Chapter II

LEGENDRE FOURIER ANALYSIS OF THE SOLAR MAGNETIC FIELD INFERRED FROM SUNSPOT GROUPS

2.1. Introduction

A number of observations suggest that the solar activity may be originating in 'waves'. For example, the well-known 'butterfly diagrams' show that the sunspot activity originates in waves with periods 11-yr (or multiples there of), which propagate from middle latitudes in each solar hemisphere towards the solar equator (Becker 1955, references therein). The observed pattern of so called 'torsional oscillation' is of the form of waves in each hemisphere traveling from high latitudes to the equator in about 22 years with an amplitude of a few ms⁻¹ (Howard & LaBonte 1980; LaBonte & Howard 1982a; Snodgrass 1985; Komm et al. 1993a). Poleward migration of weak magnetic field (Howard 1974), prominence belts and polar faculae (Makarov & Sivaraman 1983, 1986) can also be seen to be waves of one-way migrations. It could be that all these waves are in fact global, but appear confined to certain latitudes in their manifestations in the respective observations. Supposing this is true, each wave contributing to the overall pattern would be equivalent to a set of at least approximately stationary global oscillations of the Sun with appropriate phase differences. The following questions would then arise: (i) which set of oscillation modes contributes with appreciable amplitudes; (ii) what are the amplitudes and phases of these modes, (iii) how do the amplitude and phases vary in time, (iv) what is the physical nature of these oscillations, and so on. We discuss these questions in the present chapter.

We inferred sunspot 'occurrence probability' and a rough measure of the underlying 'toroidal magnetic field' during 1874-1976 using Greenwich sunspot group data (kindly provided by H. Balthasar) and Hale's Law of magnetic polarities (Section 1.2.2 of Chapter I). From spherical-harmonic-Fourier (SHF) analysis of these sunspot 'occurrence probability' and 'toroidal magnetic field' we showed that the butterfly diagrams could be a consequence of the near constancy of amplitudes and phases of axisymmetric odd-degree oscillations of period ~ 22 yr in the Sun's magnetic field (Gokhale & Javaraiah 1990a; Gokhale et al. 1990). The amplitude spectrum of these oscillations for degree up to \( l = 13 \) was shown to be similar to that derived by Stenflo & Vogel (1986) and Stenflo (1988) from magnetogram data during 1960-1985. We have also tentatively inferred that the third harmonic of the 22-yr magnetic cycle might be an artifact of some non-linearity between the field magnitude and the sunspot occurrence
probability (Gokhale & Javaraiah 1990b).

We extended the above study of global modes in the solar magnetic fields, (Gokhale et al. 1992; Gokhale & Javaraiah 1992, 1995), with a refined analysis of the data. The Legendre-Fourier (SHF order \( m = 0 \)) analyses were carried out with the highest temporal frequency resolution (\( \sim 1/103 \) yr\(^{-1} \)) allowed by the length of the whole sequence, and over an extended area of the \( \nu - l \) plane, viz., up to \( l = 36 \) and \( \nu = 55/107 \) yr\(^{-1} \), where \( l \) is the spherical harmonic degree and \( \nu \) is the temporal frequency. In this chapter I present the results of these extended studies.

In Section 2.2 we describe the data and the method of analysis. In Section 2.3 we present the LF amplitude spectrum of the 'inferred magnetic field', \( B_{m=0}(\theta, \phi, t) \), as perspective plots, and in Section 2.4 using grey-level representation. In Section 2.5 we identify coherent oscillations in the 'main power ridges'. In Section 2.6 we describe the physical reality of the coherent oscillation modes. In Section 2.7 we compare the results found by us from the sunspot data with the results from the magnetogram data (Stenflo 1988). In Section 2.8 we model the variation of the annual measure of sunspot activity. In Section 2.9 we identify four dominant modes of stationary oscillations. In Section 2.10 we conjecture a phenomenology for maintenance of the Legendre Fourier (LF) spectrum and production of activity. In Section 2.11 we draw conclusions and briefly discuss them.

The mathematical formulation of the SHF analyses of sunspot data (Gokhale & Javaraiah 1990) is given in Appendix 2A.

**2.2. Data and Method of Analysis**

2.2.1. **THE DATA**

We have used the data from Ledgers I and II of the Greenwich photographic results (GPR). A magnetic tape of this data for the years 1874–1976 was kindly provided by H. Balthasar. From this data we chose the heliographic coordinates i.e., latitude (\( \lambda \)) and longitude (\( \phi \)), and the time of observation (\( t \)) for each day observation of each sunspot group.

2.2.2. **DEFINITION OF SUNSPOT OCCURRENCE PROBABILITY**

For each time interval (\( T_1, T_2 \)) chosen for analysis (e.g., a sunspot cycle or a sequence of years/cycles) we have defined 'sunspot occurrence probability' \( p(\mu, \phi, \tau) \) as

\[
p(\mu, \phi, \tau) = \begin{cases} \frac{1}{N} \delta(\mu - \mu_i, \phi - \phi_i, \tau - \tau_i) & \text{at } (\mu_i, \phi_i, \tau_i), i = 1, 2, ..., N \\ \frac{1}{N} & \text{elsewhere in } (\mu, \phi, \tau) \text{ space} \end{cases}
\]

where \( \tau = (T_2 - T_1)/(T_2 - T_1) \). \( \mu = \cos \theta, \theta = 90^\circ - \lambda \). \( \delta \) represents a delta function i
\( \mu, \phi, \tau \), \( N = N(T_1, T_2) \) is the number of data points during the interval \((T_1, T_2)\), and \( t \) is the time of observation (in days, including the fraction of the day of the observation) from the zero hour of the first day of the interval \((T_1, T_2)\).

Owing to the non-zero (though small) spreads and uncertainties in \( \mu_i, \phi_i, \) and \( \tau_i \), the delta functions here must be considered as mathematical idealizations of properly normalized 'physical' delta functions of large finite values over small finite domains.

### 2.2.3. Definition of 'Nominal Toroidal Field' Inferred from Sunspot Data

A strong toroidal field is essential for production of sunspot activity in general. The toroidal field is not directly measurable, but from the generally bipolar nature of the fields associated with activity, it is believed that the emerging flux is toroidal. Hence, using Hale’s laws of magnetic polarities, we define a ‘nominal toroidal field’, \( B_\phi(\theta, t) \), in the following manner (we thank Professor M. Stix for this suggestion):

\[
B_\phi(\theta, t) = \begin{cases} 
\pm p(\theta, t) & \text{in the northern solar hemisphere} \\
\mp p(\theta, t) & \text{in the southern solar hemisphere},
\end{cases}
\]

the upper signs being taken during the ‘even numbered’ sunspot cycles and the lower ones during the ‘odd numbered’ cycles.

### 2.2.4. Definition of the ‘Inferred Magnetic Field’ in Terms of the ‘First Day Data’

Earlier, (Gokhale & Javaraiah 1990a; Gokhale et al. 1990; Gokhale & Javaraiah 1990b), we used the above definitions for sunspot occurrence probability and inferred magnetic field. Later, (Gokhale et al. 1992; Gokhale & Javaraiah 1992, 1995), we modified the definition of the occurrence probability \( p(\theta, \phi, t) \) during a time interval \((T_1, T_2)\) as

\[
p(\theta, \phi, t) = \begin{cases} 
\frac{n_k}{N} \delta(\mu - \mu_k, \phi - \phi_k, \tau - \tau_k) & \text{at } (\mu_k, \phi_k, \tau_k) \\
0 & \text{elsewhere},
\end{cases}
\]

where \( \theta, \phi, \tau, \) \( t \) and \( \delta \) have the same meanings as explained in Section 2.2.2. Here the arguments of the delta function \( \mu_k = \cos \theta_k, \phi_k, \) \( t_k \) are the values of \((\mu, \phi, t)\) given by the first day’s observation of the kth spot group, \( n_k \) is the overall life span of the kth spot group, in days, and

\[
N = \sum_{k=1}^{K} n_k,
\]

\( K \) being the total number of all the sunspot groups observed during the interval \((T_1, T_2)\).

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In this modified definition, each spot group is considered as a phenomenon occurring at the heliospheric position where it was first observed. As a result, the scatter due to its subsequent proper motions gets eliminated.

Moreover, in this definition the statistical weight of a spot group is proportional to its lifespan rather than the number of days of its observation. Thus, the statistical weights of spot groups get corrected for the gaps in the observations. These corrections are particularly important for the recurrent spot groups.

Similar to the magnetic field defined in Section 2.2.3, the ‘inferred’ magnetic field \( B_{\text{inf}}(\theta, \phi, t) \) is then defined here as

\[
B_{\text{inf}}(\theta, \phi, t) = \pm p(\theta, \phi, t),
\]

where the sign, plus or minus, is chosen according to Hale’s laws of magnetic polarities of bipolar sunspot groups. Care is taken to separate the data from the old and the new sunspot cycles during the periods of overlaps of successive cycles (next subsection). We find that these refinements improve the accuracy of our results by a few percent.

Another measure of the ‘inferred’ magnetic field, \( B_{\text{inf}}(\theta, \phi, t) \) is defined by replacing the life spans \( n_k \) in Equations (2.1) and (2.2) by the maximum observed areas \( A_{\text{max}, k} \) of the spot groups as statistical weights of the probability distribution \( P(\theta, \phi, t) \).

