \sigma] \] and \( g(x) \notin U[x; \sigma] \).

This implies that

\[ x^{n+m} \sigma^m(a_n)b_m + x^{n+m-1} \sigma^m(a_{n-1})b_m + x^{n+m-1} \sigma^{m-1}(a_n)b_{m-1} + \ldots + a_0 b_0 \in U[x; \sigma] \]

Now \( g(x) \notin U[x; \sigma] \) (say \( b_m \notin U \)). Now \( \sigma^m(a_n)b_m \in U \). Also \( U \) is completely prime by Theorem (3.3), therefore, \( \sigma^m(a_n) \in U \); i.e. \( a_n \in U \).

Now \( \sigma^m(a_{n-1})b_m + \sigma^{m-1}(a_n)b_{m-1} \in U \) implies that \( \sigma^m(a_{n-1})b_m \in U \). Now \( b_m \notin U \) implies that \( \sigma^m(a_{n-1}) \in U \); i.e. \( a_{n-1} \in U \).

With the same process in a finite number of steps it can be seen that \( a_i \in U \), for all \( i, 0 \leq i \leq n-2 \) also.

Therefore, \( a_i \in U \), for all \( i, 0 \leq i \leq n \); i.e. \( f(x) \in P = U[x; \sigma] \).

Thus, \( \sigma(P) = P \) and \( P \) is completely prime, for all \( P \in \text{MinSpec}(S(R)) \). Moreover \( S(R) \) is Noetherian by ([44], Theorem (1.14)). Hence by Theorem (3.3), we get that \( R[x; \sigma] \) is also a \( \sigma(*) \)-ring.

It has been also proved that if \( \sigma \) is an automorphism of \( R \), then it can be extended to an automorphism (say \( \overline{\sigma} \)) of \( R[x; \sigma] \) such that \( \sigma(x) = x \).

**Theorem 3.6.** (Theorem (2.4) of [24]) Let \( R \) be a Noetherian \( \mathbb{Q} \)-algebra, \( \sigma \)
an automorphism of \( R \) and \( \delta \) a \( \sigma \)-derivation of \( R \) such that \( \sigma(\delta(a)) = \delta(\sigma(a)) \), for all \( a \in R \). Then \( R \) is a \( \sigma(\ast) \)-ring implies that \( O(R) = R[x; \sigma, \delta] \) is a Noetherian \( \sigma(\ast) \)-ring.

**Proof.** Let \( R \) be a Noetherian ring and \( \sigma \) an automorphism of \( R \) such that \( R \) is a \( \sigma(\ast) \)-ring. We shall prove that \( O(R) = R[x; \sigma, \delta] \) is a Noetherian \( \sigma(\ast) \)-ring. For this we will show that any minimal \( P \in \text{MinSpec}(O(R)) \) is completely prime and \( \sigma(P) = P \).

Let \( P \in \text{MinSpec}(O(R)) \). Then by ([14], Lemma (2.2)) \( P \cap R \in \text{MinSpec}(R) \). Now \( R \) is a \( \sigma(\ast) \)-ring implies that \( \sigma(P \cap R) = P \cap R \) and \( P \cap R \) is a completely prime ideal of \( R \) by Theorem (3.3). Now Proposition (3.4) implies that \( \delta(P \cap R) \subseteq P \cap R \). Now ([16], Theorem (2.4)) implies that \( O(P \cap R) \) is a completely prime ideal of \( O(R) \). Now \( O(P \cap R) \subseteq P \) implies that \( O(P \cap R) = P \) as \( P \) is minimal. Now \( \sigma(P \cap R) = P \cap R \) implies that \( \sigma(P) = P \).

