CHAPTER 1

PRELIMINARIES

1.1 Introduction:

The principal goal of this chapter is to highlight some fundamental concepts, definitions and results which are absolutely necessary to carry out our investigation. These concepts and results are collected from different literature, monographs, published papers and thesis.

1.2 Dynamical system:[5]

Any system that evolves with time is called a dynamical system. Dynamical systems theory is an area of mathematics used to describe the behaviour of complex systems, usually by employing differential equations or difference equations. When differential equations are employed, the theory is called continuous dynamical systems. When difference equations are employed, the theory is called discrete dynamical systems.

A dynamical system consists of a set of possible states, together with a rule that determines the present state in terms of past state.

Let us consider the function \( f(x) = 2x \) as a rule that assigns to each number \( x \) a number twice as large. This is a simple mathematical model. We might imagine that \( x \) denotes the population of bacteria in a laboratory culture and that \( f(x) \) denotes the population one hour later. Then the rule expresses the fact that the population doubles every hour. If the culture has an initial population of 10,000 bacteria, then after one hour there will be \( f(10,000) = 20,000 \) bacteria, after two hours there will be \( f(f(10,000)) = 40,000 \) bacteria, and so on.

This is an example of a simple dynamical system whose states are population levels, that change with time under the rule \( x_n = f(x_{n-1}) = 2x_{n-1} \). Here the variable \( n \) stands for time, and \( x_n \) designates the population at time \( n \).

1.3 Autonomous dynamical system:[46,100]

An \( n^{th} \) order autonomous continuous dynamical system is defined as

\[
\dot{x} = f(x), \quad x(t_0) = x_0
\]
where $\dot{x} \equiv \frac{dx}{dt}$, $x(t) \in R^n$ is the state at time $t$ and $f: R^n \rightarrow R^n$ is called the vector field. Since the vector field does not depend on time, the initial time may always be taken as $t_0 = 0$. The solution to the above equation with the initial condition $x_0$ at time $t = 0$ is called the trajectory and is denoted by $\phi_t(x_0)$. The mapping $\phi_t: R^n \rightarrow R^n$ is called the flow of the system. The above dynamical system is linear if $f(x)$ is linear.

1.4 Non-autonomous dynamical system:[46,100]

An $n^{th}$ order non-autonomous dynamical system is defined by

$$\dot{x} = f(x, t)x(t_0) = x_0$$

The vector field $f$ depends on time and unlike the autonomous case, the initial time cannot be arbitrarily set as 0. The solution of the above equation passing through $x_0$ at time $t_0$ is $\phi_t(x_0, t_0)$. The system is linear if $f$ is linear with respect to $x$.

If there exists a $T > 0$ such that $f(x, t) = f(x, t + T)$ for all $x$ and for all $t$, then the system is said to be time periodic with period $T$. The smallest such period is called the minimal period.

An $n^{th}$ order time periodic non-autonomous system can always be converted to an $(n + 1)^{th}$ order autonomous system by taking an extra state $\theta = \frac{2\pi t}{T}$, then the autonomous system is given by

$$\dot{x} = f \left( x, \frac{\theta t}{2\pi} \right), x(0) = x_0, \dot{\theta} = \frac{2\pi}{T}, \theta(0) = \frac{2\pi t_0}{T}$$

Since $f$ is time periodic with period $T$, the new system is periodic in $\theta$ with period $2\pi$. Hence the planes $\theta = 0$ and $\theta = 2\pi$ can be identified and the state space transformed from the Euclidean space $R^n$ to the cylindrical state space $R^n \times S$ where $S = [0, 2\pi)$ is the circle. The solution in the new state space is

$$\begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix} = \begin{bmatrix} \phi_t(x_0, t_0) \\ \frac{2\pi t}{T} \mod 2\pi \end{bmatrix}$$

where the modulo function restricts $0 \leq \theta < 2\pi$. 

2
1.5 **Deterministic System:** [5, 53]

Dynamical systems are ‘deterministic’ if there is a unique consequence to every state and ‘stochastic’ or ‘random’ if there is more than one consequence chosen from some probability distribution.

A system is said to be deterministic if the values assumed by the variables or the change to the variable are predictable with certainty. For example, the model for the motion of a simple pendulum is deterministic.

In mathematics and physics, a deterministic system is a system in which no randomness is involved to the development of future states of the system. A deterministic model thus always produces the same output from a given starting condition or initial state.

Physical laws that are described by mathematical equations represent deterministic systems, even though the state of the system at a given point in time may be difficult to describe explicitly.

1.6 **Stochastic system:** [5]

A system is said to be stochastic, if the values assumed by the variables or the changes to the variables are not predictable with certainty. For example, if a rubber-ball is dropped from a given height and we try to measures the height of the bounce with sufficient accuracy, it will be found that if the same process is repeated many times, the height of the bounces are not the same every time.

Actually “stochastic” means being or having a random variable. A stochastic model is a tool for estimating probability distributions of potential outcomes by allowing for random variation in one or more inputs over time. The random variation is usually based on fluctuations observed in historical data for a selected period using standard time series techniques. Distributions of potential outcomes are derived from a large number of simulations which reflect the random variation in the input(s). Its application initially started in physics. It is now being applied in engineering, life sciences and many other fields.

1.7 **Dissipative system:** [53]

A dissipative (area decreasing) system displays the nice feature that the long-term behavior of the system is largely independent of how we “start up” the system. Thus, for
dissipative systems, we generally ignore the transient behavior associated with the startup of the system and focus our attention on the system’s long term behavior.

As the dissipative system evolves in time, the trajectory in state space will head for some final state space point, curve, area and so on.

1.8 Unimodel map: [151]

Let $f: I \to J$ be a continuous map. If $f$ has a maximum at a point $x_c$ in $I$, then $f$ is said to be a unimodal map.

1.9 Diffeomorphism:[151]

Let $A$ and $B$ be two subsets of $\mathbb{R}^n$. A $C^k$ diffeomorphism $f: A \to B$ is a mapping $f$ which is one-one and has the property that both $f$ and $f^{-1}$ are $k$-times differentiable. Clearly, it is a stronger condition than a homeomorphism.

1.10 Fixed point or Equilibrium point:[5, 53]

A fixed point of a function or transformation is a point that is mapped to itself by the function or transformation. Let $f: \mathbb{R} \to \mathbb{R}$ be a map. A real number $x$ is called a fixed point of $f$ if $f(x) = x$. For example, if $f$ is the logistic map defined by $f(x) = mx(1 - x)$ then the solution of the equation $mx(1 - x) = x$ gives the fixed points of the function $f$. For example, if $f(x) = x^2 - 2$, then $f(x) = x \Rightarrow x^2 - 2 = x \Rightarrow x = 2, -1$. Therefore, $f$ has two fixed points $2, -1$ where as for the function $f(x) = x + 3$ has not any fixed point as $f(x) = x \Rightarrow x + 3 = x \Rightarrow 3 = 0$ which is absurd.

It is not difficult to realize that geometrically, the fixed points in our example are nothing but the point of intersections of the map $f(x) = mx(1 - x)$ and the line $f(x) = x$.