**Resulting Modifications in the Formulae for Computing LF Amplitudes and Phases**

The definition of the inferred magnetic field above has led to the following modified formulae for computing the LF amplitudes \( A(l, \nu) \) and phases \( \varphi(l, \nu) \) of the axisymmetric modes in \( B_{\text{inf}}(\theta, \phi, t) \) during any specified time interval \( (T_1, T_2) \) (for determining the phases of the various modes, the time interval must be at least as long as the period of oscillation):

\[
A(l, \nu) = [P^2_c(l, \nu) + P^2_\phi(l, \nu)]^{1/2}
\]  

(2.3)

and

\[
\varphi(l, \nu) = \tan^{-1}[P_c(l, \nu)/P_\phi(l, \nu)]
\]

(2.4)

(to be chosen 0° and 360° so that \( \sin \varphi \) and \( \cos \varphi \) have the correct signs), where

\[
P_c(l, \nu) = \frac{C(l, \nu)}{N} \sum_k n_k P(l, \mu_k) \cos(2\pi \nu t_k),
\]

(2.5a)

\[
P_\phi(l, \nu) = \frac{C(l, \nu)}{N} \sum_k n_k P(l, \mu_k) \sin(2\pi \nu t_k).
\]

(2.5b)

where

\[
\gamma(l, \nu) = \begin{cases} 
(2l + 1)/2\pi & \text{if } \nu \neq 0, \\
(2l + 1)/4\pi & \text{if } \nu = 0.
\end{cases}
\]

(2.6)
2.2.5. SEPARATION OF THE DATA FROM SUCCESSIVE SUNSPOT CYCLES DURING THE PERIODS OF THEIR OVERLAPS

Earlier, (Gokhale & Javaraiah 1990a; Gokhale et al. 1990), we had separated the data of successive sunspot cycles by treating the data before and after the specified dates of sunspot minima as belonging respectively to the old and the new cycles. This had caused wrong signs to be attached in determining \( B_{\text{inf}} \) from the new cycle data before the dates of minima (and from the old cycle data after the dates of minima), during the few years of overlaps of the successive cycles. In later analyses, (Gokhale et al. 1992, Gokhale & Javaraiah 1992, 1995), we have drawn, in the latitude-time plane, straight lines in the “butterfly wings” which separate completely the data belonging to the old and the new cycles, and have attached signs strictly in accordance with Hale’s laws of sunspot magnetic polarities.

2.2.6. THE UNIT OF FREQUENCY \( (\nu_0) \)

The length of the full data set allows a frequency resolution \( \sim 1/103 \text{ yr}^{-1} \). However, from the drift rates of phases of the axisymmetric modes we have found the mean frequency of these modes to be \( 1/21.4 \text{ yr}^{-1} \). Hence, we have taken its integral quotient nearest to \( 1/103 \text{ yr}^{-1} \), viz., \( \nu_0 = 1/107 \text{ yr}^{-1} \), as the unit of frequency.

2.3. The LF Amplitude Spectrum of \( B_{\text{inf}}(\theta, \phi, t) \)

2.3.1. THE PERSPECTIVE PLOTS

In Figures 2.1(a) and 2.1(b) we give perspective plots of the SHF amplitudes of the odd and the even degree axisymmetric modes in \( B_{\text{inf}}(\theta, \phi, t) \) with respect to \( l \) and \( \nu \) up to \( l = 36 \) and \( \nu = 55\nu_0 \).

2.3.2. THE SPECTRUM FOR THE ODD DEGREE MODES

It is clear that most of the LF power is concentrated in the odd degree modes with frequencies within \( \pm \nu_0/2 \) on either side of \( \nu = 5\nu_0 \). (This corresponds to periods in the range 19.8 yr and 23.3 yr.) In this narrow frequency band the spectrum forms a smooth ‘ridge’ which has a ‘main hump’ over \( l = 1 - 11 \), with a high peak at \( l = 5, 7 \) and a ‘tail’ for \( l > 19 \).

There are low, ‘subsidiary’ ridges along \( \nu = 3\nu_0, 5\nu_0 \) and \( 9\nu_0 \). The power concentration in the ridge along \( \nu = 3\nu_0 \) is expected in view of the asymmetry of the sunspot cycle (e.g., Bracewell 1988). It was shown earlier (Gokhale & Javaraiah 1990b) that the amplitudes and phases of the modes with \( \nu = \nu_* \) are correlated to those of the modes of the same \( l \) with \( \nu = 3\nu_0 \). This indicates that the secondary ridge at \( \nu = 3\nu_0 \) could be a mathematical artifact of some nonlinear dependence of the ‘inferred field’ on the
variable representing the basic physical oscillations with frequency \( \nu_* \). Alternatively it could also result from common excitation by a single forcing oscillation. These possibilities may be true even for the other subsidiary ridges which seem to represent the set of the higher odd harmonics of \( \nu_* \).

It is important to note that in Figures 2.1(a) and 2.1(b), there is no evidence for existence of any general \( \nu - l \) relation (free oscillation modes) as hinted from magnetogram data (Stenflo \\& Vogel 1986). On the contrary power concentration in ridges along specific frequencies suggest 'forced' oscillations.

(The amplitude spectrum of the odd degree modes in the inferred field \( B_{nl}(\theta, \phi, t) \) obtained by taking in Equation (2.1) statistical weights proportional to the 'maximum observed areas of sunspot groups' instead of their 'life spans' is found to be noisier than the one obtained from \( B_{nl}(\theta, \phi, t) \). Thus, the lifespan of a sunspot group gives a clearer information on the global nature of solar cycles than that given by the maximum area.)

2.3.3. THE SPECTRUM FOR THE EVEN DEGREE MODES

The amplitude spectrum for the even degree modes is noisy and the level of power is comparable to that of the noise level in the spectrum for the odd degree modes. There are no power ridges to suggest any \( \nu - l \) relation.

2.4. The Grey-Level Representation of the Spectra

For determining whether the local power concentrations in the LF spectra are aligned along any curves in the \( \nu - l \) plane, we made image-processed grey-level representations of the power spectra corresponding to the amplitude spectra given in Figure 2.1 using the standard method of hodograph equalization. These representations are shown in Figures 2.2(a) and 2.2(b). Needless to say, equal intensities do not represent equal amplitudes in these figures.

2.4.1. 'APPARENT' EXISTENCE OF \( \nu - l \) RELATIONS

These grey level representations indicate a possibility that in each of the power spectra of the odd and the even degree modes, the peaks may lie along a set of curves in the \( \nu - l \) plane. One such set of curves, drawn subjectively, is illustrated in the right-hand side panel in each figure. Incidentally, these sets of curves are reminiscent of a similar looking set in the power spectra of the acoustic modes.

The following peculiarities raise doubts about the reality of the existence of the \( \nu - l \) relations indicated by the 'curves' in Figures 2.2(a) and 2.2(b):

(i) The power distribution does not give continuous ridges along these curves, and
Figure 2.2. Image-processed grey-level diagrams representing the LF power spectra corresponding to the amplitude spectra in Figures 2.2(a) and 2.2(b), respectively. The panel on the right side of each diagram shows a set of curves along which the LF power 'appears' to be aligned.
(ii) The curves cannot be drawn unambiguously in an objective manner. In fact some power concentrations can alternatively be considered to be lying along a set of curves with negative slopes.

2.4.2. SPECTRA OBTAINED FROM TWO ‘PARTLY RANDOM’ SIMULATED DATA SETS

In order to determine whether the apparent alignments of the power concentrations in the SHF spectra have any real significance, we compare these spectra with those obtained from two ‘partly random’ simulated data sets. In both the simulated data sets, the epochs and the lifetimes of the ‘sunspot groups’ were assumed to be the same as those in the real data, but the latitudes of sunspot groups ‘occurring’ during each year and within each butterfly wing, were redistributed within the width of the wing in a random way. Thus, the simulation is random on scales $l \geq 13$ in odd parity.

In doing so, the wings given by the new cycle data in each hemisphere, during the years of overlaps, were treated separately. Hence, the simulations are random on all scales ($l \geq 0$) in the even parity.

In simulation ‘I’ the latitudes were re-distributed with Gaussian probability distributions, with means and standard deviations the same as those in the real data.

In simulation ‘II’ the redistributions were made with uniform distributions over the full widths defined by the yearly minimum and the maximum latitudes of the real data in the respective wings.

Amplitude Spectra Given by the Simulated data sets

Both the simulated data sets SI and SII yield perspective plots of LF spectra similar to those given by the real data and, hence, those plots are not reproduced here. They are low and noisy for the even degree axisymmetric modes and show high, smooth ridges at $\nu = \nu_\epsilon = 5\nu_0$, peaking at $l = 5$, 7 (and also low ridges along a few odd harmonics of $\nu = \nu_\epsilon$) for the odd degree axisymmetric modes. The ridges given by data sets SI and SII simulate the ridge in Figure 2.1(a) up to $l = 21$ and $l = 35$, respectively. This shows that the distribution of sunspot activity within the butterfly wings is closer to reality in the uniform probability distribution than in the Gaussian probability distribution. For odd degree modes with $l \leq 11$, the similarity in the spectra of the real and the simulated data sets are a consequence of the fact that both the simulated data sets have exactly the same temporal distribution as that of the real data and also simulate the butterfly wings in the latitude time distribution.

Grey-Level Representations of the Spectra Given by the Simulated Data Sets

Power concentrations were found to be aligned along several curves even in the grey-
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Amplitude Spectra Given by the Simulated data sets

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Grey-Level Representations of the Spectra Given by the Simulated Data Sets

Power concentrations were found to be aligned along several curves even in the grey-
level representations of the spectra obtained from the simulated data sets ‘SI’ and ‘SII’. The presence of power alignments like those seen in 2.2(a) are expected in the region \( l < 13 \) of the \( \nu - l \) plane for odd degree modes since on these scales and in odd parity the simulated data sets retain the systematic properties of the butterfly diagram of the real data. But the power alignments are also reproduced for \( l \geq 0 \) in even modes and also for \( l \geq 13 \) in odd modes, on which scales the latitudes are randomly distributed in ‘SI’ and ‘SII’. Hence, the alignments in Figure 2.2 cannot be taken as evidence for any physical relation between the \( \nu \) and \( l \) of the modes in the real data.

