CHAPTER III

ASYMPTOTICS FOR LEAST SQUARES ESTIMATORS RELATING TO PURELY EXPLOSIVE MULTIPLE TIME SERIES MODELS

3.1 Introduction

In this chapter we propose to study the asymptotic properties of the (equation-wise) least squares estimators of the coefficients of a general simultaneous stochastic difference equation model given in (1.2.2), generating a vector time series under the purely explosive situation wherein the roots are real and some or all of the roots are equal.

Further in this chapter we study the asymptotic behaviour of any linear combination of centered least squares estimators.

As mentioned in Chapter II, sec. 2.5, Karthikeyan (1997), has studied the asymptotic properties of the least squares estimators associated with the model (1.2.2) under (i) partially explosive, and (ii) purely explosive situations, with the assumption that the roots are real and distinct.

The case of explosive roots being equal requires a separate study, since the results cannot be generalised nor deduced from the results of the case of distinct explosive roots.

In this thesis, we have focused our study on purely explosive vector time series
generated by a simultaneous linear stochastic difference equation of order $k$ and dimension $m$ when some or all the real characteristic roots are equal. Although we present the proofs for the case $k = 2$ and $m = 2$, the tools and arguments can easily be extended to any general $k$ and $m$.

### 3.2 Preliminary details

In this section, explicit solutions for a purely explosive vector time series which is assumed to be generated by a simultaneous linear stochastic difference equation model, are obtained. This is discussed for seven placements of explosive roots of the characteristic polynomial equation associated with the model (1.2.2) of order two and dimension two.

On focusing our attention to the case $k = 2$, $m = 2$ in model (1.2.2), the model under study is given by

$$Z(t + 2) - B_1 Z(t + 1) - B_2 Z(t) - B_0 = \epsilon(t + 2), \quad (3.2.1)$$

where $Z(t)$ is a bivariate time series. For simplicity of notation we denote the elements of $Z(t)$ as $Z(t) = (X(t), Y(t))'$; and those of $B_k$ as $(\beta_k^{(ij)})$ $i, j = 1, 2$; $k = 1, 2$ and $B_0 = (\beta_{10}, \beta_{20})'$. Here, $B_k$; $k = 1, 2$ are coefficient matrices of order $2 \times 2$ and $B_0$ is a $2 \times 1$ vector of constants.

The model (3.2.1) is governed by the assumptions

A1. $\epsilon(t) = (\epsilon_1(t), \epsilon_2(t))'$ such that $\{\epsilon_i(t); t \geq 1\}; i = 1, 2$ are independent families of i.i.d. random variables with $E(\epsilon_i(t)) = 0$, $\text{Var}(\epsilon_i(t)) = \sigma_i^2$ and $E(\epsilon_i^4(t)) \leq A_0$ ($A_0$ is used as a generic symbol to denote any majorising finite constant).

A2. For mathematical convenience we set

$$Z(t) = \epsilon(t) = 0 \quad \text{for} \ t < 0. \quad (3.2.2)$$

The characteristic polynomial associated with the model (3.2.1) is given by

$$P(z) = \det (z^2I_2 - zB_1 - B_2) \quad z \ \text{scalar.} \quad (3.2.3)$$
By eliminating the related variables in (3.2.1), which is done by adding and subtracting \( \beta_{22}^{(1)} X(t + 3) \) and \( \beta_{22}^{(2)} X(t + 2) \) in the first equation and \( \beta_{11}^{(1)} Y(t + 3) \) and \( \beta_{11}^{(2)} Y(t + 2) \) in the second equation, one gets an alternative representation of \( Z(t) \) as

\[
Z(t + 4) - D_1 Z(t + 3) - \cdots - D_4 Z(t) - D_0 = \eta(t + 4), \quad (3.2.4)
\]

with

\[
\eta(t + 4) = \epsilon(t + 4) + L_1 \epsilon(t + 3) + L_2 \epsilon(t + 2)
= (\eta_1(t + 4), \eta_2(t + 4))', \quad \text{(say)} \quad (3.2.5)
\]

where

(a) \( D_1 = (\beta_{11}^{(1)} + \beta_{22}^{(1)}) I_2 = d_1 I_2 \)
(b) \( D_2 = (\beta_{11}^{(2)} + \beta_{12}^{(1)} \beta_{21}^{(1)} - \beta_{11}^{(1)} \beta_{22}^{(1)} - \beta_{11}^{(2)} \beta_{22}^{(2)}) I_2 = d_2 I_2 \)
(c) \( D_3 = (\beta_{12}^{(1)} \beta_{21}^{(2)} + \beta_{21}^{(2)} \beta_{21}^{(1)} - \beta_{11}^{(2)} \beta_{22}^{(1)} - \beta_{11}^{(1)} \beta_{22}^{(2)}) I_2 = d_3 I_2 \)
(d) \( D_4 = (\beta_{12}^{(2)} \beta_{21}^{(2)} - \beta_{11}^{(2)} \beta_{22}^{(2)}) I_2 = d_4 I_2 \)
(e) \( D_0 = -\beta_{10} \beta_{22}^{(2)} - \beta_{10} \beta_{22}^{(1)} + \beta_{12}^{(2)} \beta_{20} + \beta_{12}^{(2)} \beta_{20} + \beta_{10} \quad d_{10} \) (say)
\[
-\beta_{10} \beta_{22}^{(2)} - \beta_{10} \beta_{22}^{(1)} + \beta_{12}^{(2)} \beta_{20} + \beta_{12}^{(2)} \beta_{20} + \beta_{10} \quad d_{10}
\]

(f) \( L_1 = \begin{pmatrix} \beta_{22}^{(1)} & \beta_{12}^{(1)} \\
\beta_{21}^{(1)} & -\beta_{11}^{(1)} \end{pmatrix}, \quad L_2 = \begin{pmatrix} -\beta_{22}^{(2)} & \beta_{12}^{(2)} \\
\beta_{21}^{(2)} & -\beta_{11}^{(2)} \end{pmatrix} \quad (3.2.6)
\]

From (3.2.4), it follows that both \( X(t) \) and \( Y(t) \) are generated by a stochastic difference equation of order four having the same characteristic polynomial, \( P(z) \), see (3.2.3). We may write

\[
P(z) = z^4 - d_1 z^3 - d_2 z^2 - d_3 z - d_4. \quad (3.2.7)
\]

Stated below are the seven placements for the real explosive roots \( \rho_1, \rho_2, \rho_3 \) and \( \rho_4 \) of \( P(z) = 0 \),
Case 1. $\rho_1 = \rho_2 = \rho_3 = \rho_4 = \rho_0; \ |\rho_0| > 1$

Case 2. $\rho_2 = \rho_3 = \rho_4 = \rho_0; \ |\rho_1| > |\rho_0| > 1$

Case 3. $\rho_3 = \rho_4 = \rho_0; \ |\rho_1| > |\rho_2| > |\rho_0| > 1$

Case 4. $\rho_1 = \rho_2 = \rho_0; \ |\rho_0| > |\rho_3| > |\rho_4| > 1$

Case 5. $\rho_1 = \rho_2 = \rho_3 = \rho_0; \ |\rho_0| > |\rho_4| > 1$

Case 6. $\rho_1 = \rho_2 = \rho_0; \ \rho_3 = \rho_4 = \rho_02; \ |\rho_0| > |\rho_02| > 1$

Case 7. $|\rho_1| > |\rho_2| > |\rho_3| > |\rho_4| > 1 \quad (3.2.8)$

Karthikeyan (1997) has discussed only Case 7 of (3.2.8). In what follows we consider all the cases and study the asymptotics of least squares estimators.

The following theorem provides the explicit solutions for $X(t)$ and $Y(t)$ of the simultaneous stochastic difference equation in (3.2.1) under the various placements of explosive roots of $P(z) = 0$.

**Theorem 3.2.1.** Let $Z(t) = (X(t), Y(t))'$ be generated by the model

$$Z(t + 2) = B_0 + B_1 Z(t + 1) + B_2 Z(t) + \epsilon(t + 2),$$

given in (3.2.1), governed by the assumptions (3.2.2). Further let all the roots of $P(z) = \det(z^2I - zB_1 - B_2) = 0$, be real and explosive.

The general form of the explicit solution for $Z(t) = (X(t), Y(t))'$ is given by

$$X(t) = \sum_{i=1}^{4} a_i(t) G_{1i} + \psi(t)$$

$$Y(t) = \sum_{i=1}^{4} a_i(t) H_{1i} + \psi(t), \quad \quad (3.2.9)$$

where $|a_1(t)| > |a_2(t)| > |a_3(t)| > |a_4(t)|$ for all $t$. Here and elsewhere, $\psi(t)$s are functions of random variables with bounded absolute expectations.
The exact expressions for \(a_4(t)\)'s, the random variables \(G_{ij}\)'s and \(H_{ij}\)'s and explicit solutions for \(X(t)\) and \(Y(t)\) for each placement of roots of \(P(z) = 0\) stated in Cases 1 to 7 in (3.2.8) are given below.

Case 1. When \(\rho_1 = \rho_2 = \rho_3 = \rho_4 = \rho_0; |\rho_0| > 1,$$

\[
X(t) = t^3 \rho_0 G_{11} + t^2 \rho_0 G_{12} + t \rho_0^2 G_{13} + \rho_0^3 G_{14} + \psi(t)
\]
\[
Y(t) = t^3 \rho_0 H_{11} + t^2 \rho_0 H_{12} + t \rho_0^2 H_{13} + \rho_0^3 H_{14} + \psi(t)
\]
\[
G_{11} = \frac{1}{6} G_{11}' ; \ G_{12} = G_{11}' - \frac{1}{2} K_{11}'
\]
\[
G_{13} = \frac{11}{6} G_{11}' - 2 K_{11}' + \frac{1}{2} P_{11}'
\]
\[
G_{14} = G_{11}' - \frac{11}{6} K_{11}' + P_{11}' - \frac{1}{6} Q_{11}'.
\]

Case 2. When \(\rho_2 = \rho_3 = \rho_4 = \rho_0; |\rho_2| > |\rho_0| > 1,$$

\[
X(t) = \rho_1 G_{11} + t^2 \rho_0 G_{12} + t \rho_0^2 G_{13} + \rho_0^3 G_{14} + \psi(t)
\]
\[
Y(t) = \rho_1 H_{11} + t^2 \rho_0 H_{12} + t \rho_0^2 H_{13} + \rho_0^3 H_{14} + \psi(t)
\]
\[
G_{11} = c_1 J_{11}' ; \ G_{12} = c_4 G_{11}' ; \ G_{13} = c_3 G_{11}' - 2 c_4 K_{11}'
\]
\[
G_{14} = c_2 G_{11}' - c_3 K_{11}' + c_4 P_{11}'.
\]

Case 3. When \(\rho_3 = \rho_4 = \rho_0; |\rho_1| > |\rho_2| > |\rho_0| > 1,$$

\[
X(t) = \rho_1 G_{11} + \rho_2 G_{12} + t \rho_0 G_{13} + \rho_0^3 G_{14} + \psi(t)
\]
\[
Y(t) = \rho_1 H_{11} + \rho_2 H_{12} + t \rho_0 H_{13} + \rho_0^3 H_{14} + \psi(t)
\]
\[
G_{11} = c_1 J_{11}' ; \ G_{12} = c_2 J_{12}' ; \ G_{13} = c_4 G_{11}'
\]
\[
G_{14} = c_3 G_{11}' - c_4 K_{11}'.
\]
Case 5. When $\rho_1 = \rho_2 = \rho_3 = \rho_0; \ |\rho_0| > |\rho_4| > 1,$

\[
X(t) = t^2 \rho_0^t G_{11} + t^2 \rho_1^t G_{12} + \rho_0^t G_{13} + \rho_4^t G_{14} + \psi(t) \\
Y(t) = t^2 \rho_0^t H_{11} + t^2 \rho_1^t H_{12} + \rho_0^t H_{13} + \rho_4^t H_{14} + \psi(t) \\
G_{11} = c_3 G_{11}'; G_{12} = c_4 G_{11} - c_2 K_{11}' \\
G_{13} = c_3 J_{13}'; G_{14} = c_4 J_{14}'.
\]

Case 6. When $\rho_1 = \rho_2 = \rho_0; \rho_3 = \rho_4 = \rho_0; \ |\rho_0| > |\rho_0| > 1,$

\[
X(t) = t \rho_0^t G_{11} + \rho_0^t G_{12} + \rho_0^t G_{13} + \rho_4^t G_{14} + \psi(t) \\
Y(t) = t \rho_0^t H_{11} + \rho_0^t H_{12} + \rho_0^t H_{13} + \rho_4^t H_{14} + \psi(t) \\
G_{11} = c_2 L_{11}'; G_{12} = c_3 L_{11} - c_2 M_{11} \\
G_{13} = c_4 L_{12}'; G_{14} = c_3 G_{11} - c_4 K_{11}'.
\]

Case 7. When $|\rho_1| > |\rho_2| > |\rho_3| > |\rho_4| > 1,$

\[
X(t) = \rho_1^t G_{11} + \rho_2^t G_{12} + \rho_3^t G_{13} + \rho_4^t G_{14} + \psi(t) \\
Y(t) = \rho_1^t H_{11} + \rho_2^t H_{12} + \rho_3^t H_{13} + \rho_4^t H_{14} + \psi(t) \\
G_{11} = c_1 J_{11}'; G_{12} = c_2 J_{12} \\
G_{13} = c_3 J_{13}'; G_{14} = c_4 J_{14}'.
\]
The notations used in the above expressions are to be interpreted as described below.

\[ G'_{11} = \sum_{u=1}^{\infty} \rho_0^{-u}\eta_1(u) \quad L'_{11} = \sum_{u=1}^{\infty} \rho_0^{-u}\eta_1(u) \]

\[ J'_{11} = \sum_{u=1}^{\infty} \rho_1^{-u}\eta_1(u) \quad J'_{12} = \sum_{u=1}^{\infty} \rho_2^{-u}\eta_1(u) \]

\[ J'_{13} = \sum_{u=1}^{\infty} \rho_3^{-u}\eta_1(u) \quad J'_{14} = \sum_{u=1}^{\infty} \rho_4^{-u}\eta_1(u) \]

\[ M'_{11} = \sum_{u=1}^{\infty} u\rho_0^{-u}\eta_1(u) \quad K'_{11} = \sum_{u=1}^{\infty} u\rho_0^{-u}\eta_1(u) \]

\[ P'_{11} = \sum_{u=1}^{\infty} u^2\rho_0^{-u}\eta_1(u) \quad Q'_{11} = \sum_{u=1}^{\infty} u^3\rho_0^{-u}\eta_1(u) \]

and \( c_i \)'s are real constants. The random variables \( H_{ij} \)'s are obtained by replacing \( \eta_1(u) \) by \( \eta_2(u) \) in the expressions for \( G_{ij} \)'s.

