ABSTRACT

Introduction:

Dynamical systems theory is an area of mathematics used to describe the behavior of complex dynamical systems, usually by employing differential equations or difference equations. When differential equations are employed, the theory is called continuous dynamical systems. When difference equations are employed, the theory is called discrete dynamical systems. This theory studies the solutions of the equations of motion of systems that are primarily mechanical in nature; although this includes planetary orbits as well as the behavior of electronic circuits and the solutions to partial differential equations that arise in biology etc.

Dynamical systems theory deals with the long-term qualitative behavior of dynamical systems. Here, the focus is not on finding precise solutions to the equations defining the dynamical system, but rather to answer questions like "Will the system settle down to a steady state in the long term, and if so, what are the possible steady states?", or "Does the long-term behavior of the system depend on its initial condition? etc."

Here, an important goal is to describe the fixed points, or steady states of a given dynamical system; these are values of the variable that don't change over time. Some of these fixed points are attractive, meaning that if the system starts out in a nearby state, it converges towards the fixed point. Similarly, one is interested in periodic points, states of the system that repeat after several time steps. Periodic points can also be attractive.

Even simple nonlinear dynamical systems often exhibit almost random, completely unpredictable behavior that has been called chaos. The branch of dynamical systems that deals with the investigation of chaos is called chaos theory.
Chaostheoryis one of the most rapidly expanding research topics of recent decades. Many fascinating results are coming out from research in the field of chaos and fractals. Deterministic chaos is a fairly active area of research in the last few decades. Simple ordinary differential equations like Lorentz system initiated from modeling of atmospheric dynamics gives rise to some fascinating structure called as chaotic attractor. For discrete chaos, there is another famous chaotic system, called logistic map, which came in to prominence as a model for studying population sizes. These simple models are widely used in the study of chaos today, while other similar models exist showing chaotic phenomenon. Till date scientists have obtained a lot of non linear systems like Lorentz system or logistic map which can produce chaotic behaviors proving that it is a common phenomenon in non linear representation of scientific observations of real world. Chaotic systems thus have become an important part of science at the theoretical as well as practical level of research.

**Objective of study and chapters:**

From above we can say that dynamical system comprises of analysis, computation and geometrical explanation of dynamical behavior of different systems. We have employed both analytical and numerical techniques (with the help of C language and MATHEMATICA) to carry out our investigation.

We set our journey as follows:

Before proceeding to the principal goal, we give a lay out of some fundamental concepts and results which will be helpful in building the material.

**Chapter 1** deals with some basic concepts and results which are closely related to our research work. These are gathered from different literature, thesis, monographs and research papers.

**Chapter 2** deals with the phenomenon of ‘reverse bifurcation’. One of the common route to chaos is the period doubling route [53, 98]. For systems that
undergo period doubling cascades, there also exists an “inverse cascade” [16] of chaotic band merging called reverse bifurcation [34, 63, 74]. This chapter investigates the period doubling route to chaos and the period doubling nature of chaotic bands using the logistic map. We have considered this map and identified the parameter values $\mu$ for which the period doubling bifurcations occur and have shown that the bifurcation points converges to an accumulation point where the chaotic situation starts. Our tool for finding such a point is with the help of establishing the ‘Feigenbaum delta’ [32] which is one of the several universalities discovered by famous particle physicist M. J. Feigenbaum. The period doubling scenario explains us how the behaviour of the model changes from regularity to a chaotic one.

Further, we have discussed about the reverse bifurcation and reverse bifurcation points called Misiurewicz points [84, 93,94,109] and established the Feigenbaum delta in that case also. This situation occurs inside the chaotic region and it unfolds some regularity even within the chaotic region.

**In chapter 3**, we have discussed about ‘Intermittency’. Intermittency is sporadic switching between two qualitatively different behaviours. The intermittent transition to turbulence was first discussed by Pomeau and Manneville [86] in connection with Lorentz model. The phenomenon of intermittency is quite common in systems where the transition from periodic to chaotic behaviour takes place through a saddle node bifurcation. In our present investigation, we have verified the power law established by Pomeau and Monneville [86, 103] which states that the number of iterations $[N(\mu)]$ inside the narrow channel is proportional to $(\mu_t - \mu)^{-\frac{1}{2}}$ in case of the logistic map. We found out the constant of proportionality in the above case for period three window and this was found approximately to be 0.377413...

**In chapter 4**, we have discussed about quasiperiodicity and mode locked states in case of a two dimensional map. In this chapter we have investigated the
emergence of quasiperiodic and mode-locked states which arises from Neimark-Sacker (NS) bifurcation in Maynard Smith map which is given by $F(x, y) = (y, ay + b - x^2)$, where $a$ and $b$ are real parameters. Analytical results are obtained near NS bifurcation using normal forms. In our investigation, we have further used the techniques of Lyapunov exponent, bifurcation diagram and phase portrait to show transition from quasiperiodic and from mode-locked states into chaotic states.

In chapter 5, we have discussed about the period doubling route to chaos in case of a three dimensional differential equation, popularly known as Rossler system.

The Rössler system is given by

$$\begin{align*}
\dot{x} &= -y - z \\
\dot{y} &= x + ay \\
\dot{z} &= b + z(x - c)
\end{align*}$$

where $a, b, c \in R$ and they are assumed to be positive and dimensionless.

The Rössler system plays an important role in the study of dynamical systems because it is one of the most elementary geometric constructions of chaos in continuous systems [70, 71]. It has the same number of terms as the Lorenz system but only a single quadratic nonlinearity, the product of $x$ and $z$ in the third equation. The Rössler system was designed with the purpose of creating a model for a strange attractor which uses only the simplest chaos generating mechanism, stretch andfold. The first two linear equations of the system are responsible for stretching and the last one which is nonlinear is responsible for folding action. The system admits chaotic solutions for $a = 0.2, b = 0.2$ and $c = 5.7$.

In this system we have discussed how the period doubling bifurcation points can be found out numerically and the Feigenbaum delta can be established, which is the universal constant established by M.J. Feigenbaum. The tools employed in
our investigation are Poincare map, phase portrait, time-series plot, bifurcation diagram and Lyapunov exponents.

**In chapter 6,** we have investigated the stability nature of Hopf bifurcation in a two dimensional nonlinear differential equation given by

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= \mu_1 + \mu_2 y + \mu_3 x^2 + \mu_4 xy
\end{align*}
\]

Interestingly, our considered model exhibits both supercritical and subcritical Hopf bifurcation for certain parameter values which marks the stability and instability of limit cycles created or destroyed in Hopf bifurcations respectively. We have used the Center manifold theorem and the technique of Normal forms in our investigation.

**In chapter 7** we have proposed some problems which can extend our investigation in our future research work on dynamical systems.

At the end, a Bibliography is given which contains books, monographs and research papers closely related to our field of research.

Submitted by:

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