2.5. Coherent Oscillations in the ‘Main Power Ridges’

2.5.1. VARIATIONS IN THE RELATIVE PHASES OF MODES IN THE ‘MAIN POWER RIDGE’

We determined the phases \( \varphi_l \) of the modes \( l = 1-35 \) in the ‘main power ridge’, (\( \nu = \nu_\odot = 1/21.4 \text{ yr}^{-1} \)), during the eighty two intervals (during 1874–1976) each of 22 yr length and displaced by one year relative to the previous interval. As expected from the earlier somewhat cruder analysis (Gokhale et al. 1990), the relative phases \( \delta_l = \varphi_l - \varphi_b \) are approximately constant. For quantitative comparison of their constancy, we give in Table 2.1 the mean values (\( \bar{\delta}_l \)) and the standard deviations (\( \Delta \delta_l \)) of \( \delta_l \) for the field inferred from the real data set (‘R’) and also for fields derived from the two simulated data sets ‘SI’ and ‘SII’.

The data samples during the successive 22-yr intervals are not statistically independent (since they overlap) and, hence, it may be thought that the r. m. s. variations should have been computed from a set of four or five non-overlapping intervals during 1874–1976. However, the mean values and the r. m. s. variations computed from the 82 intervals are equivalent to those computed from non-overlapping intervals with uniform weighting in time. Moreover, we have verified that the r.m.s. variations computed from non-overlapping intervals are actually smaller than those from the eighty-two intervals.

2.5.2. THE STATIONARY, APPROXIMATELY STATIONARY, AND NON-STATIONARY MODES

From Table 2.1 we see that for each data set, the modes in the main ridge fall automatically into three categories:

(1) Stationary Modes: viz., modes with constant phases (\( \Delta \delta_l \leq 15^\circ \)).

(2) Approximately Stationary Modes: viz., modes with approximately constant phases (\( \Delta \delta_l \leq 30^\circ \)).

(3) Non-Stationary Modes: viz., modes with large phase variations (\( \Delta \delta_l \gg 30^\circ \)).

In Table 2.2 we list the modes of each category in the data sets R, SI and SII.
Table 2.1: Mean values $\overline{\delta}_i$ (in degrees) and standard deviations $\Delta \delta_i$ (in degrees) of the relative phases $\delta_i(= \varphi_1 - \varphi_3)$ of the LF odd degree modes (SHF order $m = 0$) of frequency $1/21.4 \text{ yr}^{-1}$, in the magnetic field inferred from real data set $R$ and the simulated data sets SI and SII.

| $l$ | $R$ | | | $SI$ | | | $SII$ | | |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|     | $\overline{\delta}_i$ | $\Delta \delta_i$ | $\overline{\delta}_i$ | $\Delta \delta_i$ | $\overline{\delta}_i$ | $\Delta \delta_i$ | $\overline{\delta}_i$ | $\Delta \delta_i$ |
| 1   | 7.7 | 1.4 | 9.5 | 4.3 | 7.9 | 1.57 |         |       |
| 3   | -174.5 | 1.0 | -173.5 | 3.2 | -174.7 | 1.12 |         |       |
| 5   | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 8.0 |         |       |
| 7   | 170.4 | 2.1 | 167.4 | 8.6 | 169.3 | 2.8 |         |       |
| 9   | -27.3 | 6.0 | -30.5 | 21.8 | -32.3 | 10.0 |         |       |
| 11  | 123.0 | 10.3 | 139.6 | 27.9 | 112.9 | 16.9 |         |       |
| 13  | 93.2 | 9.6 | -35.8 | 44.7 | -99.8 | 11.5 |         |       |
| 15  | 55.4 | 9.1 | - | $\gg$ 30.0 | 75.5 | 31.9 |         |       |
| 17  | -155.5 | 10.6 | - | - | -87.3 | 77.4 |         |       |
| 19  | -9.6 | 14.5 | - | - | - | $\gg$ 30.0 |         |       |
| 21  | 142.3 | 23.1 | - | - | - | - |         |       |
| 23  | -49.2 | 32.7 | - | - | - | - |         |       |
| 25  | 132.2 | 35.2 | - | - | - | - |         |       |
| 27  | -10.5 | 25.0 | - | - | - | - |         |       |
| 29  | 190.9 | 27.5 | - | - | - | - |         |       |
| 31  | -4.1 | 70.4 | - | - | - | - |         |       |
| 33  | 147.1 | 97.6 | - | - | - | - |         |       |
| 35  | 92.0 | 77.9 | - | - | - | - |         |       |

2.5.3. PRESENCE OF FOUR MODES OF 'COHERENT GLOBAL OSCILLATIONS'

Using the complex amplitude $A_l$ (determined in Section 2.3) and relative phases $\delta_l$ we have computed the mean relative real amplitudes $a_l(= A_l \cos \delta_l)$ and $b_l(= A_l \sin \delta_l)$ of the sine and cosine (temporal) phases of the stationary and approximately stationary modes in the field inferred from the real data (‘R’) and that inferred from the simulated data set ‘SII’ (the set that simulates the real data more closely: Section 2.4.2). These are given in Table 2.3. In the same table, we also give the r. m. s. variations $\Delta a_l$ and $\Delta b_l$ of $a_l$ and $b_l$. It can be seen that in each phase, there are groups of terms characterized by a range of $l$ in which the respective ‘phase-amplitude’, $a_l$ or $b_l$, is maximum near the center and falls off to the noise level (i.e., $\leq 0.04$) at both ends. The signs of the successive phase-amplitudes in each group are alternately positive and negative, and this rule breaks at the ends of the respective $l$-ranges. The level of the r. m. s. variations $\Delta a_l$ or $\Delta b_l$ within each group are mutually similar, and different from those in the other groups.

From Table 2.3 it is clear that the field inferred from the real data contains four
Table 2.2: List of modes of each category in the three data sets

<table>
<thead>
<tr>
<th>Category</th>
<th>R (real data)</th>
<th>SI (Gaussian)</th>
<th>SII (box)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta \delta_1 \leq 15^\circ$</td>
<td>$l = 1 - 9$</td>
<td>$l = 1 - 7$</td>
<td>$l = 1 - 13$</td>
</tr>
<tr>
<td>$\Delta \delta_1 \leq 30^\circ$</td>
<td>$l = 21 - 29$</td>
<td>$l = 9 - 11$</td>
<td>$l = 15$</td>
</tr>
<tr>
<td>$\Delta \delta_1 \gg 30^\circ$</td>
<td>$l \geq 31$</td>
<td>$l \geq 13$</td>
<td>$l \geq 17$</td>
</tr>
</tbody>
</table>

...independent geometrical modes of coherent global oscillations defined by the following expressions:

$$
B_1 = \left[ \sum_{l=1}^{11} a_l P_l(\mu) \right] \sin(2\pi \nu_s t), \quad B_2 = \left[ \sum_{l=3}^{17} b_l P_l(\mu) \right] \cos(2\pi \nu_s t),
$$

$$
B_3 = \left[ \sum_{l=15}^{29} a_l P_l(\mu) \right] \sin(2\pi \nu_s t), \quad B_4 = \left[ \sum_{l=21}^{25} b_l P_l(\mu) \right] \cos(2\pi \nu_s t).
$$

(2.7)

Pending identification of the physical nature of these oscillation modes, we call them 'geometrical eigenmodes'.

2.5.4. ABSENCE OF $B_3$ AND $B_4$ IN SIMULATED DATA SETS

The simulated data sets 'SI' and 'SII' do not yield even approximately constant phases for modes beyond $l = 11$ and 15, respectively (see Table 2.1). It is obvious that the field inferred from these data sets cannot contain the geometrical eigenmodes $B_3$ and $B_4$. These data sets contain eigenmodes $B_1$ and $B_2$ in truncated forms, as illustrated for 'SII' in Table 2.3.

2.5.5. GEOMETRICAL REALITY OF THE MODES $B_1, ..., B_4$ IN DISTRIBUTION OF $B_{10f}(\theta, t)$

The simulated data set 'SI' and 'SII' are only illustrative. However, the following points emerge from the above results.

(i) The presence of $B_1$ and $B_2$, though in their truncated forms, in the simulated data sets 'SI' and 'SII' is due to the incorporation of approximately real butterfly diagrams in these data sets through prescription of the real annual means and standard deviations for latitudes of sunspot groups within each wing.

(ii) The absence of the 'higher degree' oscillations $B_3$ and $B_4$ in SI, and even in the more realistic simulation SII, illustrate the fact that these two oscillations are not
Table 2.3: Mean values \((a_l, b_l)\) and r.m.s. variations \((\Delta a_l, \Delta b_l)\) of the amplitudes of the sine and cosine components, respectively, of the LF modes of odd degrees and 21.4-yr periodicity in the magnetic field inferred from two data sets. Groups of \(a_l\) and \(b_l\) printed in bold (separated by mean values below – or nearly equal to – the level of variation together with the breaks in the alternation of the signs) represent independent coherent global oscillations (denoted by \(B_1, B_2, B_3, B_4\) in text).

<table>
<thead>
<tr>
<th>(l)</th>
<th>Real data set</th>
<th>Simulated data set (\text{‘SH’})</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(a_l)</td>
<td>(\Delta a_l)</td>
</tr>
<tr>
<td>1</td>
<td>0.247</td>
<td>0.014</td>
</tr>
<tr>
<td>3</td>
<td>-0.718</td>
<td>0.026</td>
</tr>
<tr>
<td>5</td>
<td>1.00a</td>
<td>0.028</td>
</tr>
<tr>
<td>7</td>
<td>-0.945</td>
<td>0.081</td>
</tr>
<tr>
<td>9</td>
<td>0.636</td>
<td>0.112</td>
</tr>
<tr>
<td>11</td>
<td>-0.264</td>
<td>0.095</td>
</tr>
<tr>
<td>13</td>
<td>0.017</td>
<td>0.056</td>
</tr>
<tr>
<td>15</td>
<td>-0.152</td>
<td>0.047</td>
</tr>
<tr>
<td>17</td>
<td>-0.171</td>
<td>0.052</td>
</tr>
<tr>
<td>19</td>
<td>0.137</td>
<td>0.045</td>
</tr>
<tr>
<td>21</td>
<td>-0.097</td>
<td>0.047</td>
</tr>
<tr>
<td>23</td>
<td>0.069</td>
<td>0.057</td>
</tr>
<tr>
<td>25</td>
<td>-0.061</td>
<td>0.062</td>
</tr>
<tr>
<td>27</td>
<td>0.061</td>
<td>0.035</td>
</tr>
<tr>
<td>29</td>
<td>0.060</td>
<td>0.037</td>
</tr>
<tr>
<td>31</td>
<td>0.052</td>
<td>0.060</td>
</tr>
</tbody>
</table>

By virtue of definition

present in the simulated data sets since the 'distribution of sunspot activity within the wings of the butterflies is totally random.