It may be noted that the random variables \( G_{ij} \)'s and \( H_{ij} \)'s are expressible as infinite series and they are limits in mean square of sequences of linear functions of random variables. We also assume that these limits are continuous at zero.

**Proof.** The general solution to (3.2.1) is given by

\[
X(t) = \sum_{r=0}^{t-1} \lambda(r)\eta_1(t-r) + \mu_1
\]

\[
Y(t) = \sum_{r=0}^{t-1} \lambda(r)\eta_2(t-r) + \mu_2,
\]

(3.2.10)

where \( \mu_i = d_{10}/(1 - d_1 - d_2 - d_3 - d_4); \ i = 1, 2 \). Here, \( d_{10}, d_1, d_2, d_3, d_4 \) are as in (a) to (f) of (3.2.6) and \( \eta_1(t) \) and \( \eta_2(t) \) are as in (3.2.5). Also,

\[
\lambda(r) = \begin{cases} 
0, & r < 0 \\
1, & r = 0 \\
d_1\lambda(r-1) + \cdots + d_4\lambda(r-4), & r \geq 1
\end{cases}
\]

Here we give the proof for Case 1 of the theorem.

Under this case

\[
\lambda(r) = (c_1 + c_2r + c_3r^2 + c_4r^3)\rho_0 \quad r \geq 1,
\]

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\( c_1, c_2, c_3, c_4 \) are constants which can be solved from the expressions for \( \lambda(0), \lambda(1), \lambda(2) \) and \( \lambda(3) \).

Alternatively, we can express \( \lambda(r) \) as

\[
\lambda(r) = \frac{(r+1)(r+2)(r+3) \rho_6^r}{3!} = \left(1 + \frac{11}{6} r + r^2 + \frac{1}{6} r^3\right) \rho_6^r. \tag{3.2.11}
\]

Substituting (3.2.11) in the first equation of (3.2.10), we get

\[
X(t) = \sum_{r=0}^{t-1} \left(1 + \frac{11}{6} r + r^2 + \frac{1}{6} r^3\right) \rho_0^{-u} \eta_1(t - r) + \mu_1.
\]

By putting \( t - r = u \) in the above equation, we get

\[
X(t) = \rho_0^t \sum_{u=1}^{t} \left[1 + \frac{11}{6} (t - u) + (t - u)^2 + \frac{1}{6} (t - u)^3\right] \rho_0^{-u} \eta_1(u) + \mu_1
\]

\[
= \rho_0^t v_1 \sum_{u=1}^{t} \rho_0^{-u} \eta_1(u) + \rho_0^t v_2 \sum_{u=1}^{t} u \rho_0^{-u} \eta_1(u) + \rho_0^t v_3 \sum_{u=1}^{t} u^2 \rho_0^{-u} \eta_1(u)
\]

\[
+ \rho_0^t v_4 \sum_{u=1}^{t} u^3 \rho_0^{-u} \eta_1(u) + \mu_1,
\]

where

\[
v_1 = \left(1 + \frac{11}{6} t + t^2 + \frac{1}{6} t^3\right),
\]

\[
v_2 = -\left(\frac{11}{6} + 2t + \frac{1}{2} t^2\right),
\]

\[
v_3 = 1 + \frac{1}{2} t,
\]

\[
v_4 = -\frac{1}{6}.
\]

Further,

\[
X(t) = \rho_0^t v_1 \left| \sum_{u=1}^{\infty} \rho_0^{-u} \eta_1(u) - \sum_{u=t+1}^{\infty} \rho_0^{-u} \eta_1(u) \right|
\]

\[
+ \rho_0^t v_2 \left| \sum_{u=1}^{\infty} u \rho_0^{-u} \eta_1(u) - \sum_{u=t+1}^{\infty} u \rho_0^{-u} \eta_1(u) \right|
\]

\[
+ \rho_0^t v_3 \left| \sum_{u=1}^{\infty} u^2 \rho_0^{-u} \eta_1(u) - \sum_{u=t+1}^{\infty} u^2 \rho_0^{-u} \eta_1(u) \right|
\]

\[
+ \rho_0^t v_4 \left| \sum_{u=1}^{\infty} u^3 \rho_0^{-u} \eta_1(u) - \sum_{u=t+1}^{\infty} u^3 \rho_0^{-u} \eta_1(u) \right|
\]

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We put \( u = t + r \) in all the second terms in the square brackets and denote the first terms in the square brackets by \( G'_{11}, K'_{11}, P'_{11} \) and \( Q'_{11} \) respectively, on the right hand side of the above equation. This results in

\[
X(t) = \rho_0^t v_1 G'_{11} + \rho_0^t v_2 K'_{11} + \rho_0^t v_3 P'_{11} + \rho_0^t v_4 Q'_{11} + \psi(t),
\]

where

\[
\psi(t) = -v_1 \sum_{r=1}^{\infty} \rho_0^{-r} \eta_1(t + r) - v_2 \sum_{r=1}^{\infty} (t + r) \rho_0^{-r} \eta_1(t + r) - v_3 \sum_{r=1}^{\infty} (t + r)^2 \rho_0^{-r} \eta_1(t + r) - v_4 \sum_{r=1}^{\infty} (t + r)^3 \rho_0^{-r} \eta_1(t + r) + \mu_1.
\]

\( \psi(t) \) is a random function with bounded absolute expectation.

\( \mu_1 \): The functions \( G'_{11}, K'_{11} \) and \( Q'_{11} \) are random variables which are expressed as infinite series, which are limits in mean squares of sequences of linear functions of random variables.

Finally, by substituting for \( u_i \)'s and by collecting the coefficients of \( \rho_0^i t^i; \quad i = 0, 1, 2, 3 \) in (3.2.12), we get

\[
X(t) = t^3 \rho_0 G_{11} + t^2 \rho_0 G_{12} + t \rho_0 G_{13} + \rho_0 G_{14} + \psi_{11}(t),
\]

where

\[
G_{11} = \frac{1}{6} G'_{11},
\]

\[
G_{12} = G'_{11} - \frac{1}{2} K'_{11},
\]

\[
G_{13} = \frac{11}{6} G'_{11} - 2K'_{11} + \frac{1}{2} P'_{11},
\]

\[
G_{14} = G'_{11} - \frac{11}{6} K'_{11} + P'_{11} - \frac{1}{6} Q'_{11}.
\]

Similarly we can show that

\[
Y(t) = t^3 \rho_0^t H_{11} + t^2 \rho_0^t H_{12} + t \rho_0^t H_{13} + \rho_0^t H_{14} + \psi(t),
\]

where \( \psi(t) \) is obtained from \( \psi(t) \) of (3.2.13) by replacing \( \eta_1(t + r) \) by \( \eta_2(t + r) \) and \( \mu_1 \) by \( \mu_2 \) respectively. Corresponding changes in \( G_{ij} \)'s will yield \( H_{ij} \)'s.

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The results for the Cases 2 to 7 are obtained on the same lines as above, by choosing the appropriate \( \lambda(r) \)'s which are given below

\[
\begin{align*}
\lambda(r) &= c_1 \rho_1^r + (c_2 + c_3 r + c_4 r^2) \rho_0^r \\
\lambda(r) &= c_1 \rho_1^r + c_2 \rho_2^r + (c_3 + c_4 r) \rho_0^r \\
\lambda(r) &= (c_1 + c_2 r) \rho_0^r + c_3 \rho_3^r + c_4 \rho_4^r \\
\lambda(r) &= (c_1 + c_2 r + c_3 r^2) \rho_0^r + c_4 \rho_4^r \\
\lambda(r) &= (c_1 + c_2 r + c_3 r^2) \rho_0^r + (c_3 + c_4 r) \rho_0^r \\
\lambda(r) &= c_1 \rho_1^r + c_2 \rho_2^r + c_3 \rho_3^r + c_4 \rho_4^r
\end{align*}
\]

Hence the theorem.

Note 3.2.3: Using (3.2.9) it is possible to express \( X(t+1) \) and \( Y(t+1) \) as

\[
X(t+1) = \sum_{i=1}^{4} a_i(t) J_{i1} + \psi(t) \\
Y(t+1) = \sum_{i=1}^{4} a_i(t) J_{i1} + \psi(t)
\]

\( I \) and \( J \)'s are random variables, which are expressible as infinite series being limits in mean square of sequences of linear functions of random variables. We also assume that these limits are continuous at zero.

**Auxiliary process:**

The auxiliary processes (see (2.4.10)) generated by the model in (3.2.1) are given by

\[
Z_1(t) = Z(t) \\
Z_2(t+1) = Z_1(t+1) - A_1 Z_1(t).
\]

Explicit solutions of \( Z_2(t) \) under different placements of explosive roots \( \rho_3 \) and \( \rho_4 \) are given in the following theorem.
Theorem 3.2.2. Let $Z(t)$ be generated from (3.2.1). Then, under the basic assumptions, the following statements hold

i. When $\rho_3 = \rho_4 = \rho_0$; $|\rho_0| > 1$,

$$X_2(t) = \sum_{i=3}^{4} a_i(t) G_{3i} + \psi(t)$$

$$= t \rho_0^4 G_{33} + \rho_0^4 G_{34} + \psi(t)$$

$$Y_2(t) = \sum_{i=3}^{4} a_i(t) H_{3i} + \psi(t)$$

$$= t \rho_0^4 H_{33} + \rho_0^4 H_{34} + \psi(t),$$

where

$$G_{33} = \sum_{u=1}^{\infty} \rho_0^{-u} \eta_1(u)$$

$$G_{34} = \sum_{u=1}^{\infty} \rho_0^{-u} \eta_1(u) - \sum_{u=1}^{\infty} u \rho_0^{-u} \eta_1(u)$$

$$H_{33} = \sum_{u=1}^{\infty} \rho_0^{-u} \eta_2(u)$$

$$G_{34} = \sum_{u=1}^{\infty} \rho_0^{-u} \eta_2(u) - \sum_{u=1}^{\infty} u \rho_0^{-u} \eta_2(u).$$

ii. When $|\rho_3| > |\rho_4| > 1$

$$X_2(t) = \sum_{i=3}^{4} a_i(t) G_{3i} + \psi(t) = \rho_3^4 G_{33} + \rho_4^4 G_{34} + \psi(t)$$

$$Y_2(t) = \sum_{i=3}^{4} a_i(t) H_{3i} + \psi(t) = \rho_3^4 H_{33} + \rho_4^4 H_{34} + \psi(t),$$

where

$$G_{33} = p_1 \sum_{u=1}^{\infty} \rho_3^{-u} \eta_1(u)$$

$$H_{33} = p_1 \sum_{u=1}^{\infty} \rho_3^{-u} \eta_2(u)$$

$$G_{34} = p_2 \sum_{u=1}^{\infty} \rho_4^{-u} \eta_2(u)$$

$$G_{34} = p_2 \sum_{u=1}^{\infty} \rho_4^{-u} \eta_2(u),$$

and

$p_1$ and $p_2$ are real constants.

Proof. From (2.4.11), it follows that the auxiliary process $Z_2(t)$ in (2.4.10) is generated from the linear stochastic difference equation

$$Z_2(t + 1) - A_2 Z_2(t) - B_0 = e(t + 1),$$

(3.2.16)

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where \( A_2 = \left( a_{ij}^{(2)} \right) \); \( j = 1, 2 \) is matrix root of \( P(z) = 0 \), \( P(z) \) given in (3.2.3), such that its characteristic roots are \( \rho_3 \) and \( \rho_4 \).

The model (3.2.16) has the alternative representation given by

\[
Z_2(t + 3) - D_1 Z_2(t + 2) - D_2 Z_2(t + 1) - D_0 = \eta(t + 3), \tag{3.2.17}
\]

where

\[
D_1 = \text{diag} (\tilde{d}_1, \tilde{d}_1) \quad \quad \quad D_2 = \text{diag} (\tilde{d}_2, \tilde{d}_2)
\]

\[
D_0 = (\tilde{d}_{01}, \tilde{d}_{02})' \quad \quad \quad \eta(t + 3) = (\eta_1(t + 3), \eta_2(t + 3))' \quad \text{(say)}
\]

\[
\tilde{d}_1 = \left( a_{11}^{(2)} + a_{22}^{(2)} \right) \quad \quad \quad \tilde{d}_2 = \left( a_{21}^{(2)} a_{12}^{(2)} - a_{11}^{(2)} a_{22}^{(2)} \right)
\]

\[
\tilde{d}_{01} = \left( 1 - a_{22}^{(2)} \right) \beta_{10} + a_{12}^{(2)} \beta_{20} \quad \quad \quad \tilde{d}_{02} = \left( 1 - a_{11}^{(2)} \right) \beta_{30} + a_{21}^{(2)} \beta_{10}. \tag{3.2.18}
\]

The characteristic polynomial equation associated with (3.2.18) is

\[
\bar{P}(z) = z^2 - d_1 z - d_2 = 0. \tag{3.2.19}
\]

Here \( \rho_3 \) and \( \rho_4 \) are the roots of (3.2.19).

Solution to (3.2.17) is given by

\[
X_2(t) = \sum_{r=0}^{t-1} \lambda(r) \eta_1(t-r) + \tilde{d}_{01}
\]

\[
Y_2(t) = \sum_{r=0}^{t-1} \lambda(r) \eta_2(t-r) + \tilde{d}_{02}, \tag{3.2.20}
\]

where

\[
\tilde{d}_{0i} = \tilde{d}_{0i}/(1 - \tilde{d}_1 - \tilde{d}_2); \quad i = 1, 2 \quad \text{and}
\]

\[
\lambda(r) = \begin{cases} 
0, & r < 0 \\
1, & r = 0 \\
\tilde{d}_1 \lambda(r-1) + \tilde{d}_2 (r-2), & r \geq 1.
\end{cases} \tag{3.2.21}
\]

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When \( \rho_3 = \rho_4 = \rho_0 \); \( |\rho_0| > 1 \), we have \( \lambda(r) = (1 + r)\rho_0^r \). Substituting the above \( \lambda(r) \) in the first equation of (3.2.20), we have

\[
X_2(t) = \sum_{r=0}^{t-1} (1 + r)\rho_0^r \bar{\eta}_1(t - r) + \bar{d}_{01}
\]

\[
= \rho_0^t \sum_{u=1}^{t} (1 + t - u)\rho_0^{-u} \bar{\eta}_1(u) + \bar{d}_{01} \quad \text{(by putting } t - u = r \text{)}
\]

\[
= \rho_0^t \left[ \sum_{u=1}^{\infty} (1 + t - u)\rho_0^{-u} \bar{\eta}_1(u) - \sum_{u=t+1}^{\infty} (1 + t - u)\rho_0^{-u} \bar{\eta}_1(u) \right] + \bar{d}_{01}
\]

\[
= t\rho_0^t G_{33} + \rho_0^t G_{34} - \sum_{r=1}^{\infty} (1 - r)\rho_0^{-r} \bar{\eta}_1(t + r) + \bar{d}_{01}
\]

(by putting \( u = t + r \) in the second term within the square brackets)

Thus \( X_2(t) = t\rho_0^t G_{33} + \rho_0^t G_{34} + \psi(t) \) when \( \psi(t) = \bar{d}_{01} - \sum_{r=1}^{\infty} (1 - r)\rho_0^{-r} \bar{\eta}_1(t + r) \). Similarly we can get the expression for \( Y_2(t) \) as given in statement (i) of the theorem.