Thus, \( \sigma(P) = P \) and \( P \) is completely prime, for all \( P \in \text{MinSpec}(O(R)) \). Moreover \( O(R) = R[x; \sigma, \delta] \) is Noetherian by ([44], Theorem (2.6)). Hence by Theorem (3.3), \( R[x; \sigma, \delta] \) is a \( \sigma(\ast) \)-ring. \( \square \)

**Theorem 3.7.** (Theorem (2.5) of [24]) Let \( R \) be a right Noetherian \( \mathbb{Q} \)-algebra, \( \sigma \) an automorphism of \( R \) such that \( R \) is a \( \sigma(\ast) \)-ring and \( \delta \) a \( \sigma \)-derivation of \( R \). Then

1. If \( U \) is a minimal prime ideal of \( R \), then \( O(U) \) is a minimal prime ideal of \( O(R) \) and \( O(U) \cap R = U \).
2. If \( P \) is a minimal prime ideal of \( O(R) \), then \( P \cap R \) is a minimal prime ideal of \( R \).

**Proof.** (1) Let \( U \) be a minimal prime ideal of \( R \). Then by Proposition (3.4) \( \sigma(U) = U \) and \( \delta(U) \subseteq U \). Now on the same lines as in ([41], Theorem (2.22)) we have \( O(U) \in \text{Spec}(O(R)) \). Suppose \( L \subset O(U) \) be a minimal prime ideal
of $O(R)$. Then $L \cap R \subset U$ is a prime ideal of $R$, a contradiction. Therefore, $O(U) \in \text{MinSpec}(O(R))$. Now it is easy to see that $O(U) \cap R = U$.

(2) We note that $\sigma$ can be extended to an endomorphism (say $\sigma$) of $R[x;\sigma,\delta]$ by $\sigma(\sum_{i=0}^{m} x^i a_i) = \sum_{i=0}^{m} x^i \sigma(a_i)$. Also $\delta$ can be extended to a $\sigma$-derivation (say $\delta$) of $R[x;\sigma,\delta]$ by $\delta(\sum_{i=0}^{m} x^i a_i) = \sum_{i=0}^{m} x^i \delta(a_i)$.

Now Theorem (3.6) implies that $O(R) = R[x;\sigma,\delta]$ is a Noetherian $\sigma(\ast)$-ring. Therefore, Proposition (3.4) implies that $\sigma(P) = P$ and $\delta(P) = P$. So $\sigma(P \cap R) = P \cap R$ and $\delta(P \cap R) \subseteq P \cap R$. Now it can be seen that $P \cap R \in \text{Spec}(R)$ and therefore, $O(P \cap R) \in \text{Spec}(O(R))$. Now $O(P \cap R) \subseteq P$ implies that $O(P \cap R) = P$. □

3.2 Near completely prime ideal rings of skew polynomial rings over $\sigma(\ast)$-rings

**Theorem 3.8.** (Theorem (3.6) of [32]) Let $R$ be a Noetherian ring and $\sigma$ an automorphism of $R$ such that $R$ is a $\sigma(\ast)$-ring. Then $S(R) = R[x;\sigma]$ is Noetherian near completely prime ideal ring.

**Proof.** Since $R$ is Noetherian, then by Theorem (2.1) we have that $S(R)$ is Noetherian. Let $P$ be a minimal prime ideal of $S(R) = R[x;\sigma]$. Then by Lemma(2.4) we have that $P \cap R \in \text{MinSpec}(R), \sigma(P \cap R) = P \cap R$ and $(P \cap R)[x;\sigma] = P$. By the fact that $R$ is Noetherian $\sigma(\ast)$-ring we have that $P \cap R$ is completely prime ideal by Theorem (3.3). By Theorem (2.8) we have that $P = (P \cap R)[x;\sigma]$ is completely prime. So, $R[x;\sigma]$ is near completely prime ideal ring. □
Question. Let $R$ be a Noetherian ring and $\sigma$ an automorphism of $R$ such that $R$ is a $\sigma(\ast)$-ring and $\delta$ a $\sigma$-derivation of $R$. Is $O(R) = R[x; \sigma, \delta]$ a Noetherian near completely prime ideal ring?

Question. Let $R$ be a Noetherian ring and $\sigma$ an automorphism of $R$ such that $R$ is a near completely prime ideal ring. Is $S(R) = R[x; \sigma]$ a near completely prime ideal ring?