Again, let us consider the autonomous differential equation of the form

$$\dot{x} = f(x), \text{ where } x \in \mathbb{R}^n.$$  

A critical point (equilibrium point, fixed point, stationary point) is a point that satisfies the equation $\dot{x} = f(x) = 0$. If a solution starts at this point, it remains there forever.
For example, the damped pendulum equation given by

\[ \dot{x} = y, \quad \dot{y} = -ky - \sin x \]

Here \( f(x) = (y, -ky - \sin x) \)

At the equilibrium point \( f(x) = 0 \)

\[ \Rightarrow (y, -ky - \sin x) = 0 \Rightarrow y = 0, -ky - \sin x = 0 \]

\[ \Rightarrow y = 0, \quad \sin x = 0 \Rightarrow y = 0, \quad x = m\pi, \quad m = 0, \pm 1, \pm 2, \ldots \]

Thus the equilibrium points are \((m\pi, 0), m = 0, \pm 1, \pm 2, \ldots\)

A critical point, say, \( x_0 \), of the differential equation \( \dot{x} = f(x) \) is called stable if given \( \epsilon > 0 \), there is a \( \delta > 0 \), such that for all \( t \geq t_0 \), \( \|x(t) - x_0(t)\| < \epsilon \), whenever \( \|x(t_0) - x_0(t_0)\| < \delta \), where \( x(t) \) is a solution of \( \dot{x} = f(x) \).

If a fixed point \( x^* \) is stable and

\[ \lim_{m \to \infty} f^m(x) = x^* \]

for all \( x \) in some neighbourhood of \( x^* \), the fixed point is said to be asymptotically stable. Trajectories of points near to an asymptotically stable fixed point move toward it as \( m \) increases.

1.11 **Periodic points and periodic orbit of a system:**[5, 53,76]

A point \( x \) is said to be a periodic point of period \( n \) if \( f^n(x) = x \) where \( n \) is the smallest positive integer. Here \( f^n \) means n-times iteration of the map \( f \). Note that fixed points can be included under this definition as periodic points of period one. Conversely, ‘a periodic point of period \( n \) of a map \( f \) can be viewed as the fixed point(s) of the n-th iteration of the map.

A periodic orbit of \( f \) is a finite sequence of distinct points each of which is the image of the previous one and whose first point is the image of the last. Its period \( m \) is the number of points in the sequence, which are called periodic points of period \( m \).

For a periodic orbit \( \{x_1, x_2, x_3, \ldots x_m\} \) of period \( m \), we have
\[ x_2 = f(x_1), x_3 = f(x_2), \ldots, x_m = f(x_{m-1}), x_1 = f(x_m) \]

For example, for the map \( f(x) = -x \), we have \( f(1) = -1 \) and \( f(-1) = 1 \) and so \( \{1, -1\} \) is a periodic orbit of period 2. Now \( f^2(1) = f(f(1)) = f(-1) = 1 \) and \( f^2(-1) = f(f(-1)) = f(1) = -1 \) which shows that periodic points of period 2 for the map \( f \) are the fixed points of the map \( f^2 \).

1.12 Hyperbolic and non-hyperbolic fixed point: [87, 100]

A critical point is called hyperbolic if the real part of the eigenvalues of the Jacobian matrix \( J \) of the linearized system of \( \dot{x} = f(x) \) are nonzero. On the other hand, if the real part of the either of the eigenvalues of the Jacobian matrix are equal to zero, then the critical point is called non-hyperbolic.

1.13 Limit Cycle:[53, 76]

A limit cycle is an isolated periodic solution of a nonlinear system. In two (or higher) dimensional state spaces, it is possible to have cyclic or periodic behavior. This very important kind of behavior is represented by closed loop trajectories in the state space. A trajectory point on one of these loops continues to cycle around that loop for all time. In the case where all the neighbouring trajectories approach the limit cycle as time approaches infinity, it is called a stable or attractive limit cycle. If instead all neighbouring trajectories approach it as time approaches negative infinity, it is an unstable or non-attractive limit cycle.

Stable limit cycles imply self-sustained oscillations. Any small perturbation from the closed trajectory would cause the system to return to the limit cycle, making the system stick to the limit cycle.

1.14 Saddle point:[53, 76,138]

Let \( f \) be a map on \( \mathbb{R}^m, m \geq 1 \). Let \( f(p) = p \), then the fixed point \( p \) is called saddle point if at least one of the eigenvalues of \( \text{D}f(p) \), where \( \text{D}f(p) \) is the Jacobian matrix of the map \( f \) at the fixed point \( p \) has magnitude greater than 1 and at least one eigen value has magnitude less than 1.
If \( p \) is a periodic point of period \( n \) then \( f^n(p) = p \) and \( p \) is said to be saddle if at least one eigenvalue of \( Df^n(p) \), where \( Df^n(p) \) is the Jacobian matrix of the composition function \( f \) to itself \( n \) times, has magnitude greater than 1 and at least one eigenvalue has magnitude less than 1.

### 1.15 Chaos: [53, 76, 98, 138]

Some sudden and dramatic changes in nonlinear systems may give rise to complex behavior called chaos. The noun chaos and the adjective chaotic are used to describe the time behavior.

Actually there is no generally accepted definition of chaos. From a practical point of view, chaos can be defined as bounded steady state behavior which is not an equilibrium point, not periodic, not quasiperiodic. The trajectories are, indeed bounded. An important fact about the chaotic systems is that the limit set for chaotic behavior is not a simple geometrical object like circle or torus, but is related to fractals. The following characteristics are nearly always displayed by the solutions of chaotic systems.

(i) Long-term aperiodic (non periodic) behavior
(ii) Sensitivity to initial conditions
(iii) Fractal structure

### 1.16 Quasiperiodicity and Mode-locking: [53, 87]

Quasiperiodicity is the property of a system that displays irregular periodicity. Periodic behavior is defined as recurring at regular intervals, such as “every 24 hours”. Quasiperiodic behavior is a pattern of recurrence with a component of unpredictability that does not lend itself to precise measurement.

In climatology, quasiperiodic is a term used to denote oscillations that appear to follow a regular pattern but which do not have a fixed period.

In the quasiperiodic scenario, the system begins with a limit cycle trajectory. As control parameter is changed, a second periodicity appears in the behavior of the system. This
bifurcation is a generalization of the Hopf bifurcation; so it is also called a Hopf bifurcation.

In our discussion, the "independence" of the two (or more) frequencies will be important. The following terminology is used to describe the ratio of the two frequencies, say, $f_1$ and $f_2$. If there are two (positive) integers, $p$ and $q$, that satisfy

$$\frac{f_2}{f_1} = \frac{p}{q}$$

then we say that the frequencies are commensurate or, equivalently, that the frequency ratio is rational. If there are no integers that satisfy Eq. (1), then we say that the frequencies are incommensurate or, equivalently, that the frequency ratio is irrational. If the ratio is rational, then we say that the system's behavior is periodic. If the ratio is irrational, then we say that the behavior is quasi-periodic.

The term quasi-periodic is used to describe the behavior when the two frequencies are incommensurate because, in fact, the system's behavior never exactly repeats itself in that case. Indeed, the time-behavior of a quasi-periodic system can look quite irregular.