When \( |\rho_3| > |\rho_4| > 1 \), we have \( \lambda(r) = p_1\rho_3^r + p_2\rho_4^r \), where \( p_1 = \frac{\rho_3}{(\rho_3 - \rho_4)} \) and \( p_2 = \frac{\rho_4}{(\rho_4 - \rho_3)} \).

Substituting this in (3.2.20), we get

\[
X_2(t) = \sum_{r=0}^{t-1} (p_1\rho_3^r + p_2\rho_4^r) \bar{\eta}_1(t - r) + \bar{d}_{01}
\]

\[
= p_1 \sum_{r=0}^{t-1} \rho_3^r \bar{\eta}_1(t - r) + p_2 \sum_{r=0}^{t-1} \rho_4^r \bar{\eta}_1(t - r) + \bar{d}_{01}
\]

\[
= p_1 \rho_3^t \sum_{u=1}^{t} \rho_3^{-u} \bar{\eta}_1(u) + p_2 \rho_4^t \sum_{u=1}^{t} \rho_4^{-u} \bar{\eta}_1(u) + \bar{d}_{01} \quad \text{(by putting } t - r = u \text{)}
\]

\[
= p_1 \rho_3^t G'_{33} + p_2 \rho_4^t G'_{34} - p_1 \sum_{r=1}^{\infty} \rho_3^{-r} \bar{\eta}_1(t + r) - p_2 \sum_{r=1}^{\infty} \rho_4^{-r} \bar{\eta}_1(t + r) + \bar{d}_{01}
\]

where \( G'_{33} = \sum_{u=1}^{\infty} \rho_3^{-u} \bar{\eta}_1(u) \), \( G'_{34} = \sum_{u=1}^{\infty} \rho_4^{-u} \bar{\eta}_1(u) \).

Thus \( X_2(t) = \rho_3^t G_{33} + \rho_4^t G_{34} + \psi(t) \), where \( \psi(t) = \bar{d}_{01} - p_1 \sum_{r=1}^{\infty} \rho_3^{-r} \bar{\eta}_1(t + r) - p_2 \sum_{r=1}^{\infty} \rho_4^{-r} \bar{\eta}_1(t + r) \).

On the same lines one can get the expression for \( Y_2(t) \) as given in statement (ii) of the theorem. Hence the theorem. \( \square \)
3.3 Asymptotic distribution of the least squares estimators

In this section we study the asymptotic properties of the least squares estimators of the structural parameters $B_1$, $B_2$ and $B_0$ of the model (3.2.1). This is carried out by first reparameterising the model and then studying the asymptotic behaviour of the least squares estimators of the parameters of this model. Applying these results, we then study the asymptotic behaviour of the least squares estimators of the parameters $B_1$, $B_2$ and $B_0$.

Let \( \{Z(t) = (X(t), Y(t))^\prime; \ t = 1, 2, \ldots, N\} \) be a partial realisation of \( \{Z(t); \ t \geq 1\} \). It is assumed that $Z(t)$ is generated by the model

\[
\begin{align*}
X(t + 2) - \beta_{11}^{(1)}X(t + 1) - \beta_{12}^{(1)}Y(t + 1) \\
- \beta_{11}^{(2)}X(t) - \beta_{12}^{(2)}Y(t) - \beta_{01} = \epsilon_1(t + 2) \\
Y(t + 2) - \beta_{21}^{(1)}X(t + 1) - \beta_{22}^{(1)}Y(t + 1) \\
- \beta_{21}^{(2)}X(t) - \beta_{22}^{(2)}Y(t) - \beta_{02} = \epsilon_2(t + 2)
\end{align*}
\]

As described in (2.4.8), the centered least squares estimators of the coefficients of the first equation of the model (3.2.1) are given by

\[
\begin{align*}
\hat{\beta}_{11}^{(1)} - \beta_{11}^{(1)} &= \frac{\Delta_1(1)}{\Delta_1} \\
\hat{\beta}_{12}^{(1)} - \beta_{12}^{(1)} &= \frac{\Delta_1(2)}{\Delta_1} \\
\hat{\beta}_{21}^{(2)} - \beta_{21}^{(2)} &= \frac{\Delta_1(3)}{\Delta_1} \\
\hat{\beta}_{22}^{(2)} - \beta_{22}^{(2)} &= \frac{\Delta_1(4)}{\Delta_1} \\
0 - \beta_{10}^{(1)} &= \frac{\Delta_1(5)}{\Delta_1}.
\end{align*}
\]
where $\Delta_1$ is the determinant of the matrix of sum of squares and products and is given by

$$
\begin{align*}
\sum X^2(t+1) & \quad \sum X(t+1)Y(t+1) & \quad \sum X(t+1)X(t) & \quad \sum X(t+1)Y(t) & \quad \sum X(t+1) \\
\sum Y(t+1)X(t+1) & \quad \sum Y^2(t+1) & \quad \sum Y(t+1)X(t) & \quad \sum Y(t+1)Y(t) & \quad \sum Y(t+1) \\
\sum X(t)X(t+1) & \quad \sum X(t)Y(t+1) & \quad \sum X^2(t) & \quad \sum X(t)Y(t) & \quad \sum X(t) \\
\sum Y(t)X(t+1) & \quad \sum Y(t)Y(t+1) & \quad \sum Y^2(t) & \quad \sum Y(t)X(t) & \quad \sum N-2
\end{align*}
$$

Here and else where the range of summation $\sum$ unless otherwise specified is over $t$ from $1$ to $N-2$. Further, $\Delta_1(l); \ l = 1, 2, \ldots, 5$ are determinants of $M_1(l); \ l = 1, 2, \ldots, 5$ respectively. The matrices $M_1(l); \ l = 1, 2, \ldots, 5$ are obtained from $M_1$ on replacing its $l^{th}$ column by the column vector

$$(\sum X(t+1)e_1(t+2) \ \sum Y(t+1)e_1(t+2) \ \sum X(t)e_1(t+2) \ \sum Y(t)e_1(t+2) \ \sum e_1(t+2))'$$

Similarly the centered least squares estimators of the coefficients of the second equation of (3.3.1) are given by

$$
\begin{align*}
\hat{\beta}_2^{(1)} - \beta_2^{(1)} &= \frac{\Delta_2(1)}{\Delta_2} \\
\hat{\beta}_2^{(2)} - \beta_2^{(2)} &= \frac{\Delta_2(2)}{\Delta_2} \\
\hat{\beta}_2^{(3)} - \beta_2^{(3)} &= \frac{\Delta_2(3)}{\Delta_2} \\
\hat{\beta}_2^{(4)} - \beta_2^{(4)} &= \frac{\Delta_2(4)}{\Delta_2} \\
\hat{\beta}_2^{(5)} - \beta_2^{(5)} &= \frac{\Delta_2(5)}{\Delta_2}
\end{align*}
$$

where $\Delta_2 = \Delta_1$, is the determinant of $M_2$. $\Delta_2(l); \ l = 1, 2, \ldots, 5$ are the determinants of $M_2(l); \ l = 1, 2, \ldots, 5$. The matrices $M_2(l); \ l = 1, 2, \ldots, 5$ are obtained from $M_1(l); \ l = 1, 2, \ldots, 5$ by replacing in it the vector (3.3.4) by the vector

$$(\sum X(t+1)e_2(t+2) \ \sum Y(t+1)e_2(t+2) \ \sum X(t)e_2(t+2) \ \sum Y(t)e_2(t+2) \ \sum e_2(t+2))'$$
3.3.1 Transformation to the matrices $M_l$ and $M_i(l)$, $l = 1, 2, \ldots, 5$, $i = 1, 2$.

Here we notice that the equation-wise least squares estimators of coefficients in (3.3.1) will not converge in distribution to a nondegenerate random vector, since as such (after element stabilization of) the matrix of sums of squares and products will converge in probability, as the sample size $N$ tends to infinity to a singular matrix. This is because, in explosive situation, $X(t), Y(t), X(t + 1)$ and $Y(t + 1)$ all have the same dominant time path of growth. To overcome this a well known procedure (Venkataraman (1968)) is to use some row column transformations on the relevant matrices. A useful hint in this context is to effect transformations wherein the diagonal elements of the matrices diverge to infinity at different rates, thereby identify the stabilizing factors that ensure non-zero limit in probability for its determinant value.

For establishing the boundedness in probability of least squares estimators in (3.3.1), one can adopt any of the following two elementary transformations on the matrices $M_1$ and its modifications $M_i(l)$, $l = 1, 2, \ldots, 5$, $i = 1, 2$. These two methods are essentially equivalent as will be seen later.

**Method 1:** (Successive elimination of the next higher rate of divergence of summands in $M_1$, and $M_i(l)$, $l = 1, 2, \ldots, 5$, $i = 1, 2$)

Towards elaborating these transformations we first introduce auxiliary processes, which are derived from $X(t)$ and $Y(t)$, and diverge to infinity at lower rates as $t$ tends to infinity.

From (3.2.9) we have

$$X(t) = \sum_{i=1}^{4} a_i(t) G_{1i} + \psi(t)$$

$$Y(t) = \sum_{i=1}^{4} a_i(t) H_{1i} + \psi(t)$$
using which it is possible to write

\[ X(t + 1) = \sum_{i=1}^{4} a_i(t) I_{1i} + \psi(t) \quad \text{and} \]

\[ Y(t + 1) = \sum_{i=1}^{4} a_i(t) J_{1i} + \psi(t) \quad (3.3.6) \]

Let

\[ \phi(t) = H_{11}X(t) - G_{11}Y(t) \]

\[ = \sum_{i=2}^{4} a_i(t) G_{2i} + \psi(t) \quad (3.3.7a) \]

\[ \bar{Y}(t + 1) = H_{11}Y(t + 1) - J_{11}Y(t) \]

\[ = \sum_{i=2}^{4} a_i(t) J_{2i} + \psi(t) \quad (3.3.7b) \]

\[ \bar{X}(t + 1) = H_{11}X(t + 1) - I_{11}Y(t) \]

\[ = \sum_{i=2}^{4} a_i(t) I_{2i} + \psi(t) \quad (3.3.7c) \]

\[ Y_2(t + 1) = G_{22}\bar{Y}(t + 1) - J_{22}\phi(t) \]

\[ = \sum_{i=3}^{4} a_i(t) J_{3i} + \psi(t) \quad (3.3.7d) \]

\[ X_2(t + 1) = G_{22}\bar{X}(t + 1) - I_{22}\phi(t) \]

\[ = \sum_{i=3}^{4} a_i(t) I_{3i} + \psi(t) \quad (3.3.7e) \]

\[ \phi_2(t + 1) = J_{33}X_2(t + 1) - I_{33}Y_2(t + 1) \]

\[ = a_4(t) I_{44} + \psi(t), \quad (3.3.7f) \]

where, henceforth, \( \psi(t) \) denotes generic terms with bounded absolute expectation.

It is easy to verify that as \( t \to \infty \), \( Y(t) \to \infty \) at the rate \( a_1(t) \), \( \phi(t) \to \infty \) at the rate \( a_2(t) \), \( Y_2(t) \to \infty \) at the rate \( a_3(t) \) and \( \phi_2(t) \to \infty \) at the rate \( a_4(t) \). We also note that even if the roots of \( P(z) = 0 \) are all equal, \( a_i(t) \)'s need not be equal.

Let \( R(i, j; u, v) \) denote an elementary row transformation on a matrix of adding \( u \)
times its $i^{th}$ row to $v$ times its $j^{th}$ row. In parallel, we have $C(i, j; u, v)$, referring to the columns of the matrix. It may be noted that the determinant of the matrix changes with these transformations if $v \neq 1$. Using the notation in (3.3.7), the following series of transformations (sequentially from right to left order) is made on the matrix $M_1$

$$
[C(2, 1; -J_{33}, J_{33}) C(3, 1; -I_{22}, G_{22}) C(3, 2; -I_{22}, G_{22})
C(4, 1; -I_{11}, H_{11}) C(4, 2; -J_{11}, H_{11}) C(4, 3; -G_{11}, H_{11})
R(2, 1; -I_{33}, J_{33}) R(3, 1; -I_{22}, G_{22}) R(3, 2; -I_{22}, G_{22})
R(4, 1; -I_{11}, H_{11}) R(4, 2; -J_{11}, H_{11}) R(4, 3; -G_{11}, H_{11})] M_1
$$

(3.3.8)

By virtue of (3.3.8), the value of the determinant $\Delta_1$ gets multiplied by $(H_{11})^6 (G_{22})^4 (J_{33})^2$.

The above twelve transformations cannot be carried out on $M_i(l)$, $l = 1, 2, \ldots, 5$, $i = 1, 2$ matrices, due to the presence of (3.3.4) as one of their columns. However all the six row transformations in (3.3.8) can be carried out on these matrices first. In addition to these six operations the following case specific column transformations are carried out in order to show ultimately that least squares estimators in (3.3.2) converge in distribution to a non-degenerate random vector.