We conclude that the distribution of the real sunspot activity even within the wings of butterflies is not random. At least down to scales \(l = 29\), it is defined systematically by superposition of the stationary and approximately stationary LF modes of the inferred field.

From the foregoing discussion it follows that the coherent oscillations \(B_1, B_2, B_3\) and \(B_4\) are geometrically significant.

2.6. Physical Reality of the Coherent Oscillation Modes

In this section we see how the observed global latitude-time behaviours of the Sun’s various 'magnetic’ features, considered as measures of \(B(\theta, t)\), can be reproduced by superposition of \(B_1, B_2, B_3, B_4\) not only in the sunspot or 'low' \((\leq 30^0)\) latitudes
but also in the 'medium' (30° - 75°) and the 'high' (≥ 75°) latitudes.

6.1. THE LATITUDE-TIME DISTRIBUTION OF \( B(\theta, t) \) GIVEN BY \( B_1 + B_2 \): BUTTERFLY DIAGRAMS AND THE FIELD IN HIGH LATITUDES

We plotted \( B(\theta, t) \) as given by

\[
B(\theta, t) = B_1(\theta, t) + B_2(\theta, t).
\]

We found that in latitudes ≤ 30° this produces a butterfly diagram. To some extent this is expected since \( B_1 \) and \( B_2 \) are obtained from the dominant LF modes in \( B(\theta, t) \) which is defined as \( \pm P(\theta, t) \) and \( P(\theta, t) \) has the form of the butterfly diagrams. However, it is important to note that the butterfly diagram produced by this is more realistic than the one obtained by superposing the dominant LF modes (even degrees and '11 yr' periodicity) in \( p(\theta, t) \) itself, (Gokhale & Javaraiah 1990a), in the following respects:

(i) the shape and the extent of the wings is more realistic,

(ii) the overlaps between the successive sunspot cycles near the sunspot minima are present and are of the right order (viz., ~ 3 yr).

In addition, we notice that moderate flux concentrations in the high latitudes, which may be responsible for the polar facular activity reach maxima near the 'sunspot minima' (i.e., minima of \( |B(\theta, t)| \) in the low latitudes), and vice versa (Makarov & Sivaraman 1989).

Further, these high latitude field concentrations, considered collectively in latitudes above 70°, change signs around the 'sunspot maxima', i.e., at about the same time as (and also by same sense as) the observed polar fields do.

Thus the LF modes \( B_1 \) and \( B_2 \) in the inferred magnetic field ‘\( B_{inf}(\theta, t) \)’ give a more realistic and more global description of the solar cycle phenomenon than that given by the dominant modes in the ‘sunspot occurrence probability’.

2.6.2. THE LATITUDE-TIME DISTRIBUTION OF \( B_1 + B_2 + B_3 \): MIGRATION OF NEUTRAL LINES

In the latitude-time diagram given by \( B_1(\theta, t) + B_2(\theta, t) \) (described in previous section) we have found that the 'magnetic neutral lines' do not seem to migrate up to the poles as seen in the analysis of \( H_a \), spectroheliograms by Makarov & Sivaraman (1989). Instead in the polar region, the migrations were found to be away from the poles.

The latitude-time diagram given by

\[
B(\theta, t) = B_1(\theta, t) + B_2(\theta, t) + B_3(\theta, t).
\]

gives all the afore-mentioned properties of the \( \theta - t \) distribution of \( |B_1 + B_2| \) which
agree with the observed global behaviour of the surface distribution of solar magnetic field. In addition, this also produces migrations of 'magnetic' neutral lines from the middle latitudes to poles in a way similar to those deduced by Makarov & Sivaraman (1989). To migrate from $\sim 35^\circ$ to poles these neutral lines take 15 yr, i.e., same as the real neutral lines.

2.6.3. THE LATITUDE TIME DISTRIBUTION OF $B_1 + B_2 + B_3 + B_4$

In Figure 2.3 we show the $\theta - t$ diagram given by

$$B(\theta, t) = B_1(\theta, t) + B_2(\theta, t) + B_3(\theta, t) + B_4(\theta, t).$$

This figure keeps all the 'good' properties of the $\theta - t$ distribution of $|B_1 + B_2 + B_3|$. In addition, it yields, in years around sunspot minima, a somewhat higher ratio of the polar field to the field in the middle latitudes. This is necessary to account for the presence, during such years, of facular activity in high latitudes without its presence in middle latitudes.

Figure 2.3 does not produce the detailed real trajectories of the neutral lines in the $\theta - t$ diagrams as determined by Makarov & Sivaraman (1989). However, the agreement with the observed behaviour of the neutral lines is quite satisfactory, taking into consideration: (i) the uncertainties in the amplitudes and the phases, (ii) the uncertainties in determining the global neutral lines using the $H_\alpha$ spectroheliograms, (iii) the omission of the 'non-stationary' and the higher harmonic LF modes as well as of the small-amplitude even degree modes which are also actually present in the inferred magnetic field, and (iv) the fact that the figure represents a mean pattern over all the 9 sunspot cycles since it uses amplitudes and phases averaged over 103 years.

Thus, all the four eigenmodes $B_1$ through $B_4$ are necessary and sufficient to produce a fairly satisfactory 'cleaned image model' of the observed "mean" latitude-time behaviour of the Sun's real magnetic field over nine cycles.

It is important note that this pattern in all latitudes is obtained by superposing LF terms given by spot group data which comes only from low latitudes.

2.7. Comparison with Results from the Magnetogram data

The amplitude and the relative phases of the LF modes $l = 1-13$ determined from the sunspot data are generally similar to those obtained by Stenflo (1988) from the magnetogram data during 1060-1985, which had also given a reasonably good synthetic butterfly diagram. However, the sunspot data have enabled us to study the variations or constancy of the relative phases over the 103 years. Moreover, there are some important differences. (i) The relative amplitude of the mode $l = 1$ is higher (and that of $l = 3$ is
Figure 2.3. Grey level representation of the latitude-time distribution of $B = B_1 + B_2 + B_3 + B_4$ (cf., Section 2.6.3). The different levels of magnitude of $B$ (in arbitrary units) are represented by the different levels of shades (and contours represented by continuous curves) as indicated by the right-hand-side vertical scale. The negative latitudes represent southern hemisphere.

lower) in the fields observed at the surface than in the fields inferred from the sunspot data. We attribute this difference to the fact that, on the surface, a considerable amount of magnetic flux from the 'following' polarities of the active regions spreads into the middle latitudes, thereby transferring 'SHF power' from $l = 5$ and 3 to $l = 3$ and 1, respectively. (ii) In Stenflo's synthetic butterfly diagram the synthesized 'field strength' at the high latitudes is (unrealistically) of the same order as that in the low latitudes. In contrast, the field in the high latitudes in our butterfly diagrams is (realistically) of the same order as, or only slightly stronger, than the weak field in the middle latitudes. This is because the Lf modes with $l > 13$, which are present in our synthesis, provide larger degree of interference even in the higher latitudes.
2.8. Modeling the Variation of the Annual Measure of Sunspot Activity

2.8.1. THE 'FLUX EMERGENCE RATE', $Q(\theta, t)$, INFERRED FROM DATA, AND ITS LEGENDRE-FOURIER ANALYSIS

So far the sunspot occurrence probability during any given time interval was normalized to the total amount of activity in that time interval (Section 2.2). Thus the unit of $B_{\text{inf}}$ varied from cycle to cycle, depending upon the 'size' of the cycle. This is fine for studying the latitude-time distribution of activity during any single time interval irrespective of the total amount of activity during that interval. However, for modeling the variation in the rate of production of sunspot activity the total probability in any given time interval should be proportional to the total measure of activity in that time interval. Hence, we revise the renormalization of sunspot occurrence probability in the following manner.

Consider length scales large compared to the dimensions of the spot groups and time scales large compared to the life spans of the spot groups but small compared to the durations of the sunspot cycles (e.g., length scales $\geq 10^4$ km and time scales $\geq$ a few months). On such scales we define the sunspot occurrence probability, as a function of latitude and time, as follows:

$$\varphi(\theta, t) = \begin{cases} \tau_k \delta(\theta - \theta_k, t - t_k) & \text{at } (\theta_k, t_k) \\ 0 & \text{elsewhere,} \end{cases}$$

where $\theta_k, t_k$ are the values of $\theta$ and 't' for the spot group 'k', $\tau_k$ is the life span of the spot group 'k', and $\delta$ represents the Dirac delta function. As required, the integral of $\varphi(\theta, t)$ over the unit sphere during any time interval is proportional to the measure of sunspot activity during that interval.

This probability function is related to the sunspot occurrence probability function $p(\theta, t)$ of Section 2.2.4 in the following way:

$$\varphi(\theta, t) = (\sum_k \tau_k)p(\theta, t),$$

where the summation $\sum_k$ extends over all the spot groups observed during a given time interval, or during a given sunspot cycle.

We take $\tau_k$ as a measure of the amount of magnetic flux which emerges above the photosphere (and eventually leaves the Sun), in association with appearance (and disappearance) of the spot group 'k'. From the generally bipolar nature of the fields associated with activity, it is believed that the emerging flux is toroidal.