Frequency-locking is a common phenomenon whenever two or more “oscillators” interact nonlinearly. Each oscillator is characterized by some frequency (i.e. $f_1$) for oscillator number 1 and $f_2$ for oscillator number 2. (The oscillators may be physically distinct oscillators-two different pendulum clocks, for example—or they may be different “modes of motion” within the same physical system.) If the two frequencies are commensurate over some range of control parameter values, that is, if $\frac{f_2}{f_1} = \frac{p}{q}$ ($p, q$ integers) over this parameter range, then we say that the two oscillations are frequency-locked or mode-locked or phase-locked. One frequency is locked into being a rational number times the other frequency. For example, frequency-locking explains the fixed relationship between rotation frequency and orbital frequency for the Moon’s motion around the earth (we on the earth, get to see only one side of the Moon) and the motion of Mercury about the Sun. In both the cases, tidal forces cause an interaction between the axial motion of rotation and the orbital motion. The two motions become locked together.
1.17 Stable, unstable manifold: [46, 87, 100, 151]

The stable manifold theorem is one of the most important results in the local qualitative theory of ordinary differential equations. The theorem shows that near a hyperbolic equilibrium point $x_0$, the nonlinear system

$$\dot{x} = f(x)$$  \hspace{1cm} (1)

has stable and unstable manifolds $S$ and $U$ tangent at $x_0$ to the stable and unstable subspaces $E^s$ and $E^u$ of the linearized system is

$$\dot{x} = Ax$$  \hspace{1cm} (2)

where $A = Df(x_0)$. Furthermore, $S$ and $U$ are of the same dimensions as $E^s$ and $E^u$, and if $\phi_t$ is the flow of the nonlinear system (1), then $S$ and $U$ are positively and negatively invariant under $\phi_t$ respectively and satisfy

$$\lim_{t \to \infty} \phi_t(c) = x_0 \text{ for all } c \in S$$

and

$$\lim_{t \to -\infty} \phi_t(c) = x_0 \text{ for all } c \in S$$

Example: Consider the nonlinear system

$$\dot{x}_1 = -x_1$$

$$\dot{x}_2 = -x_2 + x_1^2$$

$$\dot{x}_3 = x_3 + x_1^2$$

The only equilibrium point of this system is at the origin. The matrix

\begin{align*}
A &= \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\end{align*}
Thus, the stable and unstable subspaces $E^s$ and $E^u$ of (2) are the $x_1, x_2$ plane and the $x_3$-axis respectively. After solving the first differential equation, $\dot{x}_1 = -x_1$, the nonlinear system reduces to two uncoupled first order differential equations which are easily solved. The solution is given by

$$
\begin{align*}
x_1(t) &= c_1 e^{-t} \\
x_2(t) &= c_2 e^{-t} + c_1^2 (e^{-t} - e^{-2t}) \\
x_3(t) &= c_3 e^t + c_1^2 (e^t - e^{-2t})
\end{align*}
$$

where $c = x(0)$.

Clearly,

$$
\lim_{t \to \infty} \phi_t(c) = 0 \text{ iff } c_3 + \frac{c_1^2}{3} = 0
$$

$S = \{ c \in \mathbb{R}^3 | c_3 = -\frac{c_1^2}{3} \}$.

Similarly,

$$
\lim_{t \to -\infty} \phi_t(c) = 0 \text{ iff } c_1 = c_2 = 0
$$

and therefore

$$
U = \{ c \in \mathbb{R}^3 | c_1 = c_2 = 0 \}
$$

The stable and unstable manifolds for this system are shown in Fig. 1.2. Note that the surface $S$ is tangent $E^s$, i.e., to the $x_1, x_2$ plane at the origin and that $U = E^u$. 

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| $A = Df(0)$ | $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |

*Fig. 1.2*
The stable Manifold theorem:

Let $E$ be an open subset of $\mathbb{R}^n$ containing the origin, let $f \in C^1(E)$, and let $\phi_t$ be the flow of the non-linear system (1). Suppose that $f(0) = 0$ and that $Df(0)$ has $k$ eigenvalues with negative real part and $n - k$ eigenvalues with positive real part. Then there is a $k$-dimensional differentiable manifold $S$ tangent to the stable subspace $E^s$ of the linear system $\dot{x} = Ax$ at 0 such that for all $t \geq 0$, $\phi_t(S) \subset S$ and for all $x_0 \in S$

$$\lim_{t \to \infty} \phi_t(x_0) = 0$$

And there exists an $n - k$ dimensional differentiable manifold $U$ tangent to the unstable subspace $E^u$ of $\dot{x} = Ax$

At 0 such that for all $t \leq 0$, $\phi_t(U) \subset U$ and for all $x_0 \in U$

$$\lim_{t \to -\infty} \phi_t(x_0) = 0$$

1.18 Linear stability conditions: [53, 76]

For convenience of discussion let us consider a one dimensional iterated map $f(x)$. In general the map function depends on some parameter $p$. The iteration scheme begins with an initial value of $x$, call it $x_0$ and we generate a trajectory (or) orbit by successive applications of the map function:

$$x_1 = f(x_0)$$
$$x_2 = f(x_1)$$

and so on.

A point $x^*$ is a fixed point of the map function $f(x)$ if it satisfies

$$x^* = f(x^*)$$

Now what happens a trajectory that start near a fixed point; that is, what is the stability of the fixed point? If the trajectories approach $x^*$ as the iteration process proceeds ($n \to \infty$), we say that $x^*$ is an attracting fixed point or (equivalently) a stable fixed point or asymptotically stable. If the trajectories move away from $x^*$, then we say that $x^*$ is a repelling fixed point or (equivalently) an unstable fixed point.

We can investigate the stability of a fixed point by finding the derivative of the map function at the fixed point.
To study the stability of a fixed point we can analytically derive the stability condition. To do that we use the Taylor series expansion near $x^*$. For a trajectory starting at $x_0$ near $x^*$, we can write

$$
x_1 = f(x_0) = f(x^*) + \frac{df}{dx}\bigg|_{x=x^*} (x_0 - x^*) + \cdots
$$

$$
= x^* + \frac{df}{dx}\bigg|_{x=x^*} (x_0 - x^*) + \cdots
$$

Keeping only the first derivative term in the expansion, we write this result in terms of the difference between $x_n$ and $x^*$:

$$(x_n - x^*) = \left. \frac{df}{dx} \right|_{x=x^*} (x_{n-1} - x^*)$$

If the magnitude of the derivative at the fixed point is $< 1$ then the distance decreases with subsequent iteration and the trajectory approaches the fixed point. If the magnitude of the derivative is greater than 1, then the distance increases and the trajectory moves away from the fixed point.

The analysis in terms of the Taylor series is often called “linear stability analysis” since we are keeping only the term linear in the distance from the trajectory point to the fixed point. Then if the derivative has a magnitude less than 1, we say the fixed point is “linearly stable”. In practice this analysis tells us what happens to trajectories only in the immediate neighbourhood of the fixed point.

The fixed point $x^*$ is stable if $\left. \frac{df}{dx} \right|_{x=x^*} < 1$

The fixed point $x^*$ is unstable if $\left. \frac{df}{dx} \right|_{x=x^*} > 1$

In geometric terms, we are looking at the slope of the map function at the fixed point. In the following figure we have shown the graphical representation of the iteration process. In the immediate neighbourhood of a fixed point, the function can be approximated as straight line.
Fig. 1.3 A graphical representation of the effect of repeated use of map function $f(x)$. The map function is plotted as a dashed line. The $f(x) = x$ line is solid. (a) $0 < \frac{df}{dx} < 1$ gives a stable fixed point. (b) $\frac{df}{dx} > 1$ gives an unstable fixed point. (c) $-1 < \frac{df}{dx} < 0$ gives a stable fixed point. (d) $\frac{df}{dx} < -1$ gives an unstable fixed point.

It is to be noted that for higher dimensional maps the role of derivative explained above will be played by the eigenvalues of the Jacobian matrix near the fixed point.