1. When the vector in (3.3.4) appears as the first column, we make the following transformations

$$
[C(3, 2; -J_{22}, G_{22}) C(4, 2; -J_{11}, H_{11}) C(4, 3; -G_{11}, H_{11})]
$$

2. When the vector is (3.3.4) appears as the second column, the following transformations

$$
[C(3, 1; -I_{22}, G_{22}) C(4, 1; -I_{11}, H_{11}) C(4, 3; -G_{11}, H_{11})]
$$

3. When the vector in (3.3.4) appears as the third column, in addition to (3.3.7),
we define the auxiliary process
\[
\tilde{\phi}(t + 1) = J_{22} \tilde{X}(t + 1) - I_{22} \tilde{Y}(t + 1) = \sum_{i=3}^{4} a_i(t) \tilde{G}_{3i}
\]
and the column transformations carried out are
\[
[C \ (2,1; -I_{22}, J_{22}) \ C \ (4,1; -I_{11}, H_{11}) \ C \ (4,2; -J_{11}, H_{11})]
\]

4. When the vector in (3.3.4) appears as the fourth column, in addition to (3.3.7), we define the following auxiliary process
\[
\tilde{Y}(t + 1) = G_{11} Y(t + 1) - J_{11} X(t) = \sum_{i=2}^{4} a_i(t) \tilde{J}_{2i} + \phi(t)
\]
\[
\tilde{X}(t + 1) = G_{11} X(t + 1) - I_{11} X(t) = \sum_{i=2}^{4} a_i(t) \tilde{I}_{2i} + \phi(t)
\]
\[
\tilde{\phi}(t + 1) = J_{22} \tilde{X}(t + 1) - I_{22} \tilde{Y}(t + 1) = \sum_{i=3}^{4} a_i(t) \tilde{I}_{3i} + \psi t
\]
and the column transformations used are
\[
[C \ (2,1; -\tilde{I}_{22}, \tilde{J}_{22}) \ C \ (3,1; -I_{11}, G_{11}) \ C \ (3,2; -J_{11}, G_{11})]
\]

5. When the vector in (3.3.4) appears as the fifth column all the twelve transformations in (3.3.8) are carried out.

Some of the ratios of random functions used in each of the elementary transformations in (3.3.9), (which can eventually be seen to be functions of $G_{1i}$ and $H_{1i}$), are (almost surely) constants as can be inferred from Lemma 2 of Venkataraman (1974) and Theorem 3.2.2 of Karthikeyan (1997). In fact, it can be checked that the ratio of all the random functions used for defining the auxiliary processes in (3.3.8) and (3.3.9) are constants. These constants, which are functions of (scalar and matrix) roots of $P(z) = 0$, and the coefficients of (3.2.1), can also be used for alternative equivalent transformations.
Since the algebra gets aggravated as the order and dimension of the time series increase, we suggest the above method be used for proving the convergence in probability and hence in distribution and boundedness in probability of least squares estimators and the alternative method (given below) for identifying the nature of linear restrictions on the limiting random variables and their implications.

We note that even after these transformations, the elements of the matrices $M_1$ and $M_i(l)$ diverge to infinity. One can stabilise these elements by dividing by appropriate functions of powers of the roots of $P(z) = 0$. This method works when all the explosive roots of $P(z) = 0$ are distinct (and real), as demonstrated by Karthikeyan. Evaluating the ratios in (3.3.2), after stabilising the elements of the matrices in the numerator and the denominator in (3.3.2), one can easily identify the rate of convergence of the least squares estimates as the $N^{th}$ power of the smallest root of $P(z) = 0$, $N$ being the sample size. However, when the explosive roots are not distinct, the rate of convergence will be of the type (the $N^{th}$ power of the smallest root of $P(z) = 0$) (a specific power of $N$). To identify the specific power of $N$ in the second factor we have to evaluate the two determinants, which is cumbersome. Even the Maple software has not been effective in our investigation. Consequently, we provide a novel alternative procedure to identify the asymptotic distribution of the least squares estimates. This procedure is not only convenient but also identifies the linear restrictions among the components of the limiting random vector which have their unique implications as seen in the sequel. Generalisations to higher orders of the model appears to be easy and straightforward under this approach. This method effectively uses only one set of transformations as in (3.3.8), which can also be expressed, equivalently, in terms of the matrix and scalar roots of $P(z) = 0$, as described below.

Method 2: (an alternative to Method 1)

Here the transformations are effected only on the matrix $M_1$

Let $A_1 = \left( a_{ij}^{(1)} \right)$ and $A_2 = \left( a_{ij}^{(2)} \right)$, $(A_1 > A_2)$ be the two matrix roots of $P(z) = 0$. 

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Then the model (3.2.1) has the alternative representation, namely

\[ Z_2(t + 1) = Z(t + 1) - A_1 Z(t) \quad \text{and} \quad (3.3.10a) \]
\[ Z_2(t + 2) - A_2 Z_2(t + 1) - B_0 = e(t + 2). \quad (3.3.10b) \]

It can be shown that the components \( X_2(t + 1) = X((t + 1) - a_{11}^{(1)} X(t) - a_{12}^{(1)} Y(t) \) and \( Y_2(t + 1) = Y(t + 1) - a_{21}^{(1)} X(t) - a_{22}^{(1)} Y(t) \) of \( Z_2(t + 1) \) in (3.3.10a) are the same as those in (3.3.7d) and (3.3.7e) respectively.

Below we prove the equivalence of the two methods. We prove it for the Case 1 of (3.2.8). On similar lines one can verify the equivalence for the other cases.

Assume that \( \rho_1 = \rho_2 = \rho_3 = \rho_4 = \rho_0; |\rho_0| > 1 \)

\[ X(t) = \rho_0^4 t^3 G_{11} + t^2 \rho_0^6 G_{12} + t \rho_0^8 G_{13} + \rho_0^9 G_{14} + \psi(t) + \mu_1 \]
\[ Y(t) = \rho_0^4 t^3 H_{11} + t^2 \rho_0^6 H_{12} + t \rho_0^8 H_{13} + \rho_0^9 H_{14} + \psi(t) + \mu_2 \]
\[ \phi(t) = H_{11} X(t) - G_{11} Y(t) \]
\[ = H_{11} (X(t) - \pi_1 Y(t)) \]
\[ \Rightarrow \phi(t) = \rho_0^4 t^2 [H_{11} G_{12} - G_{11} H_{12}] + t \rho_0^6 [H_{11} G_{13} - G_{11} H_{13}] \]
\[ + \rho_0^8 [H_{11} G_{14} - G_{11} H_{14}] + \psi(t) + \mu_3 \]
\[ Y(t + 1) = \rho_0 [\rho_0^4 (t + 1)^3 H_{11} + \rho_0^6 (t + 1)^2 H_{12} + \rho_0^8 (t + 1) H_{13} + \rho_0^9 H_{14} + \psi(t) + \mu_2] \]
\[ = \rho_0 [\rho_0^4 t^3 H_{11} + t^2 \rho_0^6 (3H_1 + H_2) + t \rho_0^8 (3H_{11} + 2H_{12} + H_{13}) \]
\[ + \rho_0^9 (H_{11} + H_{12} + H_{13} + H_{14})] + \psi(t) + \mu_4. \]

Second equation of (3.3.10a) is \( Y_2(t + 1) = Y(t + 1) - aX(t) - bY(t) \) such that \( Y_2(t + 1) \) is free from \( t^3 \rho_0^4 \) and \( t^2 \rho_0^6 \) (from Theorem 3.2.2(i)) as \( Y_2(t + 1) \) is an auxiliary process of order one.

Therefore, we should have the coefficients of \( t^3 \rho_0^4 \) and \( t^2 \rho_0^6 \) in \( Y(t + 1) \) equal to sum of the coefficients of \( t^3 \rho_0^4 \) and \( t^2 \rho_0^6 \) in \( X(t) \) and \( Y(t) \) respectively on the right hand side of the above equation. This implies the following equalities

\[ \rho_0 H_{11} = a G_{11} + b H_{11} \quad \text{and} \quad \rho_0 (3H_1 + H_2) = a G_{12} + b H_{12} \]
\[ a = \frac{-3\rho_0 H_1^2}{\bar{i}_{12} H_{12} - H_{11} G_{12}} = a^{(1)}_{11} \quad \text{and} \quad b = \frac{\rho_0 (H_{11} G_{12} - G_{11} H_{12}) - 3\rho_0 H_{11} G_{11}}{H_{11} G_{12} - G_{11} H_{12}} \]

or

\[ b = \rho_0 - \frac{3\rho_0 H_{11} G_{11}}{H_{11} G_{12} - G_{11} H_{12}} = a^{(1)}_{12} \]

Alternatively, we can use \( \phi(t) \) and \( Y(t) \) to eliminate \( t^3 \rho_0^3 \) and \( t^2 \rho_0^2 \) in \( Y_2(t + 1) \) ((3.3.7d) method 1) by first considering

\[ \tilde{Y}(t + 1) = Y(t + 1) - \rho_0 Y(t) \]
\[ = \rho_0 t^2 \rho_0 (3H_{11}) + t\rho_0\rho_0 (3H_1 + H_2) \]
\[ + \rho_0 \rho_0^2 (H_{11} + H_{12} + H_{13}) + \psi(t) \quad \text{and then} \]

\[ Y_2(t + 1) = \tilde{Y}(t + 1) (H_{11} G_{12} - G_{11} H_{12}) - 3\rho_0 H_{11} \phi(t) \]
\[ = [Y(t + 1) - \rho_0 Y(t)] [H_{11} G_{12} - G_{11} H_{12}] - 3\rho_0 H_{11} [H_{11} X(t) - G_{11} Y(t)] \]
\[ = Y(t + 1) (H_{11} G_{12} - G_{11} H_{12}) - 3\rho_0 H_{11}^2 X(t) - [\rho_0 (H_{11} G_{12} - G_{11} H_{12}) \]
\[ - 3\rho_0 H_{11} G_{11}] Y(t). \]

Therefore,

\[ a = \frac{-3\rho_0 H_{11}^2}{(H_{11} G_{12} - G_{11} H_{12})} \quad \text{and} \quad b = \rho_0 - \frac{3\rho_0 H_{11} G_{11}}{H_{11} G_{12} - G_{11} H_{12}}. \]

On similar lines one can verify equivalence for Cases 2 to 7 of (3.2.8).

The following theorem finds its importance in the sequel.

In Theorem 3.3.1 below we use the following notations

\[ \frac{\rho_1 \beta_{12}^{(1)} + \beta_{12}^{(2)}}{\rho_1^2 - \rho_1 \beta_{11}^{(1)} - \beta_{11}^{(2)}} = \pi_1 \quad \frac{a^{(2)}}{\rho_3 - a^{(1)}} = \bar{\pi}_1 \]

Theorem 3.3.1. Let \( Z(t) = (X(t), Y(t))' \) satisfy (3.2.1). Then under the basic assumptions the following statements hold

a. \( \frac{X(t)}{Y(t)} \xrightarrow{P} \frac{G_{11}}{H_{11}} \quad \text{as} \quad t \to \infty; \quad P \left( \frac{G_{11}}{H_{11}} = \pi_1 \right) = 1 \)
\[ \frac{X(t + 1)}{\phi(t)} \xrightarrow{P} \frac{I_{22}}{G_{22}} \text{ as } t \to \infty; \quad P \left( \frac{I_{22}}{G_{22}} = a_{11}^{(1)} \right) = 1 \]

\[ \frac{X(t + 1)}{Y(t)} \xrightarrow{P} \frac{I_{11}}{H_{11}} \text{ as } t \to \infty; \quad P \left( \frac{I_{11}}{H_{11}} = a_{12}^{(1)} + a_{11}^{(1)} \pi_1 \right) = 1 \]

\[ \frac{Y(t + 1)}{\phi(t)} \xrightarrow{P} \frac{J_{22}}{G_{22}} \text{ as } t \to \infty; \quad P \left( \frac{J_{22}}{G_{22}} = a_{21}^{(1)} \right) = 1 \]

\[ \frac{Y(t + 1)}{Y(t)} \xrightarrow{P} \frac{J_{11}}{H_{11}} \text{ as } t \to \infty; \quad P \left( \frac{J_{11}}{H_{11}} = a_{21}^{(1)} \right) = 1 \]

\[ \frac{X_2(t + 1)}{Y_2(t + 1)} \xrightarrow{P} \frac{G_{33}}{H_{33}} \text{ as } t \to \infty; \quad P \left( \frac{G_{33}}{H_{33}} = \pi_1 \right) = 1 \]

where \( A_1 = (a_{ij}^{(1)}) \) and \( A_2 = (a_{ij}^{(2)}) \) are the two matrix roots of \( P(z) = 0 \) and \( \rho_1 \) and \( \rho_3 \) are the numerically largest characteristic roots of \( A_1 \) and \( A_2 \) respectively.

**Proof.** The first statement in (a) for all Cases 1 to 7 of placements of roots listed in (3.2.8) follow from (3.2.9) on remembering the monotonicity among \( a_i(t); \ i = 1, 2, \ldots 4. \)

In fact,

\[
\frac{X(t)}{Y(t)} = \frac{(a_1(t))^{-1} X(t)}{(a_2(t))^{-1} Y(t)}
\]

\[
= \frac{(a_1(t))^{-1} [a_1(t)G_{11} + a_2(t)G_{12} + a_3(t)G_{13} + a_4(t)G_{14} + \psi(t)]}{(a_1(t))^{-1} [a_1(t)H_{11} + a_2(t)H_{12} + a_3(t)H_{13} + a_4(t)H_{14} + \psi(t)]}
\]

\[
= \frac{G_{11} + \frac{a_3(t)}{a_1(t)} G_{12} + \frac{a_4(t)}{a_1(t)} G_{13} + \frac{a_4(t)}{a_1(t)} G_{14} + \frac{1}{a_1(t)} \psi(t)}{H_{11} + \frac{a_3(t)}{a_1(t)} H_{12} + \frac{a_4(t)}{a_1(t)} H_{13} + \frac{a_4(t)}{a_1(t)} H_{14} + \frac{1}{a_1(t)} \psi(t)}.
\]

We now allow \( t \to \infty \). On noting that \( \psi(t) \) is bounded in probability we get

\[
P \lim_{t \to \infty} \frac{X(t)}{Y(t)} = P \lim_{t \to \infty} \frac{G_{11} + \frac{a_3(t)}{a_1(t)} G_{12} + \frac{a_4(t)}{a_1(t)} G_{13} + \frac{a_4(t)}{a_1(t)} G_{14}}{H_{11} + \frac{a_3(t)}{a_1(t)} H_{12} + \frac{a_4(t)}{a_1(t)} H_{13} + \frac{a_4(t)}{a_1(t)} H_{14}}
\]

\[
= \frac{G_{11}}{H_{11}}. \quad (3.3.11)
\]

Similarly the first statements in (b), (c), (d) and (e) can be established under all placements of explosive roots on utilising (3.3.6), (3.3.7a), (3.3.7b) and (3.3.7c).
The second statement in (a) is proved for Case 1 of (3.2.8). Considering the first equation of (3.2.1) we have

\[ X(t+2) - \beta_{11}^{(1)} X(t+1) - \beta_{12}^{(1)} Y(t+1) - \beta_{12}^{(2)} X(t) - \beta_{12}^{(2)} Y(t) - \beta_{10} = \epsilon_1(t+2) \]