In analogy to $B_{\text{inf}}(\theta, t)$ in Section 2.2.4, we then define a quantity $Q(\theta, t)$ as
\[ Q(\theta, t) = \pm \varphi(\theta, t), \quad (2.8) \]

where the signs ± are chosen strictly according to Hale’s laws of magnetic polarities with care described in Section 2.5.5. Clearly, on time scales \( \geq \) a few months and length scales \( \geq 10^4 \) km, \( Q(\theta, t) \) represents a measure of ‘the amount of the toroidal magnetic flux emerging above the photosphere at (\( \theta, t \)), per unit latitude interval per unit time’. Equation (2.8) can also be written as

\[ Q(\theta, t) = (\sum_k \tau_k)B_{in}\varphi(\theta, t). \]

**Amplitudes and Phases of Legendre-Fourier Terms during Each Sunspot Cycle**

Using the method given Section 2.2.4, we have determined the amplitudes \( q(l, n) \) and the phase \( \alpha(l, n) \) of the LF terms of odd degrees, \( l = 1 \) to 29, and frequencies \( \nu = n\nu_* \) \( (n = 1, 3, 5, 7) \) in \( Q(\theta, t) \), during each of the nine sunspot cycles between 1879 and 1976. During each cycle, the spectrum of the relative amplitudes in ‘Q’ is similar to (but not the same as) that during the whole sequence of the 103 years. Owing to the above relation between \( Q \) and \( B_{in,f} \), the latter spectrum is the same as the relative amplitude spectrum of \( B_{in,f} \) Figure 2.1 (Section 2.3.1) during the 103 years.

2.8.2. A MODEL OF TOROIDAL FLUX EMERGENCE AS PROVIDED BY INTERFERENCE OF TORSIONAL MHD OSCILLATIONS

(a) Formation of Toroidal Flux Tubes and Their Emergence

Consider a set of axi-symmetric MHD oscillations (e.g., torsional), each represented, during each cycle, by a set of LF terms, and described collectively by the sets \( \{l\} \) and \( \{n\} \) of the values of \( l \) and \( n \), respectively. The toroidal magnetic field \( B_\phi \) at a point \( (r, \theta) \), at an instant ‘\( t \)’ during a cycle ‘\( i \)’, can be written as

\[ B_\phi(i; r, \theta, t) = \sum_{\{l\}} \sum_{\{n\}} b(i; l, n) f_{l,n}(r) \times P_l(\mu)\sin[2\pi n\nu_* t + \epsilon(i; l, n)], \quad (2.9) \]

where \( b(i; l, n) \) and \( \epsilon(i; l, n) \) represent the amplitudes and the ‘initial’ phases of the terms \( (l, n) \) in ‘\( B_{in,f} \)’, during the cycle ‘\( i \)’, \( f_{l,n}(r) \) is the radial eigenfunction for the mode \( (l, n) \) and \( \mu = \cos\theta \).

At \( \theta \) and ‘\( t \)’ where the interference creates a toroidal flux bundle whose magnetic buoyancy overcomes its magnetic tension, the flux bundle will emerge above the photosphere.
(b) The rate of Emergence of Toroidal Flux

Let $\tau_{\text{max}}$ be the maximum time required for any such toroidal flux bundle, after its creation, to emerge above the photosphere at all longitudes. Since the large-scale meridional flows in the Sun's radiative core are negligible, the amount of toroidal flux across any meridional section of the radiative core must remain constant. Hence, on time scales $\tau > \tau_{\text{max}}$, and less than the diffusion time scales, the amount of the toroidal flux $Q(\theta, t)$, emerging above the photosphere per unit time across a latitude interval $d\theta$, will be given by

$$Q(\theta, t) = \frac{\partial}{\partial t} \left[ \int_0^R B_\phi(r, \theta, t) r d\theta dr \right]$$

where $r = R_{bc}$ is the radius of the base of the convective envelope. Hence, for modeling the sunspot cycles, the flux emergence rate $Q_{\text{mod}}(i; \theta, t)$ during any cycle ‘i’ can be written, using Equation (2.9), as

$$Q_{\text{mod}}(i; \theta, t) = \sum_{\{l\}} \sum_{\{n\}} b(i; l, n) g_{l,n}(R_{bc}) \times 2\pi n \nu \epsilon_1(\mu) \cos[2\pi \nu \epsilon_1 t + \epsilon(i; l, n)], \quad \text{(2.10)}$$

where

$$g_{l,n}(R_{bc}) = \int_0^R f_{l,n}(r) r dr.$$

(c) Relations between the LF Terms in ‘Q’ and Those in ‘$B_\phi$’

Equation (2.10) can be rewritten as

$$Q_{\text{mod}}(i; \theta, t) = \sum_{\{l\}} \sum_{\{n\}} q(i; l, n) \times P_1(\mu) \cos[2\pi \nu \epsilon_1 t + \alpha(i; l, n)],$$

where

$$q(i; l, n) = 2\pi n \nu b(i; l, n) g_{l,n}(R_{bc}) \quad \text{(2.11a)}$$

and

$$\alpha(i; l, n) = \epsilon(i; l, n) + \pi/2. \quad \text{(2.11b)}$$

It follows that according to this model, (i) the sets of terms $\{l\}$ and $\{n\}$ of appreciable amplitudes in the LF spectrum of $B_\phi$ will be same as those obtained through the LF analysis of ‘Q’, but (ii) owing to the factor $g_{l,n}$ in Equation (2.11a), the ratios of amplitudes (‘$b$’s) of different LF terms in ‘$B_\phi$’ may not be same as those of the amplitudes (‘$q$’s) of the corresponding terms in ‘Q’, and (iii) the temporal variations and relative differences in the phases of LF terms in $B_\phi$ will be same as those in the phases of the corresponding terms in $Q(\theta, t)$. 

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The only observed large-scale flows that can, in principle, create toroidal fields are the ‘torsional waves’ of ‘11-yr’ periodicity (LaBonte & Howard 1982a), or ‘22-yr’ periodicity (Javaraiah & Gokhale 1995; Javaraiah & Komm 1998; Javaraiah 1998; see Chapters III and VI) detected in the photospheric rotation. Rotational perturbations on time scales of years seem to exist even in the solar interior (Dziembowski & Goode 1991). In the presence of even a very weak but sufficiently long-lived poloidal field with $l = 1$ and 3 (e.g., Mestel & Weiss 1987; Spruit 1990; Gokhale & Hiremath 1992; Hiremath 1995; Hiremath & Gokhale 1995), such perturbations would constitute torsional MHD waves.

Thus, if the LF terms in $B_\theta$ represent global oscillations/waves, then these must be ‘torsional’ MHD in nature.

2.8.3. METHOD TO MODEL THE ‘SHAPES’ AND ‘SIZES’ OF THE INDIVIDUAL SUNSPOT CYCLES BY RECOMBINING ANY PRESCRIBED SET $\{l, n\}$ OF LF TERMS IN ‘$Q$’

It follows from Equation (2.8) that the sunspot occurrence probability per latitude interval of unit photospheric area per unit time, at colatitude ‘$\theta$’, at time ‘$t$’, will be given by

$$\varphi_{mod}(\theta, t) = | Q_{mod}(\theta, t) |. \quad (2.12)$$

Hence, in a model combining a prescribed set $\{l, n\}$ of LF terms the sunspot occurrence probability, $\varphi_{mod}$, at the central epoch ‘$t_{ijk}$’ of the kth ‘month’ in the jth year of the ith cycle can be written as

$$\varphi_{mod}(l, n; i, j, k) = T \int_{45^\circ}^{135^\circ} |Q_{mod}(i, \theta, t_{ijk})| \sin \theta d\theta$$

$$= 2T \int_{45^\circ}^{90^\circ} |\sum_{\{l\}} \sum_{\{n\}} q(l, n) P(\mu) \sin[2\pi n \nu \tau_{t_{ijk}} + \alpha(i, j, k)]| \sin \theta d\theta. \quad (2.13)$$

Here the term ‘month’ means any given 1/12 part of a year and ‘$T$’ is its length.

The limits of integration with respect to the colatitude are chosen so as to exclude the low-level flux concentrations outside the sunspot zone which are not strong enough to be seen as sunspot activity (see Section 2.6), and the symmetry of the integrand is used for changing the limits of the integral.

The annual measure of the sunspot occurrence probability is

$$s_{mod}(l, n, i, j) = \sum_{i, j} \varphi_{mod}(l, n; i, j, k). \quad (2.14)$$

The variation of $s_{mod}(l, n; i, j)$ with ‘$j$’ gives the ‘reconstructed shape’ of the sunspot cycle ‘$i$’ for a given choice of $\{l\}$ and $\{n\}$. 
The 'size' of the cycle 'i' in the reconstructed model will then be given by

\[ S_{\text{mod}}(\{l\}, \{n\}, i) = \sum_j s_{\text{mod}}(\{l\}; \{n\}; i; j). \] (2.15)

2.8.4. IDENTIFICATION OF THE SET OF SIGNIFICANT LF TERMS IN 'Q' THAT ACCOUNT FOR MOST OF THE OBSERVED SUNSPOT ACTIVITY

According to Parceval's theorem, the size \( S_{\text{mod}}(i) \) of a sunspot cycle 'i' in the recombination model must be proportional to the total LF power in the set \([\{l\}, \{n\}]\) of terms taken in Equation (2.13). Thus

\[ P(i; \{l, n\}) = \sum_{\{l\}, \{n\}} [q(i; l, n)]^2. \]

Starting from a set of lowest \( l \) and \( n \), the correlation between \( P(i; \{l, n\}) \) and the observed size of the cycle 'i', viz., \( S_{\text{obs}}(i) \), first increases with inclusion of terms with higher and higher \( l \) and \( n \). A stage comes when the correlation starts dropping down due to larger relative errors in the amplitudes of the higher terms. From the maximum correlation we have identified \( \{l \rightarrow 1, 3, ..., 13; n = 1, 3, 5\} \) as the set of LF terms in 'Q' which are significant in modeling by recombination of LF terms. The high (\( \sim 99.98\% \)) correlation assures that the significant set is adequate to account for most of the observed sunspot activity.