Again let us consider a one dimensional continuous system $\dot{x} = f(x), x \in \mathbb{R}$. Let $x^*$ be a critical point of $\dot{x} = f(x), x \in \mathbb{R}$. Consider a small perturbation, say, $\xi(t)$, away from the critical point at $x^*$ to give $x(t) = x^* + \xi(t)$. A simple analysis is now applied to determine whether the perturbation grows or decays as time evolves. Now

$$\dot{\xi} = \dot{x} = f(x) = f(x^* + \xi),$$

And after a Taylor’s series expansion

$$\dot{\xi} = f(x^*) + \xi f'(x^*) + \frac{\xi^2}{2} f''(x^*) + \cdots$$

In order to apply a linear stability analysis, the nonlinear terms are ignored. Hence

$$\dot{\xi} = \xi f'(x^*)$$

since $f(x^*) = 0$. Therefore the perturbation $\xi(t)$ grows exponentially if $f'(x^*) > 0$ and decays exponentially if $f'(x^*) < 0$. If $f'(x^*) = 0$, then higher order derivatives must be considered to determine the stability of the critical point.
Here also, it is to be noted that for higher dimensional differential equations the role of derivative explained above will be played by the eigenvalues of the Jacobian matrix near the equilibrium point.

### 1.19 Flows:[100]

Let $E$ be an open subset of $R^n$ and let $f \in C^1(E)$. For $x_0 \in E$, let $\phi(t,x_0)$ be the solution of the initial value problem (2) defined on its maximal interval of existence $I(x_0)$. Then for $t \in I(x_0)$, the set of mappings $\phi_t$ defined by

$$
\phi_t(x_0) = \phi(t,x_0)
$$

is called the flow of the differential equation (1) or the flow defined by the differential equation (1); $\phi_t$ is also referred to as the flow of the vector field $f(x)$.

### 1.20 Poincare-Benedixson theorem: [138]

Suppose that $R$ is closed, bounded subset of the plane.

1. $\dot{x} = f(x)$ is continuously differentiable vector field on an open set containing $R$
2. $R$ does not contain any fixed point and
3. There exists a trajectory $C$ that is confined in $R$ in the sense that it starts in $R$ and stays in $R$ for all future time.

Then either $C$ is a closed orbit, or it spirals towards a closed orbit $t \to \infty$.

### 1.21 Fundamental theorem of Linear System:[100]

Let $A$ be an $n \times n$ matrix. Then for a given $x_0 \in R^n$, the initial value problem

$$
\dot{x} = Ax
$$

$$
x(0) = x_0
$$

has a unique solution given by

$$
x(t) = e^{At}x_0
$$

### 1.22 Hartman-Grobman Theorem: [46, 100]

The Hartman-Grobman Theorem is another very important result in the local qualitative theory of ordinary differential equations. The theorem shows that near a hyperbolic equilibrium point $x_0$, the nonlinear system

$$
\dot{x} = f(x)
$$

where
has the same qualitative structure as the linear system

\[ \dot{x} = Ax \]

with \( A = Df(x_0) \). It is assume that the equilibrium point \( x_0 \) has been translated to the origin.

### 1.23 The Poincare Map:[5, 53, 76, 96]

Probably the most basic tool for studying the stability and bifurcations of periodic orbits is the Poincare map or first return map, defined by Henri Poincare in 1881. A Poincare section is a surface in phase space that cuts across the flow of a given system. It replaces the flow of a continuous-time dynamical system with a map called the Poincare map.

Poincare maps are useful in highlighting what solution the dynamical system is portraying through time. If, for example, the system is being attracted to a limit cycle, we will see the dots converging to a stationary point or points depending on the period of the solution. A few limit cycles are shown below alongwith the relative Poincare map for the limit cycle.

![Poincare section of periodic orbits](image)

**Fig. 1.4 Poincare section of periodic orbits**

The definitions of the Poincare map are slightly different for autonomous and non-autonomous systems, and the two cases are presented separately.
It should be noted that maps arising from a Poincare section of a flow are necessarily invertible, because the flow has a unique solution through any point in phase space—the solution is unique both forward and backward in time.

1.24 The Poincare Map for Nonautonomous systems: [5, 53, 96]

It was stated earlier that a n-th order non-autonomous system with period $T$ may be transformed into an (n+1)–dimensional autonomous system in cylindrical state space $R^n \times S$.

Let us consider the n dimensional hyper plane $\Sigma$ in $R^n \times S$ defined by $\Sigma = \{(x, \theta) \in R^n \times S|\theta = \theta_0\}$.

Every $T$ seconds, the trajectory 
\[
\begin{bmatrix}
    x(t) \\
    \theta(t)
\end{bmatrix} = \begin{bmatrix}
    \phi_t(x_0, t_0) \\
    \frac{2\pi t}{T} \mod 2\pi
\end{bmatrix}
\]

intersects $\Sigma$ (see the adjacent figure).

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{poincare_map.png}
\caption{The Poincare map of a first order non-autonomous system[96]}
\end{figure}
Thus a map \( P: \Sigma \to \Sigma: \mathbb{R}^n \to \mathbb{R}^n \) is defined by \( P(x) = \phi_T(x, t_0) \). Here \( P \) is called the Poincare map. Note that \( \phi_T \) is a diffeomorphism and, therefore, \( P \) is one-to-one and differentiable.

\( P \) may be thought of in two ways:

(i) \( P(x) \) indicates where the flow takes \( x \) after \( T \) seconds. This is called \( T \)-advancing mapping.

(ii) The orbit \( \{P^k(x)\}_{k=1}^{\infty} \) is a sampling of a single trajectory every \( T \) seconds; that is

\[
P^k(x_0) = \phi_{kT}(x_0, t_0) \quad \text{for} \quad k = 0, 1, \ldots
\]

1.25 **The Poincare Map for Autonomous systems:**[5,46, 53, 92, 96]

Consider the \( n \)-th order autonomous system with the limit cycle \( \Gamma \) shown in figure 1.6. Let \( \Sigma \) be a point on the limit cycle and let \( T \) be the minimal period of the limit cycle. Let us take an \( (n - 1) \)-dimensional hyperplane \( \Sigma \) transverse (in this setting, transverse means not tangent) to \( \Gamma \) at \( x^* \). The trajectory emanating from \( x^* \) will hit \( \Sigma \) at \( x^* \) in \( T \) seconds. Due to the continuity of \( \phi_t \) with respect to the initial condition, trajectories starting on \( \Sigma \) in a sufficiently small neighbourhood of \( x^* \) will in approximately \( T \) seconds, intersect \( \Sigma \) in the vicinity of \( x^* \). Hence, \( f \) and \( \Sigma \) define mapping \( P \) of some neighbourhood \( U \subset \Sigma \) of \( x^* \) onto another neighbourhood \( V \subset \Sigma \) of \( x^* \). Here \( P \) is the Poincare map or First-Return map.

![Diagram of the Poincare map for autonomous systems](image)

*Fig. 1.6 The Poincare map for a autonomous system is defined by a limit cycle and a cross section. [96]*
It is important to realize that for autonomous systems, the Poincare map is defined only in a neighbourhood of $x^*$. Unlike the non-autonomous case, it is not guaranteed that the trajectory emanating from any point on $\Sigma$ will intersect $\Sigma$ again. As in the nonautonomous case, it can be shown that $P$ is a diffeomorphism.

### 1.26 Time series: [53,76, 137]

One of the key tools used for quantifying chaotic behavior is the notion of a time-series of data for the system. By observing data over a period of time, one can easily understand what changes have taken place in the past. Such an analysis is extremely helpful in predicting the future dynamical behavior.

A time series is a collection of observations of well-defined data items obtained through repeated measurements over time. For example, measuring the value of retail sales each month of the year would comprise a time series. This is because sales revenue is well defined, and consistently measured at equally spaced intervals. Data collected irregularly or only once are not time series.