On substituting the explicit solutions for \( X(t) \) and \( Y(t) \) provided by Theorem 3.2.1-C, in the above equation we get

\[
\rho_0^t t^3 \left\{ G_{11} \left( \rho_0^2 - \beta_{11}^{(1)} \rho_0 - \beta_{11}^{(2)} \right) - H_{11} \left( \beta_{12}^{(2)} \rho_0 + \beta_{12}^{(2)} \right) \right\} = \\
- \left\{ \rho_0^t t^2 \left[ \rho_0^2 \left( 6G_{11} + G_{12} \right) - \beta_{11}^{(1)} \rho_0 \left( 3G_{11} + G_{12} \right) \right] \\
- \beta_{12}^{(2)} G_{12} - \beta_{12}^{(2)} \rho_0 \left( 3H_{11} + H_{12} \right) - \beta_{12}^{(2)} H_{12} \right\} \\
+ \rho_0^t \left[ \rho_0^2 \left( 8G_{11} + 4G_{12} + G_{13} \right) - \beta_{11}^{(1)} \rho_0 \left( 3G_{11} + 2G_{12} + G_{13} \right) \right] \\
- \beta_{12}^{(2)} G_{13} - \beta_{12}^{(1)} \rho_0 \left( 3H_{11} + 2H_{12} + H_{13} \right) - \beta_{12}^{(2)} H_{12} \right\} \\
+ \rho_0^t \left[ \rho_0^2 \left( 8G_{11} + 4G_{12} + 2G_{13} + G_{14} \right) - \beta_{11}^{(1)} \rho_0 \left( G_{11} + G_{12} + G_{13} + G_{14} \right) \right] \\
- \beta_{12}^{(2)} G_{14} - \beta_{12}^{(1)} \rho_0 \left( G_{11} + G_{12} + G_{13} + G_{14} \right) - \beta_{12}^{(2)} H_{14} \right\} \psi(t) \right\} + \epsilon_1(t+2)
\]

On dividing by \( \rho_0^t t^2 \) the right hand side is bounded in probability, and therefore the expression within brackets on the left hand side should vanish (through Lemma 6.1 of Bhat (1999)), which implies

\[
P \left( \frac{G_{11}}{H_{11}} = \frac{\rho_0 \beta_{12}^{(1)} - \beta_{12}^{(2)}}{\rho_0^2 - \rho_0 \beta_{11}^{(1)} - \beta_{11}^{(2)}} \right) = 1 \quad (3.3.12)
\]

Similarly we can prove the above result for Cases 2 to 7 of (3.2.8) on replacing \( \rho_0 \) in (3.3.12) by the largest characteristic root of matrix \( A_1 \).

To prove the second statements in (b) and (c), consider the first equation of (3.3.10a) given by

\[
X(t+1) - a_{11}^{(1)} X(t) - a_{12}^{(1)} Y(t) = X_2(t+1).
\]

Now, on using (3.3.7e), (3.3.7c) and (3.3.7a) we can express \( X_2(t+1) \) in terms of \( X(t+1), X(t) \) and \( Y(t) \) leading to

\[
X(t+1) - a_{11}^{(1)} X(t) - a_{12}^{(1)} Y(t) = X(t+1) - \frac{I_{22}}{G_{22}} X(t) - \left\{ \frac{I_{11}}{H_{11}} - \frac{I_{22}}{G_{22}} \pi_1 \right\} Y(t). \quad (3.3.13)
\]

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Equating the like coefficients, we get
\[ \frac{I_{22}}{G_{22}} = a_{11}^{(1)} = c_{11} \text{ (say)} \quad \text{and} \quad \frac{I_{11}}{H_{11}} = a_{12}^{(1)} + a_{11}^{(1)} \pi_1. \] (3.3.14)

Similarly by using the second equation of (3.3.10a) and (3.3.7d), (3.3.7b) and (3.3.7a) we get
\[ Y(t+1) - a_{21}^{(1)} X(t) - a_{22}^{(1)} Y(t) = Y(t+1) - \frac{J_{22}}{G_{22}} X(t) - \left\{ \frac{J_{11}}{H_{11}} - \frac{J_{22}}{G_{22}} \pi_1 \right\} Y(t). \] (3.3.15)

Equating the like coefficients we get (d) and (e).

Finally, to prove the first statement in (f), we use the first equation of (3.3.10a) and substitute the general representation for \( X_2(t) \) and \( Y_2(t) \) from Theorem 3.2.2 leading to
\[
\frac{X_2(t + 1)}{Y_2(t + 1)} = \frac{(a_3(t + 1))^{-1} X_2(t + 1)}{(a_3(t + 1))^{-1} Y_2(t + 1)} = \frac{(a_3(t + 1))^{-1} [a_3(t + 1) G_{33} + a_4(t + 1) G_{34} + \psi(t)]}{(a_3(t + 1))^{-1} [a_3(t + 1) H_{33} + a_4(t + 1) H_{34} + \psi(t)]}
\]
\[
= \frac{G_{33} + \frac{a_4(t+1)}{a_3(t+1)} G_{34} + \frac{1}{a_3(t+1)} \psi(t)}{H_{33} + \frac{a_4(t+1)}{a_3(t+1)} H_{34} + \frac{1}{a_3(t+1)} \psi(t)}
\]

We now allow \( t \to \infty \). On noting that \( |a_3(t)| > |a_4(t)| \) and \( \psi(t) \) bounded in probability, we get
\[ \frac{X_2(t + 1)}{Y_2(t + 1)} \overset{P}{\to} \frac{G_{33}}{H_{33}} \] (3.3.16)

Further, to prove the second statement in (f), we consider the first equation of (3.3.10b), given by
\[ X_2(t + 2) - a_{11}^{(2)} X(t) - a_{12}^{(2)} Y(t) - \beta_{10} = \epsilon_1(t + 2). \]

Now on substituting the explicit solutions for \( X_2(t) \) and \( Y_2(t) \) given in Theorem 3.2.2-(i) under Case 1 of (3.2.8) we get
\[
X_2(t + 2) \rho_0^{t+2} G_{33} + \rho_0^{t+2} G_{34} + \psi(t) - a_{11}^{(2)} [(t + 1) \rho_0^{t+1} G_{33} + \rho_0^{t+1} G_{34} + \psi(t)] - a_{12}^{(2)} [(t + 1) \rho_0^{t+1} H_{33} + \rho_0^{t+1} H_{34} + \psi(t)] - \beta_{10} = \epsilon_1(t + 2).
\]

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Further
\[
\begin{align*}
t \rho_0^{t+1} \left\{ G_{33} \left( \rho_0 - a_{11}^{(3)} \right) - a_{12}^{(2)} H_{33} \right\} &= - \left\{ \rho_0^{t+1} [ \rho_0 (2G_{33} + G_{34}) - a_{11}^{(2)} (G_{33} + G_{34}) - a_{12}^{(2)} (H_{33} + H_{34}) ] \right\} + \psi(t) + \epsilon_1(t+2)
\end{align*}
\]
On dividing by $\rho_0^{t+1}$, the right hand side is bounded in probability and therefore the expression in the brackets on the left hand side should vanish, which implies that
\[
P \left( \frac{G_{33}}{H_{33}} = \frac{a_{12}^{(2)}}{\rho_0 - a_{11}^{(2)}} \right) = 1.
\tag{3.3.17}
\]
Similarly we can prove the second statement of (f) for Cases 2 to 7 of (3.2.8) which results in (3.2.17) with $\rho_0$ replaced by the largest characteristic root of $A_2$. Hence the theorem.

Note 3.3.1:

1. The above theorem can also be proved using the substitution based arguments as in Lemma 2 of Venkataraman (1974).

2. For (a) and (e) of Theorem 3.2.3 we have used only the first equation of (3.2.1) and that of (3.3.10). We can also utilise the information in the second equations to get
\[
\begin{align*}
\frac{\rho_1 \beta_{12}^{(1)} + \beta_{12}^{(2)}}{\rho_1 - \rho_1 \beta_{11}^{(1)} - \beta_{11}^{(2)}} &= \pi_1 = \frac{\rho_1 - \rho_1 \beta_{22}^{(1)} - \beta_{22}^{(2)}}{\rho_1 \beta_{21}^{(1)} - \beta_{21}^{(2)}} \\
\frac{a_{12}^{(1)}}{\rho_3 - a_{11}^{(1)}} &= \pi_1 = \frac{\rho_3 - a_{22}^{(1)}}{a_{21}^{(1)}}
\end{align*}
\]

3. It is pertinent to note that the above ratios of components of an explosive vector time series converge in probability to a constant as $t$ approaches $\infty$, in contrast to the autoregressive situation where this limit is a random variable. This is a characteristic property which distinguishes an explosive model from an autoregressive model. Though the first statements in (a) to (f) of the theorem are not directly used for proving the theorems in this sequel, we have established these to highlight this special feature of explosive time series.
3.3.2 Reparametrisation of the model in (3.2.1)

In the context of Theorem 3.3.1, the transformations (3.3.7) can be written as

\[ \phi(t) = X(t) - \frac{G_{11}}{H_{11}} Y(t) = X(t) - \pi_1 Y(t) \]

\[ Y_2(t + 1) = \tilde{Y}(t + 1) - \frac{J_{22}}{G_{22}} \phi(t) \]

\[ = \tilde{Y}(t + 1) - a_{21}^{(1)} \phi(t) \]

\[ = \left( Y(t + 1) - \frac{J_{11}}{H_{11}} Y(t) \right) - a_{21}^{(1)} \left( X(t) - \pi_1 Y(t) \right) \]

\[ = Y(t + 1) - a_{21}^{(1)} X(t) - \left( \frac{J_{11}}{H_{11}} - a_{21}^{(1)} \pi_1 \right) Y(t) \]

\[ = Y(t + 1) - a_{21}^{(1)} X(t) - \left( a_{22}^{(1)} + a_{21}^{(1)} \pi_1 - a_{21}^{(1)} \pi_1 \right) Y(t) \]

\[ = Y(t + 1) - a_{21}^{(1)} X(t) - a_{22}^{(1)} Y(t) \]

\[ \phi_2(t + 1) = X_2(t + 1) - \frac{J_{33}}{J_{33}} Y_2(t + 1) \]

\[ = X_2(t + 1) - \pi_1 Y_2(t + 1) \]

\[ = \left( X(t + 1) - a_{11}^{(1)} X(t) - a_{12}^{(1)} Y(t) \right) \]

\[ - \pi_1 \left( Y(t + 1) - a_{21}^{(1)} X(t) - a_{22}^{(1)} Y(t) \right) \]

\[ = X(t + 1) - \pi_1 Y(t + 1) - \left( a_{11}^{(1)} + \pi_1 a_{21}^{(1)} \right) X(t) - \left( a_{12}^{(1)} + \pi_1 a_{22}^{(1)} \right) Y(t) \]

\[ Y(t) = Y(t) \quad (3.3.18) \]

(3.3.18) can be compactly written as

\[
\begin{bmatrix}
\phi_2(t + 1) \\
y_2(t + 1) \\
\phi(t) \\
y(t)
\end{bmatrix} =
\begin{bmatrix}
1 & -\pi_1 & -\left( a_{11}^{(1)} + \pi_1 a_{21}^{(1)} \right) & -\left( a_{12}^{(1)} + \pi_1 a_{22}^{(1)} \right) \\
0 & 1 & -a_{21}^{(1)} & -a_{22}^{(1)} \\
0 & 0 & 1 & -\pi_1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x(t + 1) \\
y(t + 1) \\
x(t) \\
y(t)
\end{bmatrix} \quad (3.3.19)
\]

The above transformations can be seen as reparametrisation of the model (3.2.1) as follows.
Model (3.2.1) is given by

\[ X(t + 2) = (X(t + 1), Y(t + 1), X(t), Y(t), 1) \beta + \epsilon_1(t + 2) \]  

(3.3.20a)

\[ Y(t + 2) = (X(t + 1), Y(t + 1), X(t), Y(t), 1) \beta + \epsilon_2(t + 2), \]  

(3.3.20b)

where

\[ \beta = \left( \beta_{11}, \beta_{12}, \beta_{1s}, \beta_{s1}, \beta_{s}, \beta_{10} \right) \]

and

\[ \beta = \left( \beta_{s1}, \beta_{s2}, \beta_{s2}, \beta_{s2}, \beta_{s2} \right) \].