2.8.5. COMPARISON BETWEEN THE MODELED AND THE OBSERVED 'SHAPE'

For each cycle 'i' we have modeled the 'shape' \( s_{\text{mod}}(i) \) with various choices of \( \{l\} \) in the range 1 to 13 and \( \{n\} \) in the range 1, 3, 5. This must be compared with the 'observed shape'.

For determining the 'shape' of a cycle 'i' we take the sum \( s_{\text{obs}}(i, j) \) of the life-spans (in days) of all the spot groups born during the jth year of the cycle 'i' as the measure of the sunspot activity during that year.

For each of several choices of the sets \( \{l\} \) and \( \{n\} \), we have computed the coefficients of correlation between \( s_{\text{mod}}(\{l\}, \{n\}; i, j) \) and \( s_{\text{obs}}(i, j) \) during each cycle 'i'. We have also computed the correlation between \( S_{\text{mod}}(\{l\}, \{n\}, i) \) and \( S_{\text{obs}}(i) \) for \( i = 1 \) to 9. The results are given in the next two subsections.

**Importance of The Terms with \( n = 1, 3, 5 \)**

We have found that during each cycle 'i' the correlation between the modeled 'shape' \( s_{\text{mod}}(\{l\}, \{n\}; i, j) \) and the observed 'shape' \( s_{\text{obs}}(i, j) \), for \( j = 1 \) to 11, is > 80% for any choice of \( \{l\} \) in the range \( l = 1 \) to 13 with \( n = 1 \). As expected from the asymmetries
of the sunspot cycles (Bracewell 1988), inclusion of corresponding terms with \( n = 3 \)
increases the correlations substantially (e.g., by > 4\%). Still further improvement by
inclusion of higher terms (\( n > 7 \)) will be negligible since their amplitudes are quite
small. Thus, terms of \( l = 1, \ldots, 13 \) and \( n = 1, 3 \) (or \( n = 1, 3, 5 \)), are adequate for
modeling the shapes of the cycles.

**The Optimal Set \( \{l, n\} \) That Give the best Correlation**

Among the high correlations given by the different choices of \( \{l, n\} \) with \( l \) in the range
\( l = 1, \ldots, 13 \), and \( n = 1, 3 \), the subset \( \{l = 3; n = 1, 3\} \) or \( \{l = 3, 5; n = 1, 3\} \) gives
the highest correlation (\( \sim 0.97\% \)) between \( s_{\text{mod}} \) and \( s_{\text{obs}} \) during each of the nine cycles.
The latter subset gives the highest average correlation between the observed and the
modeled shapes over the whole sequence of the nine cycles (see Figure 2.4). However,
the differences between the correlations given by the two subsets seem too small to be
significant (see Table 2.4).

Hence, during every cycle the shape consisting of the eleven observed points (ten
relative values) can be equally well reproduced by specifying **only four parameters**, viz.,
the amplitudes and phases of \( \{l = 3; n = 1, 3\} \).

![Graph](image_url)

**Figure 2.4.** Variation of \( s_{\text{obs}}(i, j) \) (dashed curve), and that of \( s_{\text{mod}}(i, j) \) (continuous curve), both
normalized to their values in 1958, during sunspot cycles \( i = 12 \) to \( 20 \). For each cycle the model uses
amplitudes and phases of only \( \{l = 3, 5; n = 1, 3\} \). Agreement from the model using
\( \{l = 3; n = 1, 3\} \) only will be almost equally good (see Section 2.8.5).
The high correlations between the modeled and the observed shapes for subset \( \{l\} = \{1, 3, ..., 13\} \) noted in Section 2.8.5 suggest the presence of high mutual correlations among the amplitudes and phases of the LF terms in this range of \( l \) for each \( n \). Such high mutual correlations are indeed seen in Figures 2.5 and 2.6 showing, for \( n = 1 \), the variations in amplitudes \( q(l, n) \) and phases \( \alpha(l, n) \) respectively, determined from 11-yr intervals successively displaced by 1 yr.

### 2.9. The Stationary Oscillations

#### 2.9.1. HIGH MUTUAL CORRELATIONS BETWEEN THE AMPLITUDES AND PHASES OF TERMS WITH DIFFERENT \( l \) BUT SAME \( n \)

The high correlations between the modeled and the observed shapes for subset \( \{l\} = \{1, 3, ..., 13\} \) noted in Section 2.8.5 suggest the presence of high mutual correlations among the amplitudes and phases of the LF terms in this range of \( l \) for each \( n \). Such high mutual correlations are indeed seen in Figures 2.5 and 2.6 showing, for \( n = 1 \), the variations in amplitudes \( q(l, n) \) and phases \( \alpha(l, n) \) respectively, determined from 11-yr intervals successively displaced by 1 yr.

#### 2.9.2. EXISTENCE OF RESONANT AND APPROXIMATELY STATIONARY GLOBAL OSCILLATIONS IN \( B_\phi \)

In view of Equation (2.11b), the phase variations in Figures 2.6 and 2.7 can be considered also as the variations in the phases \( \epsilon(l, n) \) of LF terms in \( B_\phi \).

Further, we see in Figures 2.6 and 2.7 that the terms \( l = 1, 3, ..., 15, n = 1, 3 \) form four separate groups \((l = 1, 3, 5, 7; n = 1), (l = 9, 11, 13, 15; n = 1), (l = 1, 3, 5, 7; n = 3)\), and \((l = 9, 11, 13, 15; n = 3)\), such that terms within each group \( \approx \pm 180^\circ \), and a certain common degree of ‘phase constancy’.

Hence, the expression on the right-hand side of Equation (2.9) can be written as four sums, each collecting the terms in each group, and representing a stationary global
Figure 2.5. Temporal variation of the amplitude $q(l, n)$, for $n = 1$, as represented by values during 11-yr intervals successively displaced by 1 yr. The symbols $\triangle, \Box, \ast, \cdot, \times, \Diamond$, and $\bullet$, represent $l = 1, 3, 5, 7, 9, 11$ and 13, respectively. The continuous curve represents the values of the amount of observed sunspot activity ($S_{obs}$) during the respective intervals.

Figure 2.6. Temporal variation of the phases $\alpha(l, n)$, or $\varepsilon(l, n)$ of the terms $l = 1, 3, \ldots, 15$, all with $n = 1$, during the 11-yr intervals successively displaced by 1 yr.
Figure 2.7(a).

Figure 2.7(b).

Figure 2.7a-b. Temporal variation of the phases $\alpha(l, n)$ or $\varepsilon(l, n)$ of the third harmonic ($n = 3$) terms as represented by values during 11-yr intervals successively displaced by 1 yr. In (a), symbols $+$, $\times$, $\circ$, and $\triangle$ represent $l = 1, 3, 5$ and 7, respectively. In (b) they represent $l = 9, 11, 13$, and 15, respectively.
Mutual correlations are also seen in the similarly determined variations of the phase \(\alpha(l,n)\) of the terms of different \(l\) for \(n = 3\) shown in Figures 2.7(a) and 2.7(b).

2.10. A possible Phenomenology for Maintenance of the LF Spectrum and Production of Activity

Here I present a phenomenology which we conjectured for understanding the above results.

2.10.1. POSSIBILITY OF EXISTENCE OF CASCADE OF ENERGY IN THE LF SPECTRUM

In the Sun, the density falls rapidly near the photosphere. The intensity of the background field may not vary much (e.g., Gokhale & Hiremath 1993). Hence, if torsional MHD waves are excited inside the Sun, their phase speed would increase rapidly as they approach the photosphere traveling along the field lines. Therefore the waves will be trapped inside the Sun by total internal reflections at the photosphere. In general the angles of incidence will be non-zero, and hence the reflected waves will have different values of \(l\) than the incident waves. This transfer of energy will continue during successive reflections provided there is a continued supply of energy at the original ‘\(l\)’. In the whole process, appreciable energy will be stored only in the normal modes of global oscillations (for which higher \(l\) corresponds to higher \(\nu\)). Since the dissipation occurs at high \(l\) and \(\nu\), the overall transfer of energy will constitute a cascade from modes of lower \(l\), \(\nu\) to those of higher \(l\), \(\nu\).

2.10.2. EVIDENCE FOR EXISTENCE OF CASCADE OF ENERGY IN THE LF SPECTRUM

There are high correlations among the phases (and also among the amplitudes) of LF terms of low and high values of \(l\) (Section 2.9.1) and among the terms of low and high values of \(n\) (Gokhale & Javaraiah 1990b).

If we examine Figure 2.6 carefully, we find increments and decrements of the initial phases of terms in \(n = 1\) occurring during intervals of lengths \(\sim 7\) yr and \(\sim 4-5\) yr. These variations imply decelerations and accelerations, respectively, in the phase speeds of the waves of the frequency \(\nu_0\) on time scales corresponding to \(3\nu_0\) and \(5\nu_0\). Accelerations and decelerations in the phase speeds of the terms in \(\nu = 3\nu_0\) in Figure 2.7 are also seen to occur during exactly the same intervals of time. Thus, the correlations which exist for each \(l\) in the phase variations of the lower and the higher \(n\) (e.g., \(\sim 85-90\%\) between \(n = 1\) and 3; Gokhale & Javaraiah 1990b), seem to be due to the simultaneous phase accelerations and phase decelerations of the waves of the lower and the higher \(n\). So, the phase variations in Figures 2.6 and 2.7 indicate transfer of energy from LF terms
of $3\nu_*$ and $5\nu_*$.

Similarly the mutual correlations between the amplitudes and phases of LF terms of different $l$, and same $n$, imply transfer of energy from waves of lower $'l'$ to those of higher $l$, with the same $\nu$.

Whenever the flux bundles formed by interference of MHD waves emerge above the photosphere by the process envisaged in Section 2.8.2, they will be seen as ‘surface fields’. The dissipation of the emerged flux bundles in the atmosphere will produce ‘activity’ of various types on various scales.