In statistics, signal processing, pattern recognition, econometrics, mathematical finance, Weather forecasting, Earthquake prediction, Electroencephalography, Control engineering and Communication engineering etc., a time series is a sequence of data points, measured typically at successive time instants spaced at uniform time intervals. Examples of time series are the daily closing value of the Sensex or the annual flow volume of the Brahmaputra River in Assam. Time series analysis comprises methods for analyzing time series data in order to extract meaningful statistics and other characteristics of the data. Time series forecasting is the use of a model to predict future values based on previously observed values. Time series are very frequently plotted via line charts.

Time series data have a natural temporal ordering. This makes time series analysis distinct from other common data analysis problems, in which there is no natural ordering of the observations (e.g. explaining people's wages by reference to their respective education levels, where the individuals' data could be entered in any order). Time series analysis is also distinct from spatial data analysis where the observations typically relate to geographical locations (e.g. accounting for house prices by the
location as well as the intrinsic characteristics of the houses). A stochastic model for a time series will generally reflect the fact that observations close together in time will be more closely related than observations further apart. In addition, time series models will often make use of the natural one-way ordering of time so that values for a given period will be expressed as deriving in some way from past values, rather than from future values. Time series analysis can be applied to:

(i) real-valued, continuous data  
(ii) discrete numeric data  
(iii) discrete symbolic data (i.e. sequences of characters, such as letters and words in English language).

There are several types of data analysis available for time series which are appropriate for different purposes. In the context of statistics, econometrics, quantitative finance, seismology, meteorology, geophysics the primary goal of time series analysis is forecasting, in the context of signal processing, control engineering and communication engineering it is used for signal detection and estimation while in the context of data mining, pattern recognition and machine learning time series analysis can be used for clustering, classification, query by content, anomaly detection as well as forecasting.

For quantifying chaotic behavior in a nonlinear system is the notion of a time series of data for the system. For the sake of clarification, we have shown regular and chaotic dynamical behavior in the well known logistic map [53, 83, 84] with the help of time series.

The logistic map is given by $x_{n+1} = r x_n (1 - x_n)$, where $r$ is the growth parameter (control parameter) and $x_n$ is the percentage of population at time $n$.  

Horizontal axis represents the number of iteration
Vertical axis represents the iterated values of the logistic map

**Fig. 1.7** Time series plot of population ($x_n$) Vs time (no. of iteration, $n$) $a r = 2$

The above time series plot shows period-1 behavior i.e. for the parameter value $r = 2$ the population predicted by the Logistic map in the long run settles to a steady state (fixed point).

Horizontal axis represents the number of iteration
Vertical axis represents the iterated values of the logistic map

**Fig. 1.8** Time series plot of population ($x_n$) Vs time (no. of iteration, $n$) $a r = 3.2$

The above time series plot shows period-2 behavior i.e. for the parameter value $r = 3.2$ the population predicted by the Logistic map in the long run oscillate between two values (periodic points).
The above time series plot shows period-4 behavior i.e. for the parameter value $r = 3.4$ the population predicted by the Logistic map in the long run oscillate between four values (periodic points).

The above time series plot shows chaotic behavior i.e. for the parameter value $r = 3.8$ the population predicted by the Logistic map in the long run does not show any pattern but oscillates randomly.
From the above figures of time series plot we found that in between the parameter values $r = 2$ and $r = 3.2$ there is a value of the parameter at which a bifurcation from period-1 to period-2 occurs. Similar situation takes place between parameter values $r = 3.2$ and $r = 3.4$ where a bifurcation from period-2 to period-4 takes place. The time series plot shows that at $r = 4$, the system behaves chaotically.

Thus, from time series plot we can estimate or identify which type of behavior the system is showing as the control parameter is varied.

1.27 Graphical iteration and Cobweb diagram: [53, 76]

Graphical iteration (or graphical analysis as it is sometimes called) is a way to study orbits of a function $f$ graphically by following the orbit as it moves along the diagonal line $y = x$ which is drawn along with the graph of the function. For example, fixed points $p$ of $f$ are the $x$-coordinates of the points of intersection of the graph of $f$ and the line $y = x$ i.e. the points which satisfy $f(x) = x$. Notice that the orbit of a fixed point $p$ is $\{p, p, p \ldots \}$. The next thing to ask is what is the stability of the fixed point. We say that a fixed point is stable if orbits that start nearby the fixed point converge to the fixed point as the number of iterations tends to infinity. We call the fixed point unstable if at least one nearby point has an orbit that does not converge to the fixed point. The stability of a fixed point refers to whether it is stable or not.

It is easy to determine the stability of a fixed point using graphical analysis. The stability of a fixed point can also be determined analytically; the fixed point $p$ is stable if $|f'(p)| < 1$ and is unstable if $|f'(p)| > 1$. In the case $|f'(p)| = 1$, the fixed point could either be stable, unstable, or neutral. So in this case one has to make a careful sketch of the graph of $f$ near the fixed point to determine the stability. All one needs to know in order to determine the stability of a fixed point $p$ from graphical analysis is how the graph of $f$ lies with respect to the diagonal line $y = x$ near $p$. Periodic orbits of (prime) period greater than one are not easy to detect graphically with $f$. Here we should apply graphical analysis to $f^k$ if we want to study periodic points of $f$ of period-$k$. So we would first look for fixed points of $f^k$, i.e., we would solve $f^k(x) = x$. Graphically, these would be the $x$-coordinates of points of intersection between the graph of $f^k$ and the line $y = x$. Using graphical analysis we could then determine the stability of these
fixed points $f^k$. We call a periodic point $q$ stable if $|(f^k(q))'| < 1$ and unstable if $|(f^k(q))'| > 1$. The stability of a periodic point $q$ of $f$ of period $k$ is then just the stability of the fixed point $q$ of $f^k$. So if $q$ is a stable periodic point of $f^k$ of period $k$, then the orbits starting at points near $q$ will converge to the periodic orbit of $q$. And if $q$ is unstable then at least one nearby point will have an orbit that does not converge to the periodic orbit of $q$.

A cobweb plot, or Verhulst diagram is a visual tool used in the dynamical systems to investigate the qualitative behavior of one dimensional iterated functions, such as the logistic map. Using a cobweb plot, it is possible to infer the long term status of an initial condition under repeated application of a map. On the cobweb plot, a stable fixed point corresponds to an inward spiral, while an unstable fixed point is an outward one. It follows from the definition of a fixed point that these spirals will center at a point where the diagonal $x = x$ line crosses the function graph. A period-2 orbit is represented by a rectangle, while greater period cycles produce further, more complex closed loops. A chaotic orbit would show a 'filled out' area, indicating an infinite number of non-repeating values.

![Cobweb Diagram](image)

*Horizontal axis represents the initial point
Vertical axis represents the orbit of the initial point*

**Fig 1.11** Cobweb diagram (or graphical iteration) of the logistic map for the parameter value $a = 1 + 2\sqrt{2}$ and at the initial point $x_0 = 0.5$. The diagram shows period-3 behavior.
Horizontal axis represents the initial point
Vertical axis represents the orbit of the initial point

Fig 1.12 Cobweb diagram (or graphical iteration) of the logistic map for the parameter value \( a = 4 \) and at the initial point \( x_0 = 0.2 \). The graph fills the whole area and thereby showing an orbit of infinite order and hence shows the chaotic behavior.