We can express \( X(t + 2) \) and \( Y(t + 2) \) in terms of the transformed variables \( \phi_2(t + 1), Y_2(t + 1), \phi(t), Y(t) \) using the transformation matrix

\[
T = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
-\pi & 1 & 0 & 0 & 0 \\
-a_{11}^{(1)} + a_{21}^{(1)} \pi & -a_{21}^{(1)} & 1 & 0 & 0 \\
-a_{12}^{(1)} + a_{22}^{(1)} \pi & -a_{22}^{(1)} & -\pi & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

(3.3.21)

So that (3.3.20a) and (3.3.20b) can be written as

\[
X(t + 2) = [X(t + 1) \quad Y(t + 1) \quad X(t) \quad Y(t)] TT^{-1} \beta_1 + \epsilon_1(t + 2)
\]

\[
Y(t + 2) = [X(t + 1) \quad Y(t + 1) \quad X(t) \quad Y(t)] TT^{-1} \beta_2 + \epsilon_2(t + 2).
\]

Denoting \( T^{-1} \beta_1 = \theta_1 \) and \( T^{-1} \beta_2 = \theta_2 \), where \( \theta_1 \) and \( \theta_2 \) are the new parameters,

\[
X(t + 2) = (\phi_2(t + 1), Y_2(t + 1), \phi(t), Y(t), 1) \theta_1 + \epsilon_1(t + 2),
\]

(3.3.22a)

\[
Y(t + 2) = (\phi_2(t + 1), Y_2(t + 1), \phi(t), Y(t), 1) \theta_2 + \epsilon_2(t + 2).
\]

(3.3.22b)

Using the partial realisation \( \{Z(t) = (X(t), Y(t))' \}; t = 1, 2, \ldots, N \} \) of \( \{Z(t) = (X(t), Y(t))'; t \geq 1 \} \), (3.3.20a), (3.3.20b), (3.3.22a) and (3.3.22b) can be
written as

\[ P_1 = L \beta_1 + \epsilon_1, \quad \text{(3.3.23a)} \]
\[ P_2 = L \beta_2 + \epsilon_2, \quad \text{(3.3.23b)} \]
\[ P_1 = L^* \theta_1 + \epsilon_1, \quad \text{(3.3.23c)} \]
\[ P_2 = L^* \theta_2 + \epsilon_2, \quad \text{(3.3.23d)} \]

where

\[ P_1 = (X(3) \cdots X(N))' \]
\[ P_2 = (Y(3) \cdots Y(N))' \]

\[
\begin{align*}
L &= \begin{bmatrix}
X(2) & Y(2) & X(1) & Y(1) & 1 \\
X(N-1) & Y(N-1) & X(N-2) & Y(N-2) & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & 1
\end{bmatrix}

L^* &= \begin{bmatrix}
\phi_2(2) & Y_2(2) & \phi(1) & Y(1) & 1 \\
\phi_2(N-1) & Y_2(N-1) & \phi(N-2) & Y(N-2) & 1
\end{bmatrix}
= LT
\]

\[ \epsilon_1 = (\epsilon_1(3), \ldots, \epsilon_1(N))' \]
\[ \epsilon_2 = (\epsilon_2(3), \ldots, \epsilon_2(N))' \]

\[ \theta_1 = (\theta_{11}, \theta_{12}, \theta_{13}, \theta_{14}, \theta_{15})' \]
\[ = (\beta_{11}^{(1)}, \beta_{12}^{(1)} - \pi_1 \beta_{11}^{(1)}, \beta_{12}^{(2)} - a_{21} \beta_{12}^{(1)} + (-a_{11}^{(1)} + a_{21}^{(1)} \pi_1) \beta_{11}^{(1)}, \beta_{12}^{(2)} - \pi_1 \beta_{11}^{(2)} - a_{22} \beta_{12}^{(1)} + (-a_{12}^{(1)} + a_{22}^{(1)} \pi_1) \beta_{11}^{(1)}, \beta_{10})' \]

\[ \theta_2 = (\theta_{21}, \theta_{22}, \theta_{23}, \theta_{24}, \theta_{25})' \]
\[ = (\beta_{21}^{(1)}, \beta_{22}^{(1)} - \pi_1 \beta_{21}^{(1)}, \beta_{21}^{(2)} - a_{21} \beta_{22}^{(1)} + (-a_{11}^{(1)} + a_{21}^{(1)} \pi_1) \beta_{21}^{(1)}, \beta_{22}^{(2)} - \pi_1 \beta_{21}^{(2)} - a_{22} \beta_{22}^{(1)} + (-a_{12}^{(1)} + a_{22}^{(1)} \pi_1) \beta_{21}^{(1)}, \beta_{20})' \]
3.3.3 Relationship between centered least squares estimators of $\theta$ and that of $\beta$

Minimising the sum of squares $\varepsilon_1^t\varepsilon_1$ in (3.3.23a) with respect to $\beta_1$ gives centered least squares estimators of $\beta_1$ as

$$\hat{\beta}_1 - \beta_1 = (L'L)^{-1} L'\varepsilon_1$$  \hspace{1cm} (3.3.24)

On minimising $\varepsilon_2^t\varepsilon_2$ in (3.3.23b) with respect to $\beta_2$ gives centered least squares estimators of $\beta_2$ as

$$\hat{\beta}_2 - \beta_2 = (L'L)^{-1} L'\varepsilon_2$$  \hspace{1cm} (3.3.25)

Now, on minimising $\varepsilon_1^t\varepsilon_1$ in (3.3.23c) with respect to $\theta_1$ and $\varepsilon_2^t\varepsilon_2$ in (3.3.23d) with respect to $\theta_2$ gives centered least squares estimators of $\theta_1$ and $\theta_2$ as

$$\hat{\theta}_1 - \theta_1 = (L^*L^*)^{-1} L^*\varepsilon_1$$  \hspace{1cm} (3.3.26a)

$$\hat{\theta}_2 - \theta_2 = (L^*L^*)^{-1} L^*\varepsilon_2.$$  \hspace{1cm} (3.3.26b)

Substituting for $L = L^*T^{-1}$ in (3.3.24) and (3.3.25), we get

$$\hat{\beta}_1 - \beta_1 = \left(T^{-1'}L^*L^*T^{-1}\right)^{-1} (T^{-1})' L^*\varepsilon_1$$

$$= T \left(L^*L^*\right)^{-1} T' (T^{-1})' L^*\varepsilon_1$$

$$\hat{\beta}_1 - \beta_1 = T \left(\hat{\theta}_1 - \theta_1\right)$$  \hspace{1cm} (3.3.27)

Similarly, we have

$$\hat{\beta}_2 - \beta_2 = T \left(\hat{\theta}_2 - \theta_2\right)$$  \hspace{1cm} (3.3.28)

Relationships (3.3.27) and (3.3.28) are used later, to study the rate of convergence in probability of $\hat{\theta}_i - \theta_i; \ i = 1, 2$, after studying the rate of convergence in probability of $\hat{\theta}_i - \theta_i; \ i = 1, 2$.

We first study the asymptotic properties of $\hat{\theta}_i - \theta_i; \ i = 1, 2$. 

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The centered least squares estimators \( \hat{\theta}_i - \theta_i; i = 1, 2 \) can be expressed as ratios of two determinants. This follows from (3.3.26a) and (3.3.26b). We have

\[
\hat{\theta}_{1j} - \theta_{1j} = \frac{D_1(j)}{D} \quad j = 1, 2, \ldots, 5, \quad \text{(3.3.29a)}
\]

\[
\hat{\theta}_{2j} - \theta_{2j} = \frac{D_2(j)}{D} \quad j = 1, 2, \ldots, 5, \quad \text{(3.3.29b)}
\]

where \( D \) is the determinant of the matrix of sums of squares and products of the transformed variables, given by

\[
M_1^* = \begin{bmatrix}
\sum \phi(t)Y_2(t+1) & \sum \phi(t)Y_3(t+1) & \sum \phi(t)Y(t) & \sum \phi(t)Y(t) & \sum \phi(t)
\sum Y(t)Y_2(t+1) & \sum Y(t)Y_3(t+1) & \sum Y(t)Y(t) & \sum Y(t)Y(t) & \sum Y(t)
\sum \phi_2(t+1) & \sum \phi_2(t+1) & \sum \phi_2(t+1) & \sum \phi_2(t+1) & \sum \phi_2(t+1)
\sum Y(t)\phi_2(t+1) & \sum Y(t)Y_3(t+1) & \sum Y(t)Y(t) & \sum Y(t)Y(t) & \sum Y(t)
\sum \phi_2(t+1) & \sum Y(t)Y_2(t+1) & \sum \phi_2(t+1) & \sum \phi_2(t+1) & \sum \phi_2(t+1)
\end{bmatrix}
\]

\[
M_1^*(1) = \begin{bmatrix}
\sum \phi_2(t+1)Y_1(t+2) & \sum \phi_2(t+1)Y_2(t+1) & \sum \phi_2(t+1)Y(t) & \sum \phi_2(t+1)Y(t) & \sum \phi_2(t+1)
\sum Y_2(t+1)Y_1(t+2) & \sum Y_2(t+1)Y_2(t+1) & \sum Y_2(t+1)Y(t) & \sum Y_2(t+1)Y(t) & \sum Y_2(t+1)
\sum \phi_2(t+1) & \sum \phi_2(t+1) & \sum \phi_2(t+1) & \sum \phi_2(t+1) & \sum \phi_2(t+1)
\sum Y(t)Y_1(t+2) & \sum Y(t)Y_2(t+1) & \sum Y(t)Y(t) & \sum Y(t)Y(t) & \sum Y(t)
\sum \phi_2(t+1) & \sum \phi_2(t+1) & \sum \phi_2(t+1) & \sum \phi_2(t+1) & \sum \phi_2(t+1)
\end{bmatrix}
\]

For \( j = 1, 2, \ldots, 5 \) \( D_1(j) \) is the determinant of the matrix \( M_1^*(j) \), where \( M_1^*(j) \) is obtained from \( M_1^* \) by replacing the \( j^{th} \) column of \( M_1^* \) by the vector

\[
(\sum \phi_2(t+1)Y_1(t+2), \sum Y_2(t+1)Y_1(t+2), \sum \phi_2(t+1)Y(t), \sum Y(t)Y(t), \sum Y(t))
\]

The matrices \( M_1^*(j); j = 1, 2, \ldots, 5 \) are

\[
M_1^*(1) = \begin{bmatrix}
\sum \phi_2(t+1)Y_1(t+2) & \sum \phi_2(t+1)Y_2(t+1) & \sum \phi_2(t+1)Y(t) & \sum \phi_2(t+1)Y(t) & \sum \phi_2(t+1)
\sum Y_2(t+1)Y_1(t+2) & \sum Y_2(t+1)Y_2(t+1) & \sum Y_2(t+1)Y(t) & \sum Y_2(t+1)Y(t) & \sum Y_2(t+1)
\sum \phi_2(t+1) & \sum \phi_2(t+1) & \sum \phi_2(t+1) & \sum \phi_2(t+1) & \sum \phi_2(t+1)
\sum Y(t)Y_1(t+2) & \sum Y(t)Y_2(t+1) & \sum Y(t)Y(t) & \sum Y(t)Y(t) & \sum Y(t)
\sum \phi_2(t+1) & \sum \phi_2(t+1) & \sum \phi_2(t+1) & \sum \phi_2(t+1) & \sum \phi_2(t+1)
\end{bmatrix}
\]
Similarly $D_2(j)$ is the determinant of the matrix $M_2^*(j)$ obtained from $M_1^*$ by replacing its $j^{th}$ column by the vector

$$(\sum \phi_2(t+2), \Sigma Y_2(t+2), \Sigma g(t+2), \Sigma Y(t+2), \Sigma Y(t+2), \Sigma g(t+2))^t$$

### 3.3.4 The main theorems on limit distribution of $\hat{\beta} - \beta$ through model reparametrisation

We are now set to prove the following series of theorems, which are case specific and relate to the asymptotic behaviour of $\hat{\theta}$ and $\hat{\beta}$. In the theorems that follow, we have assumed that all the roots of $P(z) = 0$ are real.

**Case 1:** $\rho_1 = \rho_2 = \rho_3 = \rho_4 = 0_0; \ |\rho_0| > 1$.

**Theorem 3.3.2.** Let \( \{Z(t) = (X(t), Y(t))^t\} \) be generated from the model (3.2.1) and \( (\hat{\theta}_1, \hat{\theta}_2) = (\hat{\theta}_{11}, \hat{\theta}_{12}, \hat{\theta}_{13}, \hat{\theta}_{14}, \hat{\theta}_{21}, \hat{\theta}_{22}, \hat{\theta}_{23}, \hat{\theta}_{24}, \hat{\theta}_{25}) \) be the least squares estimator of the parameter vector \( (\theta'_1, \theta'_2) \) of the reparametrised model in (3.3.22). Then, under the assumptions of (3.2.2), governed by the model, the following statements hold under the placement $\rho_1 = \rho_2 = \rho_3 = \rho_4 = 0_0; \ |\rho_0| > 1$ relating to the roots of $P(z) = 0$.

**a.**

\[
\frac{\rho_0^N}{N^3} (\hat{\theta}_{11} - \theta_{11}), \frac{\rho_0^N}{N^2} (\hat{\theta}_{12} - \theta_{12}), \frac{\rho_0^N}{N} (\hat{\theta}_{13} - \theta_{13}), \frac{\rho_0^N}{N} (\hat{\theta}_{14} - \theta_{14}), \frac{\rho_0^N}{N^3} (\hat{\theta}_{21} - \theta_{21}), \frac{\rho_0^N}{N^2} (\hat{\theta}_{22} - \theta_{22}), \frac{\rho_0^N}{N} (\hat{\theta}_{23} - \theta_{23}), \frac{\rho_0^N}{N} (\hat{\theta}_{24} - \theta_{24})
\]

converges in probability, as $N \to \infty$, to the random vector

\[
(\xi_{11}(1), \xi_{12}(1), \xi_{13}(1), \xi_{14}(1), \xi_{21}(1), \xi_{22}(1), \xi_{23}(1), \xi_{24}(1)) \quad (\text{say})
\]

**b.** Each $\xi_{ij}(1)$ is distributed like a ratio of two random variables of the type $\sum_{i=1}^{4} (c_i \pi_{ij}) / J$, wherein $\pi_{ij}$ and $J$ are independent and are distributed like specific linear combinations of $\epsilon_i(t); \ i = 1, 2; \ t = 1, 2, \ldots$.

**c.**

\[
(\sqrt{N} (\hat{\theta}_{15} - \theta_{15}), \sqrt{N} (\hat{\theta}_{25} - \theta_{25}))
\]

converges in distribution to a bivariate normal random vector with mean vector zero and a diagonal covariance matrix.
Proof. The explicit solutions of $X(t)$ and $Y(t)$ when Case 1 holds are given by Theorem 3.2.1 to be

$$X(t) = t^3 \rho_0^t H_{11} + t^2 \rho_0^t G_{12} + t \rho_0^t G_{13} + \rho_0^t G_{14} + \psi(t),$$

$$Y(t) = t^3 \rho_0^t H_{11} + t^2 \rho_0^t H_{12} + t \rho_0^t H_{13} + \rho_0^t H_{14} + \psi(t).$$

Using the above in (3.3.7), we get

$$\phi(t) = t^2 \rho_0^t G_{22} + t \rho_0^t G_{23} + \rho_0^t G_{24} + \psi(t),$$

$$X_2(t + 1) = t \rho_0^t I_{33} + \rho_0^t I_{34},$$

$$Y_2(t + 1) = t \rho_0^t J_{33} + \rho_0^t J_{34},$$

$$\phi_2(t + 1) = \rho_0^t I_{44} + \psi(t),$$

where

$$G_{22} = H_{11} G_{12} - G_{11} H_{12} \quad G_{24} = H_{11} G_{14} - G_{11} H_{14}$$

$$G_{23} = H_{11} G_{13} - G_{11} H_{13} \quad I_{44} = J_{33} I_{34} - I_{33} J_{34}$$

$$I_{33} = \rho_0 G_{33} \quad I_{34} = \rho_0 (G_{33} + G_{34})$$

$$J_{33} = \rho_0 H_{33} \quad J_{34} = \rho_0 (H_{33} + H_{34})$$

We use the following notations in evaluating the determinant of the matrix $M^*$

Define

$$s_p(2) = \sum_{u=1}^{N-2} u^p \rho_0^{-2u},$$

$$\overline{s}_p(2) = \sum_{u=1}^{\infty} u^p \rho_0^{-2u}.$$
Let \( q = \frac{1}{2} \) so that

\[
\begin{align*}
\tilde{s}_0(2) &= \frac{1}{q - 1} \\
\tilde{s}_1(2) &= \frac{q}{(1 - q)^2} \\
\tilde{s}_2(2) &= \frac{2! q^2}{(1 - q)^3} + \tilde{s}_1(2) \\
\tilde{s}_3(2) &= \frac{3! q^3}{(1 - q)^4} + 3\tilde{s}_2(2) - 2\tilde{s}_1(2) \\
\tilde{s}_4(2) &= \frac{4! q^4}{(1 - q)^5} - 6\tilde{s}_3(2) + 11\tilde{s}_2(2) - 6\tilde{s}_1(2) \\
\tilde{s}_5(2) &= \frac{5! q^5}{(1 - q)^6} + 10\tilde{s}_4(2) - 35\tilde{s}_3(2) + 50\tilde{s}_2(2) - 24\tilde{s}_1(2) \\
\tilde{s}_6(2) &= \frac{6! q^6}{(1 - q)^7} + 15\tilde{s}_5(2) - 85\tilde{s}_4(2) + 225\tilde{s}_3(2) - 274\tilde{s}_2(2) + 120\tilde{s}_1(2)
\end{align*}
\]