The ‘toroidal flux bundles’ given by interference of the waves with non-random phase variations will be regularly distributed in latitude and time. Thus the modes with $\nu = \nu_*, 3\nu_*$, and $5\nu_*$, would yield the observed ‘photospheric fields’ and ‘activity’ that are distributed regularly in latitudes and in time, viz., (i) sunspot activity distributed in ‘butterfly diagrams’, (ii) ‘weak’ fields appearing to migrate towards the poles, and (iii) the ‘reversing’ polar fields. (This is already shown, up to the terms in $\nu_*$ alone, for $l = 1$ to 13 by Stenflo (1988), and for $l = 1$ to 29 by us in Section 2.6.)

Since the waves with $n > 5$ have random phases (Section 2.10.3), the flux tubes formed by their interference may be producing the small-scale fields and activity distributed randomly on the surface, and in time (may be, e.g., ‘bright points’ in X-ray and EUV emissions).

The emergence of flux tubes as in Section 2.8.2 implies sudden removal of energy, viz., the magnetic energy of the flux tubes, from the interfering waves. Since this occurs on time scales ($\tau_{\text{max}}$ : Section 2.8.2) much shorter than the wave periods, this would lead to ‘shifts’ in the phases of the respective terms in the LF spectrum.

We have determined the cycle-to-cycle ‘shifts’ in the phase, $\epsilon_{\text{in}}$, from changes in $\alpha_{\text{in}}$, using Equation (2.11b). These are $< 30^\circ$ for $n = 1$, $\sim 30^\circ - 90^\circ$ for $n = 3$, and $90^\circ - 120^\circ$ for $n = 5$. For $n > 5$ the phase changes are $> 120^\circ$ and hence, essentially random.

For each LF terms, the observed cycle-to-cycle phase change will be the net result of the energy received from terms of lower $l$, $n$, energy contributed to the emerging flux tubes, and energy passed on to the terms of higher $l$, $n$.

The foregoing discussion suggests that the LF spectrum of the global oscillations is a net result of (i) input of energy at some low $l$, $\nu$, (ii) cascade of energy from lower $l$, $\nu$ to higher $l$, $\nu$, and (iii) intermittent removal of energy from the waves in the form of toroidal flux tubes formed by interference.
The dominant LF Terms in the Basically Excited Oscillation and the Approximate Balance Between Inputs and Outputs of Energy

In Figure 2.5 we also see that the amplitudes \( q(l, n) \) for \( n = 1 \) and \( l = 1-13 \) determined from 11-yr long time intervals successively displaced by 1 yr show high correlations with the measure of sunspot activity, \( S \), during those intervals. Actually, these correlations are expected from the definition of \( q(l, n) \). However, among these, the best correlation is given by amplitudes of \( l = 3 \) and 5 not by the largest two amplitudes in \( 'Q' \), viz., of \( l = 5 \) and 7 (see Figure 2.1). Thus, the energy in \( \{l = 3, 5; n = 1\} \) seems to control the variation of the amount of sunspot activity even on time scales \( \geq 11 \) yr. This, along with the result of 2.10.3, shows that \( \{l = 3, 5; n = 1\} \) may be the dominant terms in the basically excited waves.

The same correlation also suggests that on time scales \( \geq 11 \) yr there may be a fairly good balance between the rate of energy input into \( \{l = 3, 5; n = 1\} \) and that of energy disposal through sunspot activity. [Evidence for the Associated ‘Torsion’: It may be noted that the terms \( \{l = 3, 5; n = 1\} \) belong to the mode \( \{l = 1, 3, 5, 7; n = 1\} \) in \( B^* \) which corresponds to \( \{l = 2, 4, 6; n = 1\} \) in the rotational angular velocity (i.e., the ‘torsional oscillation’ of ‘22-yr periodicity’). In the analysis of surface rotation the presence of such an oscillation is indicated by that of 22-yr periodicity in the coefficient of \( \sin^2 \theta \) (Javaraiah & Gokhale 1995, see Chapters III and VI)].

2.10.3. VERIFICATION OF THE PHENOMENOLOGY

**Correlation Expected between the ‘Cycle Size’ and the ‘Phase-Changes’**

Since the changes in the phases of LF terms with \( n = 1 \) and 3 are not large (see Section 2.10.3), these phase changes can serve as measures of the net effect of the gains and losses of energy by these terms. Hence, according to the foregoing model of energy cascade and production of activity (Sections 2.10.2 and 2.10.3), the following correlations should exist.

**Test 1:** the size of a sunspot cycle should be proportional to the amount of energy by all those waves whose interference creates the sunspot activity during that cycle. Hence, \( S_{\text{obs}}(i) \) should be correlated to the phase changes of the corresponding LF terms from cycle ‘\( i - 1 \)’ to cycle ‘\( i \)’.

**Test 2:** also, the change in the cycle size from one cycle to the next cycle should be correlated to the phase shifts of the terms representing those waves of \( n = 1 \) into which the energy is input during some earlier cycles.
Verification of the ‘Test 1’

For verifying Test 1 we have determined the coefficients of correlations of \( S_{\text{obs}}(i) \) with the sums of the changes \( \Delta \epsilon_{l,n}(i - 1, i) \) in the phases of the terms \((i, n)\), taken in different combinations, during the cycle \( i - 1 \) to those during the \( i \).

We find the correlations between

\[
S_{\text{obs}}(i) \text{ and } \sum_{n=1,3,5} \Delta \epsilon_{5,n}(i - 1, i) \text{ equal to } 90\%,
\]

\[
S_{\text{obs}}(i) \text{ and } \sum_{l=1,3,5,7} \Delta \epsilon_{l,5}(i - 1, i) \text{ equal to } 94\%,
\]

where the combination given in the summation is the one that gives the maximum correlation (of the value given).

Thus, we find: (a) the size of a sunspot cycle is highly correlated to the energy lost by a set of waves during the current cycle, and (b) the maximum correlation is with the energy lost by the wave corresponding to the term \( l = 5 \) through \( n = 1, 3, 5 \), and also to the energy lost by the waves corresponding to the terms \( l = 1, 3, 5, 7 \) through \( n = 5 \).

Verification of the ‘Test 2’

For verifying Test 2 and identifying the terms of the fundamental frequency in which the energy is input, we have determined the correlations between the changes in the cycle size:

\[
\Delta S_{\text{obs}}(i - 1, i) = S_{\text{obs}}(i) - S_{\text{obs}}(i - 1)
\]

and the sums of phase shifts, in combination of terms with \( n = 1 \), occurring between the previous one or two cycles.

In the notation used earlier, the maximum correlations are:

\[
\Delta S_{\text{obs}}(i - 1, i) \text{ and } \Delta \epsilon_{5,1}(i - 1, i) : 87\%
\]

and

\[
\Delta S_{\text{obs}}(i - 1, i) \text{ and } \Delta \epsilon_{11,1}(i - 2, i - 1) :
\]

Thus the change in the cycle size from \( i - 1 \) to \( i \) is well correlated with the amounts of energy input into the waves of frequency \( \nu \) during the cycles \( i - 2 \), \( i - 1 \), and the maximum correlations are with energy inputs into: (a) the wave \([l = 5, n = 1]\) during either \( i - 1 \) or cycle \( i \), (b) the wave \([l = 11, n = 1]\) during the cycles \( i - 2 \) and \( i - 1 \). [Note: the above correlations imply that the time lapse between the input of energy and its loss through interference is longer for \( l = 11 \) than for \( l = 5 \). This means that the phase difference between \( l = 5 \) and \( l = 11 \) seen in Figure 2.7 should not be considered as lag of 240° for \( l = 11 \) rather than a lead of 120°. This is to point out
that the result ‘(b)’ need not be interpreted as energy-input in $l = 11$ occurring earlier than in $l = 5$."

Scope for Forecasting the ‘Cycle Size’

In Figure 2.8 we compare the observed cycle sizes $S_{obs}$ with those ‘predicted’ using the second correlation. It is clear that such a forecast can be satisfactory.

---

Figure 2.8. The observed cycle sizes $S_{obs}(i)$, represented by ‘+’ and those ‘predicted’ (‘*’) on the basis of the 90% correlation of $S_{obs}(i)$ to $\Delta \varepsilon_{11,1}(i - 2, i - 1)$ (see Section 2.10.3).
2.11. Conclusions and Discussion

Conclusions:

The refined analysis of the Sun's magnetic field 'inferred' from the sunspot data, we draw the following conclusions (interpretations of the results):

(1) This study confirm the earlier result (Gokhale & Javaraiah 1990a; Gokhale et al. 1990) that the sunspot data can be used to study, even quantitatively, the global behaviour of the solar magnetic field during several cycles before the beginning of the regular magnetogram observations, at least on large scales.

(2) There is no convincing evidence for existence of any relation between the frequency (v) and the degree (l) of either odd or even degree axisymmetric modes, whatever be the physical nature of the modes.

(3) The narrow band widths of the ridges at v, 3v, 5v, etc. in Figure 2.1(a) provide the following clues for the theoretical modeling. One possibility is that the internal thermal and magnetic field of the Sun are so structured that the frequencies of all the admissible modes lie in band widths ~ ν0 around ν (~ 1/21.4 yr⁻¹), 3ν, 5ν,...etc. The other, simpler interpretation of Figure 2.1(a) is that the basic oscillations may be 'forced' (e.g., through boundary conditions imposed on the 'dynamo' by some 'clock' in the deep solar interior as suggested by Dicke 1979). The approximate constancy of the band width of the 'ridges' at ν, 3ν, etc. in Figure 2.1(a) suggests that the band width of the 'forcing frequency' is much less than ν0, i.e., ≪ 1/107 yr⁻¹ (≪ 0.3 nHz), if the high frequencies are harmonics of ν0 and not independent forcing frequencies.

(4) The modes in the 'main power ridge' of the SHF spectrum of the Sun's magnetic field as inferred from sunspot data constitute at least four independent distinct coherent global oscillations B₁,...,B₄. Superposition of all the four modes, B₁,...,B₄, is necessary and sufficient to reproduce important observed properties of the latitude-time distribution of the real solar magnetic field, not only in the 'sunspot zone' (from where the data comes), but also in the middle (35° - 75°) and the high (≥ 75°) latitudes, with appropriate relative orders of magnitudes and phases (Section 2.6). Thus, B₁,...,B₄ seem to represent really existing global oscillations in the Sun's magnetic field.