1.28 Final state or Bifurcation Diagram: [53, 76, 128, 134]

If we are considering a function \( f_a(x) \) that depends on a parameter \( a \), then since the graph of \( f \) will (perhaps) change as the parameter \( a \) changes, we expect that the periodic points of \( f \) will also change with \( a \). Change here could mean that the periodic point changes location as \( a \) changes and also its stability could change. In addition, a periodic point can appear or disappear as \( a \) changes. When the qualitative behavior of the orbits of \( f_a \) changes at a certain parameter value, say \( a = a_0 \), then we call that value of \( "a" \) a bifurcation point of \( f \) and say that " \( f_a \) undergoes a bifurcation at \( a = a_0 \)." For example, if \( f_a(x) = ax(1-x) \), the logistic function, then \( f_a \) undergoes a bifurcation at \( a = 3 \) because the fixed point of \( f_a \) changes from stable to unstable at \( a = 3 \). That's enough to call \( a = 3 \), a bifurcation point, but in addition a period two orbit appears, which also changes the qualitative behavior of the orbits of \( f_a \).

A bifurcation diagram is the plot of the locations of the fixed points versus the parameter value \( a \). In general, bifurcation diagrams are difficult to draw because one has to solve the equations \( f_a^k(x) = x \) (and then plot the curves of \( x \) as a function of \( a \)). A similar diagram that can be drawn using computer generated orbits of \( f_a \) is the final-
statediagram. Here one plots the 'final state' of an orbit \(x_n\) of \(f_a\); say \(x_{101} \to x_{200}\). If the orbit is periodic, or is converging to a periodic orbit, the final state will be that orbit. If a periodic orbit attracts (essentially) all the orbits no matter where they begin, then the final state will be the same no matter where the orbit starts. Now changing "a" a little and plotting \(x_{101} \to x_{200}\), etc., for a range of \(a\) values one sees then are the stable periodic orbits. If \(f_a\) happens not to have any stable periodic orbits, then the final state of an orbit will be points spread out over a portion of an interval ([0,1], in case of the Logistic map) instead of being concentrated in just a few. Below, we have shown the final state diagram for the Logistic map.

![Final-state diagram for the logistic function between the parameter values \(a = 1\) and \(a = 4\).](image)

*Fig. 1.13 The final-state diagram for the logistic function between the parameter values \(a = 1\) and \(a = 4\).*

The final-state diagram of the logistic function for parameter values \(a\) between 0 and 4 has remarkable and complex patterns. The above figure shows period-doubling bifurcation to chaos. The sequence of period doublings ends at about \(a = 3.5699456\ldots\) and the system becomes chaotic. Also it can be seen from the diagram that there are portions inside the chaotic region where the system returns back to periodic behavior which are marked by white spaces. These regions are called periodic windows.
1.29 Lyapunov exponent: [5, 53, 132, 141, 152, 153]

A Lyapunov exponent is a measure of the rate of attraction to or repulsion from a fixed point in state space. We can also apply this notion to the divergence of nearby trajectories in general at any point in state space.

(For a dissipative, one dimensional system, we know that this average Lyapunov exponent must be negative.)

The Lyapunov exponent or Lyapunov characteristic exponent of a dynamical system is a quantity that characterizes the rate of separation of infinitesimally close trajectories. As chaotic systems are very much sensitive to the initial conditions, this becomes an important measure to identify whether a given system is chaotic or not.

For a one-dimensional state space, let \( x_0 \) be one initial point and \( x \) a nearby initial point. Let \( x_0(t) \) be the trajectory that arises from the first initial point, while \( x(t) \) is the trajectory arising from the other initial point.

The time development equation is assumed to be \( \dot{x}(t) = f(x) \)

Since we assume that \( x \) is close to \( x_0 \), we can use a Taylor series expansion to write

\[
f(x) = f(x_0) + \left. \frac{df(x)}{dx} \right|_{x_0} (x - x_0) + \ldots
\]

The rate of change of distance between the two trajectories is given by

\[
\dot{s} = \dot{x} - \dot{x}_0 = f(x) - f(x_0) = \left. \frac{df(x)}{ds} \right|_{x_0} (x - x_0)
\]

where we have neglected the second and higher order derivative terms in the Taylor series expansion of \( f(x) \). Since we expect the distance to change exponentially in time, the Lyapunov exponent \( \lambda \) can be introduced as the quantity that satisfies

\[
s(t) = s_0 e^{\lambda t}
\]

Differentiating the above with respect to time, we find

\[
\dot{s} = \lambda s_0 e^{\lambda t} = \lambda s
\]

\((**\))
Comparing equation (**) and equation (*) yields

\[ \lambda \equiv \left. \frac{df(x)}{dx} \right|_{x_0} \]

Thus, we see that if \( \lambda \) is positive, then the two trajectories diverge; if \( \lambda \) is negative, the two trajectories converge.

In practice, the derivative of the time evolution function generally varies with \( x \); therefore, we want to find an average of \( \lambda \) over the history of a trajectory. If we know the time evolution function, we simply evaluate the derivative of the time evaluation function along the trajectory and find the average value. So, we define the Lyapunov exponent \( \lambda \) by

\[ \lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \ln|f'(x_i)| \]

Clearly, if \( \lambda > 0 \) neighbouring trajectories diverge from each other at large, which corresponds to chaos. However, if trajectories converge to a fixed point or a limit cycle they will get closer together, which corresponds to \( \lambda < 0 \). Hence, we can determine whether or not the system is chaotic by the sign of the Lyapunov exponent.

For a one-dimensional iterated map function or time-series of data let us label the series \( x(t_0), x(t_1), x(t_2), \ldots \) as \( x_0, x_1, x_2, \ldots \) and let time intervals between samples are all equal. Thus we can write \( t_n - t_0 = n \tau \), where \( \tau \) is the time interval between samples.

If the system is chaotic, the divergence of nearby trajectories will manifest itself in the following way: if we select some value from the sequence \( \{x_n\} \), say \( x_i \), and then search the sequence for another \( x \) value, say \( x_j \), that is close to \( x_i \), then the sequence of differences

\[ \delta_0 = |x_j - x_i|, \delta_0 = |x_{j+1} - x_{i+1}|, \delta_0 = |x_{j+2} - x_{i+2}|, \ldots, \delta_n = |x_{j+n} - x_{i+n}|, \]

increases exponentially, at least on the average, as the number of iterations \( n \) increases i.e. \( \delta_n = \delta_0 e^{\lambda n} \) or \( \lambda = \frac{1}{n} \ln \frac{\delta_n}{\delta_0} \), where \( \lambda \) is the Lyapunov exponent. However, again as earlier the value of \( \lambda \) depends on the initial value \( x_i \), so we should write \( \lambda(x_i) \). So to
characterize the attractor, we should calculate the average value of $\lambda$ which can be found by calculating $\lambda(x_i)$ for a large number $n$ of initial values distributed over the attractor. The average Lyapunov exponent is then calculated from

$$\lambda = \frac{1}{n} \sum_{i=1}^{n} \lambda(x_i)$$

The rate of separation can be different for different orientations of initial separation vector. Thus, there is a spectrum of Lyapunov exponents—equal in number to the dimensionality of the phase space. It is common to refer to the largest one as the Maximal Lyapunov exponent (MLE), because it determines a notion of predictability for a dynamical system. A positive MLE is usually taken as an indication that the system is chaotic (provided some other conditions are met, e.g., phase space compactness). Note that an arbitrary initial separation vector will typically contain some component in the direction associated with the MLE, and because of the exponential growth rate, the effect of the other exponents will be obliterated over time.

The maximal Lyapunov exponent can be defined as follows:

$$\lambda = \lim_{t \to \infty} \lim_{\delta Z_0 \to 0} \frac{1}{t} \ln \left| \frac{\delta Z(t)}{\delta Z_0} \right|$$

The limit $\delta Z_0 \to 0$ ensures the validity of the linear approximation at any time.