Let

\[
\begin{align*}
p_i(1) &= \sum_{u=1}^{N-2} u^r \rho_0^{-u} \epsilon_i(N - u + 1) & i = 1, 2 \\
n_i &= \sum_{u=1}^{N-2} (N - u - 1)^i \rho_0^{-2u} & i = 0, 1, \ldots, 6 \\
u_i &= \sum_{u=1}^{N-2} (N - u - 1)^i \rho_0^{-u} \epsilon_1(N - u + 1) & i = 0, 1, \ldots, 3 \\
v_0(2) &= \sum_{u=1}^{N-2} \rho_1^{-2u} \\
w_0(2) &= \sum_{u=1}^{N-2} \rho_2^{-2u} \\
x_p(1) &= \sum_{u=1}^{N-2} u^p \rho_0^{-u} \rho_1^{-u} & p = 0, 1, \ldots, 3 \\
y_p(1) &= \sum_{u=1}^{N-2} u^p \rho_0^{-u} \rho_2^{-u} & p = 0, 1, \ldots, 3 \\
z_p(1) &= \sum_{u=1}^{N-2} u^p \rho_1^{-u} \rho_2^{-u} & p = 0, 1, \ldots, 3
\end{align*}
\]
\[ l_{40}(1) = \sum_{u=1}^{N-2} \rho_1^{-u} \varepsilon_i (N - u + 1) \quad i = 1, 2 \]

\[ m_{40}(1) = \sum_{u=1}^{N-2} \rho_2^{-u} \varepsilon_i (N - u + 1) \quad i = 1, 2 \]  \hspace{1cm} (3.3.36)

A typical \( n_i \), say \( n_3 \), is given by

\[ n_3 = \sum_{u=1}^{N-2} (N - u - 1)^3 \rho_0^{-2u} = N^3 s_0(2) - 3N^2 s_1(2) + 3Ns_2(2) - s_3(2) - 3N^2 s_0(2) \]

\[ + 6Ns_1(2) - 3s_2(2) + 3Ns_0(2) - s_1(2) - s_0(2) \]

Now, using the explicit solutions for \( X(t) \), \( Y(t) \) and the auxiliary processes given in (3.3.7), and notations in (3.3.36), we expand the elements of the matrix \( M_i \), which gives the following expressions.

\[ \sum \phi_2^2(t + 1) = \sum (\rho_0^2 I_{44} + \psi(t))^2 \]

\[ = I_{44}/9 \rho_0^{-2} \sum_{u=1}^{N-2} \rho_0^{-2u} + o_P \left( \rho_0^{2N} \right) \quad \text{(by putting } t = N - u - 1) \]

Similarly, we put \( t = N - u - 1 \) in the following expressions

\[ \sum \phi_2(t + 1)Y_2(t + 1) = I_{44}J_{33} \rho_0^{-2} \sum_{u=1}^{N-2} (N - n - 1) \rho_0^{-2u} + I_{44}J_{34} \rho_0^{-2} \sum_{u=1}^{N-2} \rho_0^{-2u} \]

\[ + o_P \left( \rho_0^{2N} \right) \]

\[ \sum \phi_2(t + 1)\phi(t) = \rho_0^{-2} I_{44} \left( G_{22}n_2 + G_{23}n_1 + G_{24}n_0 \right) + o_P \left( \rho_0^{2N^2} \right) \]

\[ \sum \phi_2(t + 1)Y(t) = \rho_0^{-2} I_{44} \left( H_{11}n_3 + H_{12}n_2 + H_{13}n_1 + H_{14}n_0 \right) + o_P \left( \rho_0^{2N^3} \right) \]

\[ \sum Y_2^2(t + 1) = \rho_0^{-2} \left( J_{33}^2 n_2 + 2J_{33}J_{34}n_1 + J_{34}^2 n_0 \right) + o_P \left( \rho_0^{2N^2} \right) \]

\[ \sum \phi(t)Y_2(t + 1) = \rho_0^{-2} \left( J_{33} \left( G_{22}n_3 + G_{23}n_2 + G_{24}n_1 \right) + J_{34} \left( G_{22}n_2 + G_{23}n_1 + G_{24}n_0 \right) \right) + o_P \left( \rho_0^{2N^3} \right) \]

\[ \sum Y(t)Y_2(t + 1) = \rho_0^{-2} \left( J_{33} \left( H_{11}n_4 + H_{12}n_3 + H_{13}n_2 + H_{14}n_1 \right) + J_{34} \left( H_{11}n_3 + H_{12}n_2 + H_{13}n_1 + H_{14}n_0 \right) \right) + o_P \left( \rho_0^{2N^4} \right) \]

\[ \sum \phi^2(t) = \rho_0^{-2} \left( G_{22}^2 n_4 + G_{23}^2 n_2 + G_{24}^2 n_0 + 2G_{22}G_{23}n_3 \right) \]
\[ +2G_{22}G_{24}n_2 + 2G_{23}G_{24}n_1 + o_P (\rho_0^N N^4) \]

\[ \sum \phi(t) Y(t) = \rho_0^N \rho_0^{-2} \left\{ G_{22} [H_{11} n_5 + H_{12} n_4 + H_{13} n_3 + H_{14} n_2] + G_{23} [H_{11} n_4 + H_{12} n_3 + H_{13} n_2 + H_{14} n_1] + G_{24} [H_{11} n_3 + H_{12} n_2 + H_{13} n_1] \right\} + o_P (\rho_0^N N^5) \]

\[ \sum Y^2(t) = \rho_0^N \rho_0^{-2} \left\{ H_{11}^2 n_6 + H_{12}^2 n_4 + H_{13}^2 n_2 + H_{14}^2 n_0 + 2H_{11} [H_{12} n_5 + H_{13} n_4 + H_{14} n_3] + 2H_{12} [H_{13} n_3 + H_{14} n_2] + 2H_{13} H_{14} n_1 \right\} + o_P (\rho_0^N N^6) \]

\[ \sum \phi_2(t + 1) = \rho_0^{-1} \rho_0^N I_{44}^0(1) + o_P (\rho_0^N) \]

\[ \sum Y_2(t + 1) = \rho_0^{-1} \rho_0^N \left\{ J_{33} (N s_0(1) - s_1(1) - s_0(1)) + J_{34} s_0(1) \right\} + o_P (\rho_0^N N) \]

\[ \sum \phi(t) = \rho_0^N \rho_0^{-1} \left\{ G_{22} (N^2 s_0(1) + s_2(1) + s_0(1) - 2N s_1(1) - 2Ns_0(1) + 2s_1(1)) + G_{23} (N s_0(1) - s_1(1) - s_0(1)) + G_{24} s_0(1) \right\} + o_P (\rho_0^N N^2) \]

\[ \sum Y(t) = \rho_0^N \rho_0^{-1} \left\{ H_{11} \left[ N^3 s_0(1) - 3N^2 s_1(1) + 3Ns_2(1) - s_3(1) - 3N^2 s_0(1) \right] + 6N s_1(1) - 3s_2(1) + 3Ns_0(1) - 3s_1(1) - s_0(1)] + H_{12} \left[ N^2 s_0(1) + s_2(1) + s_0(1) - 2N s_1(1) - 2Ns_0(1) + 2s_1(1)] \right\} H_{13} \left[ N s_0(1) - s_1(1) - s_0(1) \right] + H_{14} s_0(1) \right\} + o_P (\rho_0^N N^3) \] \hspace{1cm} (3.3.37)

To evaluate determinant of \( M_*(l); \quad l = 1, 2, \ldots, 5 \), we need the following expressions in addition to (3.3.37).

\[ \sum \phi_2(t + 1) \epsilon_1(t + 2) = \sum (\rho_0^N I_{44} + \psi(t)) \epsilon_1(t + 2) \]

\[ = I_{44} \rho_0^{-1} \sum_{u=1}^{N-2} \rho_0^{-1} \epsilon_1(N - u + 1) + \sum_{u=1}^{N-2} \psi(t) \epsilon_1(N - u + 1) \]

\[ = I_{44} \rho_0^N \rho_0^{-1} \rho_0(1) + o_P (\rho_0^N N) \]

\[ \sum Y_2(t + 1) \epsilon_1(t + 2) = \rho_0^N \rho_0^{-1} (J_{33} u_1 + J_{34} u_0) + o_P (\rho_0^N N) \]

\[ \sum \phi(t) \epsilon_1(t + 2) = \rho_0^N \rho_0^{-1} (G_{22} u_2 + G_{23} u_1 + G_{24} u_0) + o_P (\rho_0^N N^2) \]

\[ \sum Y(t) \epsilon_1(t + 2) = \rho_0^N \rho_0^{-1} (H_{11} u_3 + H_{12} u_2 + H_{13} u_1 + H_{14} u_0) + o_P (\rho_0^N N^3). \]

\hspace{1cm} (3.3.38)
After substituting for the elements of the matrices $M^*$ and $M^*_j(l); l = 1, 2, \ldots 5; j = 1, 2$ the expressions given in (3.3.37) and (3.3.38), the determinants $D, D^*_j(l); l = 1, 2, \ldots 5; j = 1, 2$ are evaluated using MAPLE software.

The determinants so obtained are

1. $\rho_0^{-8N} D = \frac{144}{(\rho_0^2 - 1)} \rho_0^{10} (\bar{\pi}_2 - \bar{\pi}_1)^2 (\pi_2 - \pi_1)^2 H_{11}^4 H_{33}^2 H_{12}^2 H_{34} + o_P(1)$

Note that $\rho_0^{-8N} D \overset{P}{\rightarrow} \xi_0$ (say), a random variable, such that $P(\xi_0 = 0) = 0$ (a.s.). This implies that the matrix $M^*$ converges in probability, as $N \to \infty$ to a non-singular matrix.

2. $\rho_0^{-7N} N^{-3} D_1(1) = (\pi_2 - \pi_1)(\pi_2 - \pi_1)^2 H_{11}^4 H_{33}^2 H_{12}^2 H_{34} \sum_{i=0}^{3} a_i p_{1i}(1) + o_P(1)$

3. $\rho_0^{-7N} N^{-2} D_2(1) = (\bar{\pi}_2 - \bar{\pi}_1)^2 (\pi_2 - \pi_1)^2 H_{11}^4 H_{33}^2 H_{12}^2 H_{34} \sum_{i=0}^{3} -3\rho_0 a_i p_{1i}(1) + o_P(1)$

4. $\rho_0^{-7N} N^{-1} D_3(1) = (\bar{\pi}_2 - \bar{\pi}_1)^2 (\pi_2 - \pi_1)^2 H_{11}^3 H_{33}^3 H_{12} H_{34} \sum_{i=0}^{3} 3\rho_0 a_i p_{1i}(1) + o_P(1)$

5. $\rho_0^{-7N} D_4(1) = (\bar{\pi}_2 - \bar{\pi}_1)^2 (\pi_2 - \pi_1)^2 H_{11}^3 H_{33}^3 H_{12}^2 H_{34} \sum_{i=0}^{3} (\rho_0^2) a_i p_{1i}(1) + o_P(1)$

6. $\rho_0^{-7N} N^{-3} D_1(2) = (\pi_2 - \pi_1)(\pi_2 - \pi_1)^2 H_{11}^4 H_{33}^2 H_{12}^2 H_{34} \sum_{i=0}^{3} a_i p_{2i}(1) + o_P(1)$

7. $\rho_0^{-7N} N^{-2} D_2(2) = (\bar{\pi}_2 - \bar{\pi}_1)^2 (\pi_2 - \pi_1)^2 H_{11}^4 H_{33}^2 H_{12}^2 H_{34} \sum_{i=0}^{3} -3\rho_0 a_i p_{2i}(1) + o_P(1)$

8. $\rho_0^{-7N} N^{-1} D_3(2) = (\bar{\pi}_2 - \bar{\pi}_1)^2 (\pi_2 - \pi_1)^2 H_{11}^3 H_{33}^3 H_{12} H_{34} \sum_{i=0}^{3} 3\rho_0 a_i p_{2i}(1) + o_P(1)$

9. $\rho_0^{-7N} D_4(2) = (\bar{\pi}_2 - \bar{\pi}_1)^2 (\pi_2 - \pi_1)^2 H_{11}^3 H_{33}^3 H_{12}^2 H_{34} \sum_{i=0}^{3} (\rho_0^2) a_i p_{2i}(1)+o_P(1)$  (3.3.39)

10. $D_5(1) = N^{-1} D \sum \epsilon_1(t) + o_P(1)$

11. $D_5(2) = N^{-1} D \sum \epsilon_2(t) + o_P(1)$  (3.3.40)
where

\[
\begin{align*}
  a_0 &= -\frac{24\rho_0^6}{(\rho_0^2 - 1)^2}, \\
  a_1 &= \frac{4\rho_0^3 (11\rho_0^6 + 5\rho_0^2 + 2)}{(\rho_0^2 - 1)^{11}}, \\
  a_2 &= -\frac{12\rho_0^5 (2\rho_0^2 + 1)}{(\rho_0^2 - 1)^{10}}.
\end{align*}
\]

and from Theorem 3.3.1, we have

\[
\begin{align*}
  P\left( \frac{G_{33}}{H_{33}} = \bar{\pi}_1 \right) &= 1 \quad \text{and} \\
  P\left( \frac{G_{11}}{H_{11}} = \pi_1 \right) &= 1.
\end{align*}
\]