(5) As an interpretation of the results of analysis in Sections 2.8, 2.9, and 2.10.1-2.10.2 we have suggested in Sections 2.10.3 and 2.10.4 the following phenomenological model for production of sunspot activity and maintenance of the 'approximately steady' LF spectrum of the global MHD waves.

(i) The primary input of fresh energy into an existing spectrum of torsional MHD waves occurs mainly at l = 3, 5; ν = ν₀ (by some unidentified forcing process),
(ii) This energy cascades to the waves of higher spatial and temporal frequencies, maintaining the oscillations described in Section 2.9.2, and the waves of higher $l$, $n$ (presumably due to reflections of the waves at the boundaries such as the photosphere and the base of the convective envelope).

(iii) The cascading energy keeps on leaking out intermittently in the form of critical buoyant toroidal flux bundles (created by superposition of waves) whose emergence produces surface fields and activity on various scales (e.g., sunspot activity from interference of waves represented by $\{l = 1, 3, \ldots, 13; \nu = 1, 3, 5\}$, and 'non-sunspot activity' at higher $l$ and $n$).

(iv) 'Inferred rate' does yield relative amplitudes and phases for LF terms, at least up to $= 13$ (Section 2.7), similar to those derived from the directly 'observed' poloidal flux distribution at the photosphere (Stenflo & Vogel 1986; Stenflo 1988). This is expected from model in Section 2.10 if, on the time scales and length scales of the model, the 'instantaneously' observed photospheric poloidal field at $(\theta, t)$ is proportional to $Q(\theta, t)$, the 'rate of emergence of the toroidal flux per unit latitude interval per unit time'.

Discussion:

Section 2.8.5 presents one possible model of sunspot cycle as arising from superposition of the LF terms in the 'rate of emergence of toroidal magnetic field', not only qualitatively in terms of the latitude-time distribution, but also quantitatively in terms of the shapes and sizes of the successive sunspot cycles.

Here the rate of emergence of magnetic field is not directly measured but is inferred from the sunspot data itself. However, Conclusion (4) shows that the phenomenological model in Section 2.10 is not based on a trivial consequence of the forward and the backward LF transforms.

This brings us back to the questions: (a) what is the physical nature of these waves and oscillations?, (b) what kind of steady field in the Sun's interior can sustain such oscillations?

The answers to these questions can hardly be expected purely from a data analysis. However, for the sake of completeness of the model, on the grounds given in Sections 2.8.2 and 2.10.4 it is expected that the LF terms in $B_\phi$ represent the toroidal magnetic component of the 'torsional MHD oscillations'. As for the question (b), we note that in a recent model of the 'steady' part of the Sun's internal poloidal field, 'the best fit' for its iso-rotation with the helio-seismologically determined internal rotation of the Sun is given by terms only up to $l = 3$ (Gokhale & Hiremath 1993; Hiremath 1995; Hiremath & Gokhale 1995). This model of the 'steady' field is constrained to an
asymptotically uniform finite field at large distance. The alfvén travel time along all field lines is nearly same so that the model can provide the necessary 'steady framework' for the oscillations. The strength of this 'steady' field (required for the 22-yr periodicity of the torsional MHD oscillation) is $\sim 10^{-2}$ G, which would not be detectable in the presence of the periodically reversing surface fields produced by emergence of toroidal flux tubes (2.10.3 to 2.10.3). Hence, the presence of the necessary 'steady' background field of primordial origin, is not ruled out.

It is also shown in Hiremath (1995) that the 'residuals' of the fit indicate the presence of deviation from isorotation (i.e., time-dependent perturbations) with $l = 5$ as the dominant term and with a time scale in the range 1–100 yr. These properties of the 'deviation from isorotation' are in agreement with the present analysis and interpretation.

The phenomenological model in Section 2.10.3 describes a possible way in which toroidal magnetic flux tubes could be produced. The time scales of their rise to the surface are assumed to be smaller than the smallest ($\sim 1$ yr) resolution used in modeling the 'shapes' of the cycles. This is in accordance with computations by Choudhury & D'Silva (1990) for radial travel of flux tubes and with the observational estimate of Howard & LaBonte (1981), and the rising rates of sunspot magnetic structures estimated by us (Chapter V). However, a detailed mathematical modeling of torsional MHD waves (e.g., in a steady field such as in the model mentioned above), and their interference, will be necessary for (i) ascertaining the reality of the phenomenology developed in this study and for (ii) exploring the possibility of sound predictions of the 'shapes' and 'sizes' of future sunspot cycles. It will be important to model the emergence of the toroidal flux bundles, especially the separation of their identities from the ambient field and their distribution in longitude.

The phenomenological model of the energy cascade indicates that the overall sunspot cycle phenomenon resembles a 'relaxation oscillation' (mentioned by Bracewell, 1988). Here the 'negative damping' corresponds to the energy input into the waves of $\nu = \nu_*$ and the 'positive damping' to the loss of energy in the form of flux tubes leaving the main body of the Sun and dissipating in the atmosphere. Therefore, the most important task will be to model the process that perpetually excites the waves at $\nu = \nu_*$.

At present the only mechanism of perpetual excitation at frequencies near $\nu_*$ which we can think of is a resonance coupling to the Sun's motion about the center of mass of the solar system. These torques will depend upon the orbital motions of the planets, whose configurations are known to have some dominant periodicities common to sunspot activity (e.g., review by Solumure et al. 1992) and to the solar differential rotation (see Chapter IV). However, the energetics of such a mechanism need to be worked out.
APPENDIX 2A.

2A.1 Formulae for SHF components

The harmonic components $H_{cc}(l, m, n \setminus T_1, T_2)$, $H_{cs}(l, m, n \setminus T_1, T_2)$, $H_{sc}(l, m, n \setminus T_1, T_2)$, $H_{ss}(l, m, n \setminus T_1, T_2)$ in the expansion

$$p(\mu, \phi, \tau) = \sum_{\alpha, l, m, n} H_\alpha(l, m, nT_1, T_2) \times P^m_l(\mu) \cos(m\phi) \cos(2\pi n\tau)$$  \hspace{1cm} (2A.1)

during the interval $(T_1, T_2)$ are given by

$H_\alpha(l, m, n \setminus T_1, T_2) = C(l, m, n) \int_0^1 d\tau \int_0^{2\pi} d\phi \int_{-1}^{+1} du(\theta, \phi, \tau) \times P^m_l(\mu) \cos(m\phi) \cos(2\pi n\tau)$  \hspace{1cm} (2A.2)

$$= \frac{C(l, m, n)!}{N!} \sum_i P^m_i(\mu) \cos(m\phi) \cos(2\pi n\tau),$$  \hspace{1cm} (2A.3)

where $\alpha$ is a symbol representing the subscript 'cc', 'cs', 'sc' or 'ss', depending upon the combination of the cosines or sines of $(m\phi)$ and $(2\pi n\tau)$ in the respective term, and

$$C(l, m, n) = \Lambda \frac{(l - m)!(2l + 1)}{(l + m)!\pi}$$

with

$$\Lambda = \begin{cases} 
1 & \text{for } m \neq 0 \text{ and } n \neq 0, \\
\frac{1}{2} & \text{if } m = 0 \text{ or } n = 0, \\
\frac{1}{4} & \text{if } m = 0 \text{ and } n = 0.
\end{cases}$$

2A.2. Determination of Amplitudes and Phases

2A.2.1. REFERRED TO $t = T_1$ AS ZERO EPOCH

The amplitudes $A_c(l, m, n)$ and $A_s(l, m, n)$ of the modes $P^m_l(\cos\theta) \cos(m\phi)e^{2\pi in\tau}$ and $P^m_l(\cos\theta) \sin(m\phi)e^{2\pi in\tau}$ during $(T_1, T_2)$ are given by

$$A_c(l, m, n) = [H_{cc}^2(l, m, n) + H_{cs}^2(l, m, n)]^{1/2},$$

$$A_s(l, m, n) = [H_{sc}^2(l, m, n) + H_{ss}^2(l, m, n)]^{1/2}.$$  \hspace{1cm} (2A.4)

We also define the rms amplitude of $P^m_l(\mu)$ term as

$$A(l, m, n) = [A^2_c(l, m, n) + A^2_s(l, m, n)]^{1/2}.$$
For $m = 0$ : $A(l, m, n) = A_c(l, m, n)$.

By phases $\varphi_c(l, m, n)$ and $\varphi_s(l, m, n)$ of the above modes during $(T_1, T_2)$ ‘referred to $t = T_1$ as the zero epoch’ we mean the values of $\varphi$ in their time dependence expressed as

$$\sin[2\pi\nu(t - T_1) + \varphi], \quad \text{where } \nu = n/(T_1 - T_2).$$

These phases are given by

$$\varphi_c(l, m, n) = \tan^{-1}[H_{cc}(l, m, n)/H_{cc}(l, m, n)] + 0 \text{ or } \pi$$

and

$$\varphi_s(l, m, n) = \tan^{-1}[H_{ss}(l, m, n)/H_{ss}(l, m, n)] + 0 \text{ or } \pi,$$

(2A.5)

where 0 or $\phi$ is chosen to ensure the correct signs for the sine and cosines.

For axisymmetric ($m = 0$) modes $\varphi_s$ and the symbol $\varphi_c$ will be replaced by $\varphi$.

2A.2.2. AMPLITUDES AND PHASES REFERRED TO OTHER ZERO EPOCHS

It can be shown that the above formulae also give the amplitudes and phases referred to any epochs $T_0$ other than $T_1$ as zero epoch if in equation (2A.3) one takes

$$\tau = (t - T_0)/(T_2 - T_1)$$

instead of $(t - T_1)/(T_2 - T_1)$.

In such a shift of zero epoch, the amplitudes remain invariant and the phases shift by $2\pi\nu(T_1 - T_0)/(T_2 - T_1)$.