Lyapunov exponent is actually a quantitative test for chaotic behavior which can distinguish chaotic behavior from noisy behavior due to random and external influences. Also it gives a quantitative measure for the degree of chaoticity. The basic property of Lyapunov exponent is that if the system is conservative (i.e. there is no dissipation), a volume element of the phase space will stay the same along a trajectory and thus the sum of all Lyapunov exponents must be zero. If the system is dissipative, the sum of Lyapunov exponents is negative. Again if the system is a flow and the trajectory does not converge to a single point then it happens that one of the Lyapunov exponent is always zero.

For a dynamical system with evolution equation $f^t$ in an $n$–dimensional phase space, the spectrum of Lyapunov exponents $\{\lambda_1, \lambda_2, ..., \lambda_n\}$ in general, depends on the starting
point $x_0$. However, we will usually be interested in the attractor (or attractors) of a dynamical system, and there will normally be one set of exponents associated with each attractor. The choice of starting point may determine which attractor the system ends up on, if there is more than one. Note that as Hamiltonian systems do not have attractors, so this particular discussion does not apply to them. The Lyapunov exponents describe the behavior of vectors in the tangent space of the phase space and are defined from the Jacobian matrix $J^t(x_0) = \frac{df^t(x)}{dx} \bigg|_{x_0}$.

The $J^t$ matrix describes how a small change at the point $x_0$ propagates to the final point $f^t(x_0)$.

The limit
\[
\lim_{t \to \infty} (J^t \cdot \text{Transpose}(J^t))^{1/2t}
\]
defines a matrix $L(x_0)$ (the conditions for the existence of the limit are given by the Oseledec theorem. If $\Lambda_i(x_0)$ are the eigenvalues of $L(x_0)$, then the Lyapunov exponents $\lambda_i$ are defined by
\[
\lambda_i(x_0) = \log \Lambda_i(x_0)
\]

The signs of the Lyapunov exponents provide a qualitative picture of a system's dynamics. One dimensional maps are characterized by a single Lyapunov exponent which is positive for chaos, zero for a marginally stable orbit, and negative for a periodic orbit. In a three-dimensional continuous dissipative dynamical system the only possible spectra, and the attractors they describe, are as follows: $(+, 0, -)$, a strange attractor; $(0, 0, -)$, a two-torus; $(0, - , -)$, a limit cycle; and $(-, - , -)$, a fixed point [152]. The Maximal Lyapunov exponents can be used to distinguish between chaotic, quasi-periodic and periodic behaviour. In two or more dimensional dynamical systems, there are two or more Lyapunov exponent and in this case the maximal Lyapunov exponent tells us the dynamics of the system. A positive maximal Lyapunov exponent indicates the divergence of nearby trajectories i.e. sensitive dependence on initial conditions and hence chaos. Maximal Lyapunov exponent equal to zero indicates quasi-periodic behaviour and the attractor is a torus. A negative maximal Lyapunov exponent represents a periodic or regular behaviour.
Generally the calculation of Lyapunov exponents, as defined above, cannot be carried out analytically, and in most cases one must resort to numerical techniques. We used the computer software C-Language and Mathematica to determine the Lyapunov exponents. As we are relying on numerical schemes, so, we can not expect accurate result but the result becomes more accurate as the number of iteration goes on increasing.

1.30 Bifurcations Theory: [5, 53, 76, 138]

The qualitative structure of the flow can change as control parameters are varied. Particularly fixed points may be created or destroyed or their stability may be changed. These qualitative changes in the dynamics are called bifurcations [138], and the parameter values at which they occur are called bifurcation points.

Most commonly applied to the mathematical study of dynamical systems, a bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameter) of a system causes a sudden ‘qualitative’ or topological change in its behaviour. Bifurcation occurs in both continuous systems and discrete systems. It is useful to divide bifurcation into two principal classes:

(i) **Local bifurcation**, which can be analysed entirely through changes in the local stability properties of equilibria, periodic orbits or other invariant sets as parameter cross through critical thresholds; and

(ii) **Global bifurcation**, which often occurs when larger invariant sets of the system ‘collide’ with each other, or with equilibria of the system. They can not be detected purely by a stability analysis of the equilibria (fixed points).

It is to be noted that period doubling bifurcation, saddle node (tangent) bifurcation, pitchfork bifurcation, transcritical bifurcation, Neimark-Sacker bifurcation, Hopf bifurcation etc. falls under the category of local bifurcation whereas events like intermittency, crisis, Homoclinic bifurcation, Heteroclinic bifurcation etc. falls under the category of global bifurcation.
1.31 Period doubling bifurcation:[53]

In mathematics, a period doubling bifurcation in a discrete dynamical system is a
bifurcation in which the system switches to a new behaviour with twice the period of
the original system. That is, there exists two points such that applying the dynamics to
each of the points yields the other point. Period doubling bifurcation can also occur in
continuous dynamical systems, namely when a new limit cycle emerges from an
existing limit cycle, and the period of the new limit cycle is twice that of the old one.

When a stable period-$k$ cycle ($k \geq 1$) loses its stability varying one of its parameter $\mu$
from a particular value $\mu = \mu_0$ and when a stable period-$2k$ cycle appears at $\mu = \mu_0$, then
we generally have a period doubling bifurcation. The variation of this system
parameter $\mu$ in a larger interval can highlight this phenomenon several times: this is
called a cascade of period doubling bifurcation. We can observe this phenomenon in
many fields like medicine, demography, stock market study, sociology, mechanics etc.

![Horizontal axis represents the bifurcation parameter
Vertical axis represents the final state of the function variable](image)

_Fig 1.14_ First figure (on the left) shows period-doubling bifurcation from period-1 cycle to
period-2 cycle and the second figure (on the right) shows period-doubling cascade.

1.32 Saddle node (or Tangent) bifurcation:[53, 138]

The saddle node bifurcation is the basic mechanism by which fixed points are created
and destroyed. As a parameter is varied, two fixed points move toward each other,
collide and mutually annihilate.

For example: We consider the following one dimensional dynamical system depending
on one parameter:
Let us assume that \( \dot{x} = \alpha + x^2 = f(x, \alpha) \). For \( \alpha < 0 \), the system has two fixed points one of which is stable and the other is unstable. For \( \alpha = 0 \) the two fixed points become one and then disappear when \( \alpha > 0 \).

**Fig: 1.15** \( \alpha < 0 \)(abscissa represents x while ordinate represents f)

In Fig: 1.15, fixed points are those points of the x-axis which are cut by the curve. It can be observed that left part of the curve is decreasing, so the fixed point lying in the left part has derivative less than 0, so it is a stable fixed point while the fixed point lying in the right part has derivative greater than 0, hence it is an unstable fixed point.

**Fig: 1.16** \( \alpha = 0 \)(abscissa represents x and ordinate represents f)

In Fig 1.16 the first derivative of the fixed point is 0 and hence bifurcation occurs at the parameter value \( \alpha = 0 \).

**Fig: 1.17** \( \alpha > 0 \)(abscissa represents x and ordinate represents f)
In fig 1.17 we can see that fixed points disappear for $\alpha > 0$.

1.33 Transcritical bifurcation:[53, 98]

There are certain scientific situations where a fixed point exists for all values of a parameter and can never be destroyed. However such a fixed point may change its stability as the parameter is varied. The transcritical bifurcation is the standard mechanism for such changes in stability.

For example: let $x' = r x - x^2$, $f(x, r) = r x - x^2$ where $r$ is a parameter. Clearly $x = 0$ is a fixed point. When $r < 0$, $x = 0$ is a stable fixed point and $x = r$ is an unstable fixed point. For $r = 0$, the two fixed points become the same and for $r > 0$, $x = r$ becomes stable and $x = 0$ becomes unstable. We may say that in the whole process the stability has been exchanged between the two fixed points. However like the saddle node bifurcation, the fixed points are not vanished.

*Fig: 1.17* \( r < 0 \) (abscissa represents $x$ and ordinate represents $f$)

*Fig: 1.18* \( r = 0 \) (abscissa represents $x$ and ordinate represents $f$)
1.34 Pitchfork bifurcation:[53,98]

This bifurcation generally appears in the physical problems which have symmetry. There are two types of pitch fork bifurcations, one is supercritical and the other subcritical.

For example, let us consider the differential equation $\dot{x} = r x - x^3$, $f(x, r) = r x - x^3$, we can see that the equation is invariant, if we replace $x$ by $-x$. For $r < 0$, the fixed point $x = 0$ is the only fixed point which is stable. On increasing $r$ when it becomes equal to 0, $x = 0$ is still a stable fixed point but is very much weak. That means convergence to the fixed point 0 is very much slow, which may be termed as “critical slowing down”. For $r > 0$, the fixed point $x = 0$ becomes unstable and two new fixed points appear on either side of the origin in a symmetry.
In Fig: 1.22, $x = 0$ becomes unstable and two new fixed points are produced at equal distance of $x = 0$. Near both the fixed points on two sides of $x=0$, $f$ is decreasing means both of them are stable. Near $x=0$, $f$ is increasing. Hence 0 is unstable.

Another example, let $\dot{x} = rx + x^3$, $f(x, r) = rx + x^3$ gives subcritical pitch fork bifurcation. When $r < 0$, the two fixed points $x = \pm \sqrt{-r}$ are unstable but 0 is still stable and when $r > 0$, $x = 0$ becomes unstable as $f$ near 0 is increasing function.
1.35 **Neimark-Sacker bifurcation:**[88, 116, 117]

Let $F$ be a two dimensional map from $\mathbb{R}^2 \to \mathbb{R}^2$. The NS bifurcation occurs if the Jacobian matrix $J$ of the linearised system of the map $F$ have a complex pair of eigenvalues $\lambda_{1,2}$ so that the following conditions are satisfied

(i) $|\lambda_{1,2}(a_{NS})| = 1$, but

(ii) $\lambda_{1,2}^j(a_{NS}) \neq 1$ for $j = 1,2,3,4,$

(iii) $\frac{d}{da} (|\lambda_{1,2}(a_{NS})|) = d > 0$

where $a_{NS}$ is the bifurcation parameter calculated at the bifurcation point, and $d$ is a constant.

The conditions for NS bifurcation were first derived independently by Neimark in 1959 [88], Sacker in 1964 [117] and by Ruelle and Takens in 1971[116]. Landford in 1973 [69] includes the condition $\lambda^5 \neq 1$. A modification to deal with this case may be found.
in Iooss [59] who gives more precise details of the differentiability conditions required and the regularity of the bifurcating circles.

1.36 Hopf bifurcation: [46, 53, 76, 87, 151]

(i) If two complex conjugate eigenvalues simultaneously cross the imaginary axis into the right half plane, Hopf bifurcation occurs.

Two types of Hopf bifurcations are:

a) Supercritical hopf bifurcation

\[
\begin{array}{c}
\beta < 0 \\
\beta = 0 \\
\beta > 0
\end{array}
\]

*Fig: 1.26 Supercritical Hopf bifurcation at the parameter value $\beta = 0$*

b) Subcritical hopf bifurcation

\[
\begin{array}{c}
\beta < 0 \\
\beta = 0 \\
\beta > 0
\end{array}
\]

*Fig: 1.27 Subcritical Hopf bifurcation at the parameter value $\beta = 0$*

1.37 Intermittency:[53, 98, 138]

In dynamical systems, intermittency is the irregular occurrence of bursts of chaotic behavior interspersed with intervals of apparently periodic behavior. As some control parameter of the system is changed, the chaotic bursts become longer and occur more frequently until, eventually, the entire time record is chaotic.

Pomeau and Manneville described three routes to intermittency where a nearly periodic system show irregularly spaced bursts of chaos. These (type I, II, III) correspond to the
approach to a saddle node bifurcation, a subcritical Hopf bifurcation, or an inverse period doubling bifurcation. The type I intermittency leads to irregularly occurring bursts of periodic and chaotic behavior. However, during these bursts, the amplitudes of the motion are stable (on the average). We call this stable intermittency or tangent bifurcation intermittency since the bifurcation event is a tangent bifurcation or a saddle node bifurcation. In type II intermittency, the limit cycle associated with the second frequency becomes unstable, and we observe bursts of two-frequency behavior mixed with intervals of chaotic behavior. Thus, type II intermittency is a type of Hopf event. In type III intermittency event, the amplitude of the sub-harmonic behavior created at the bifurcation point grows, while the amplitude of the motion associated with the original period decreases. This periodic behavior, however, is interrupted by bursts of chaotic behavior.

![Horizontal axis represents the number of iterations
Vertical axis represents the iterated values of the logistic map](image)

**Fig 1.28** Time series plot for \( f \) and \( f^3 \), at \( \mu = 3.82832712474619 \) near the period 3 window. Intermittent behaviour is evident. Parts of the behaviour appears to be periodic with period-3 and other parts appear to be chaotic.

### 1.38 Crisis:[53, 98]

In the theory of dynamical systems, a *crisis* is the sudden appearance or disappearance of a strange attractor as the parameters of a dynamical system are varied. This global bifurcation occurs when a chaotic attractor comes into contact with an unstable periodic orbit or its stable manifold. As the orbit approaches the unstable orbit it will diverge away from the previous attractor, leading to a qualitatively different behavior. Crises can produce intermittent behavior.
Grebowi, Ott, Romeiras, and Yorke distinguished between three types of crises as given below:

- The first type, a boundary or an exterior crisis, the attractor is suddenly destroyed as the parameters are varied. In the post-bifurcation state the motion is transiently chaotic, moving chaotically along the former attractor before being attracted to a fixed point, periodic orbit, quasi-periodic orbit, another strange attractor, or diverging to infinity.
- In the second type of crisis, an interior crisis, the size of the chaotic attractor suddenly increases. The attractor encounters an unstable fixed point or periodic solution that is inside the basin of attraction.
- In the third type, an attractor merging crisis, two or more chaotic attractors merge to form a single attractor as the critical parameter value is passed. The reverse case (i.e. the sudden appearance, shrinking or splitting of attractors) can also occur. In the bifurcation diagram of logistic map function we can identify all the three types of crisis.

\[ \mu \in [3.5, 4.0] \]

Horizontal axis represents the bifurcation parameter
Vertical axis represents the final state of the logistic map

**Fig 1.29** Bifurcation diagram for the logistic map between $a = 3.5$ and $4.0$

Referring to the above figure, the most prominent attractor-merging crisis occurs at $a = 3.68$ where the two-piece chaotic attractor merges smoothly at a wedge-shaped point, sometimes called the Misiurewicz point. The most notable interior crisis is near at
\( a = 3.86 \) in the period-3 window where a 3-piece chaotic attractor abruptly blows up to fill the entire space between the three pieces. The only example of boundary crisis in the logistic map occurs at \( a = 4.0 \) beyond which the chaotic attractor vanishes and all orbits tend to infinity.