From a recursive use of the statements in Theorem 3.3.1 or otherwise one can show that

\[
\begin{align*}
  P\left( \frac{G_{34}}{H_{34}} = \bar{\pi}_2 \ (\text{say}) \right) &= 1 \quad \text{and} \\
  P\left( \frac{G_{12}}{H_{12}} = \pi_2 \ (\text{say}) \right) &= 1.
\end{align*}
\]

Now, in the definition of \( \hat{\theta}_{il} - \theta_{il} \); \( i = 1, 2; \ l = 1, 2, \ldots, 5 \) given in (3.3.29a) and (3.3.29b), the values of the determinants \( D, D_i(l) \); \( l = 1, 2, \ldots, 5; \ i = 1, 2 \) so obtained in (3.3.39) and (3.3.40) are substituted. This leads to

1. \( \rho_0^N N^{-3} \left( \hat{\theta}_{11} - \theta_{11} \right) = \rho_0^N N^{-3} \left( \frac{D_1(1)}{D} \right) = \frac{1}{(\bar{\pi}_2 - \bar{\pi}_1)} \sum_{i=0}^{3} c_i p_{i1}(1) \frac{1}{H_{33} H_{34}} + o_P(1) \)

2. \( \rho_0^N N^{-2} \left( \hat{\theta}_{12} - \theta_{12} \right) = \rho_0^N N^{-2} \left( \frac{D_2(1)}{D} \right) = \frac{1}{H_{33}} \sum_{i=0}^{3} c_i p_{i1}(1) + o_P(1) \)

3. \( \rho_0^N N^{-1} \left( \hat{\theta}_{13} - \theta_{13} \right) = \rho_0^N N^{-1} \left( \frac{D_3(1)}{D} \right) = \frac{1}{(\bar{\pi}_2 - \bar{\pi}_1)} \sum_{i=0}^{3} 3\rho_0^3 c_i p_{i1}(1) \frac{1}{H_{11} H_{12}} + o_P(1) \)
4. $\rho_0^N (\hat{\theta}_{14} - \theta_{14}) = \rho_0^N \left( \frac{D_4(1)}{D} \right) = \frac{\sum_{i=0}^{3} -\rho_0^2 c_i p_{1i}(1)}{H_{11}} + o_P(1)$

5. $\rho_0^N N^{-3} (\hat{\theta}_{21} - \theta_{21}) = \rho_0^N N^{-3} \left( \frac{D_1(2)}{D} \right) = \frac{1}{(\pi_2 - \pi_1)} \frac{\sum_{i=0}^{3} c_i p_{2i}(1)}{H_{33} H_{34}} + o_P(1)$

6. $\rho_0^N N^{-2} (\hat{\theta}_{22} - \theta_{22}) = \rho_0^N N^{-2} \left( \frac{D_2(2)}{D} \right) = \frac{\sum_{i=0}^{3} c_i p_{2i}(1)}{H_{33}} + o_P(1)$

7. $\rho_0^N N^{-1} (\hat{\theta}_{23} - \theta_{23}) = \rho_0^N N^{-1} \left( \frac{D_3(2)}{D} \right) = \frac{1}{(\pi_2 - \pi_1)} \frac{\sum_{i=0}^{3} 3\rho_0^2 c_i p_{2i}(1)}{H_{11} H_{12}} + o_P(1)$

8. $\rho_0^N (\hat{\theta}_{24} - \theta_{24}) = \rho_0^N \left( \frac{D_4(2)}{D} \right) = \frac{\sum_{i=0}^{3} -\rho_0^2 c_i p_{2i}(1)}{H_{11}} + o_P(1)$ (3.3.41)

9. $\sqrt{N}(\hat{\theta}_{15} - \theta_{15}) = N^{-1/2} \sum_{t=1}^{N-2} \epsilon_1(t) + o_P(1)$

10. $\sqrt{N}(\hat{\theta}_{25} - \theta_{25}) = N^{-1/2} \sum_{t=1}^{N-2} \epsilon_2(t) + o_P(1)$ (3.3.42)

where the constants are

$$c_0 = -\frac{1}{6} \rho_0^{-1} \left( \rho_0^2 - 1 \right)^4$$
$$c_1 = \frac{1}{36} \rho_0^{-7} \left( 11\rho_0^4 + 5\rho_0^3 + 2 \right) \left( \rho_0^2 - 1 \right)^5$$
$$c_2 = -\frac{1}{12} \rho_0^{-7} \left( 2\rho_0^2 + 1 \right) \left( \rho_0^2 - 1 \right)^6$$
$$c_3 = \frac{1}{36} \rho_0^{-7} \left( \rho_0^2 - 1 \right)^7$$

In 1 to 8 of (3.3.41), we make the following observations

1. $\left( \rho_0^N N^{-3} (\hat{\theta}_{11} - \theta_{11}) \rho_0^N N^{-2} (\hat{\theta}_{12} - \theta_{12}) \rho_0^N N^{-1} (\hat{\theta}_{13} - \theta_{13}) \rho_0^N (\hat{\theta}_{14} - \theta_{14}) \rho_0^N N^{-3} (\hat{\theta}_{21} - \theta_{21}) \rho_0^N N^{-2} (\hat{\theta}_{22} - \theta_{22}) \rho_0^N N^{-1} (\hat{\theta}_{23} - \theta_{23}) \rho_0^N (\hat{\theta}_{24} - \theta_{24}) \right) \Rightarrow (\xi_{11}, \xi_{12}, \xi_{13}, \xi_{14}, \xi_{21}, \xi_{22}, \xi_{23}, \xi_{24})$ (say), as $N \to \infty$.

This is statement (a) of the theorem.
2. The random variables in the denominators of (3.3.41) are of the type
\[ k_1 \sum_{u=1}^{\infty} \rho_0^{-u} \epsilon_1(u) + k_2 \sum_{u=1}^{\infty} \rho_0^{-u} \epsilon_2(u). \]

Recall that
\[ H_{33} = \rho_0 \sum_{u=1}^{\infty} \rho_0^{-u} \eta_2(u) \]
\[ = \rho_0 \sum_{u=1}^{\infty} \rho_0^{-u} \left[ \epsilon_2(u) - a_{11}^{(2)} \epsilon_2(u-1) + a_{21}^{(2)} \epsilon_1(u-1) \right] \]
\[ = \rho_0 \left( 1 - a_{11}^{(2)} \right) \sum_{u=1}^{\infty} \rho_0^{-u} \epsilon_2(u) + \rho_0 a_{21}^{(2)} \sum_{u=1}^{\infty} \rho_0^{-u} \epsilon_1(u), \quad (3.3.43) \]
on remembering that \( \epsilon_j(t) = 0 \) for \( t < 0; \ j = 1,2. \)

Similar expressions can be obtained for \( H_{34}, H_{11} \) and \( H_{12}. \)

Further each term on the right hand side of (3.3.43) can be written in the form
\[ \sum_{u=1}^{\infty} \rho_0^{-u} \epsilon_j(u) = \sum_{u=1}^{n} \rho_0^{-u} \epsilon_j(u) + \sum_{n+1}^{\infty} \rho_0^{-u} \epsilon_j(u), \quad (3.3.44) \]
where the first term is a Borel function of \( \epsilon_j(1), \epsilon_j(2), \ldots, \epsilon_j(n) \) and the second term converges in probability as \( N \rightarrow \infty, \) to zero.

Now, notice that for \( n \geq 1 \) and \( N \) sufficiently large, it is possible to write the random variables \( p_{ij}(1); \ i = 1,2; \ j = 0,1,\ldots,3 \) in the numerators of (3.3.41) as
\[ \sum_{u=1}^{N-2} u^i \rho_0^{-u} \epsilon_i(N-u+1) = \sum_{u=1}^{N-2-n} u^i \rho_0^{-u} \epsilon_i(N-u+1) + \sum_{(N-2)-(n+1)}^{N-2} u^i \rho_0^{-u} \epsilon_i(N-u+1). \quad (3.3.45) \]
The first term is a linear combination of \( \epsilon_j(n+1), \epsilon_j(n+2) \) etcetera, and the second term \( T(N,n) \) (say), satisfies the requirement that
\[ \lim_{n \rightarrow \infty} \lim_{N \rightarrow \infty} \sup P \{ |T(N,n)| \geq \delta \} = 0 \quad \text{for any } \delta > 0. \quad (3.3.46) \]
The observations made in (3.3.43) to (3.3.46) together with an appeal to Lemmas (1.7.1) and (1.7.2) would yield statement (b) on remembering that \( \{ \epsilon_j(t) \} \) are i.i.d.
sequences. To prove (c) of the theorem, we apply the limiting behaviour of the first two components of the basic limit Theorem 1.7.1 to (9) and (10) in (3.3.42). Hence the theorem.

We now state and prove the main theorem under Case 1 of (3.2.8) relating to the least squares estimators of the coefficients of the model (3.2.1).

**Theorem 3.3.3.** Let \( \hat{\beta}' = (\beta_\epsilon', \beta_{10}, \beta_{20}) = (\hat{\beta}_1^{(1)}, \hat{\beta}_{12}^{(1)}, \hat{\beta}_{11}^{(1)}, \hat{\beta}_{21}^{(1)}, \hat{\beta}_{22}^{(1)}, \hat{\beta}_{21}^{(2)}, \hat{\beta}_{22}^{(2)}, \beta_{10}, \beta_{20}) \) be the least squares estimators of \( \beta' = (\beta'_\epsilon, \beta_{10}, \beta_{20}) = (\beta_1^{(1)}, \beta_{12}^{(1)}, \beta_{11}^{(1)}, \beta_{21}^{(2)}, \beta_{22}^{(1)}, \beta_{21}^{(2)}, \beta_{22}^{(2)}, \beta_{10}, \beta_{20}) \), the vector of structural parameters of the model in (3.2.1) satisfying the assumptions of (3.2.2). Then the following statements hold under the placement \( \rho_1 = \rho_2 = \rho_3 = \rho_4 = \rho_0; |\rho_0| > 1 \) relating to the real roots of \( P(z) = 0 \).

a. \( \left( (\rho_0^N/N^3) \left( \hat{\beta}_\epsilon - \beta_\epsilon \right) \right) \) converges in probability as \( N \to \infty \), to a random vector \((\zeta_{11}(1), \zeta_{12}(1), \zeta_{13}(1), \zeta_{14}(1), \zeta_{21}(1), \zeta_{22}(1), \zeta_{23}(1), \zeta_{24}(1))\) (say).

b. Each \( \zeta_{ij}(1) \) is distributed like a ratio of two random variables of the type \( \sum_{i=1}^{4} (c_0 \pi_{ij})/J \), wherein \( \pi_{ij} \) and \( J \) are independent and are distributed like specific linear combinations of \( \epsilon_i(t); i = 1, 2; t = 1, 2, \ldots \).

c. \( \left( N^{1/2} \left( \hat{\beta}_{10} - \beta_{10} \right), N^{1/2} \left( \hat{\beta}_{20} - \beta_{20} \right) \right) \) converges in distribution, as \( N \to \infty \), to a bivariate random vector with mean vector a null vector and a diagonal covariance matrix.

d. (i) \( \pi_2 \zeta_{11}(1) + \zeta_{12}(1) = 0 \) a.s.

(ii) \( a_{11}^{(1)} \zeta_{11}(1) + a_{21}^{(1)} \zeta_{12}(1) + \zeta_{13}(1) = 0 \) a.s.

(iii) \( \left( a_{12}^{(1)} + \pi_1 a_{11}^{(1)} \right) \zeta_{11}(1) + \left( a_{21}^{(1)} \pi_1 + a_{22}^{(1)} \right) \zeta_{12}(1) + \pi_1 \zeta_{13}(1) + \zeta_{14}(1) = 0 \) a.s.

(iv) \( \pi_1 \zeta_{21}(1) + \zeta_{22}(1) = 0 \) a.s.

(v) \( a_{11}^{(1)} \zeta_{21}(1) + a_{21}^{(1)} \zeta_{22}(1) + \zeta_{23}(1) = 0 \) a.s.

(vi) \( \pi_1 a_{11}^{(1)} \zeta_{21}(1) + \left( a_{21}^{(1)} \pi_1 + a_{22}^{(1)} \right) \zeta_{22}(1) + \pi_1 \zeta_{23}(1) + \zeta_{24}(1) = 0 \) a.s.
Proof. Recalling the relationship between \( \hat{\theta} \) and \( \hat{\beta} \) given in (3.3.27) and (3.3.28), we have

\[
\begin{align*}
\beta_{11}^{(1)} - \beta_{11}^{(1)} & = \hat{\beta}_{11} - \theta_{11} & \text{(3.3.47a)} \\
\beta_{12}^{(1)} - \beta_{12}^{(1)} & = \hat{\beta}_{12} - \theta_{12} - \hat{\pi}_1 (\hat{\beta}_{11} - \theta_{11}) & \text{(3.3.47b)} \\
\beta_{11}^{(2)} - \beta_{11}^{(2)} & = \hat{\beta}_{13} - \theta_{13} - a_{21}^{(1)} (\hat{\beta}_{12} - \theta_{12}) + (-a_{11}^{(1)} + a_{21}^{(1)} \hat{\pi}_1) (\hat{\beta}_{11} - \theta_{11}) & \text{(3.3.47c)} \\
\beta_{12}^{(2)} - \beta_{12}^{(2)} & = \hat{\beta}_{14} - \theta_{14} - \pi_1 (\hat{\beta}_{13} - \theta_{13}) - a_{22}^{(1)} (\hat{\beta}_{12} - \theta_{12}) \\
& \quad + (-a_{12}^{(1)} + a_{22}^{(1)} \hat{\pi}_2) (\hat{\beta}_{11} - \theta_{11}) & \text{(3.3.47d)} \\
\beta_{10} - \beta_{10} & = \hat{\beta}_{15} - \theta_{15} & \text{(3.3.47e)} \\
\beta_{21}^{(1)} - \beta_{21}^{(1)} & = \hat{\beta}_{21} - \theta_{21} & \text{(3.3.47f)} \\
\beta_{22}^{(1)} - \beta_{22}^{(1)} & = (\hat{\beta}_{22} - \theta_{22}) - \hat{\pi}_1 (\hat{\beta}_{21} - \theta_{21}) & \text{(3.3.47g)} \\
\beta_{21}^{(2)} - \beta_{21}^{(2)} & = (\hat{\beta}_{23} - \theta_{23}) - a_{21}^{(1)} (\hat{\beta}_{22} - \theta_{22}) + (-a_{11}^{(1)} + a_{21}^{(1)} \hat{\pi}_1) (\hat{\beta}_{21} - \theta_{21}) & \text{(3.3.47h)} \\
\beta_{22}^{(2)} - \beta_{22}^{(2)} & = \hat{\beta}_{24} - \theta_{24} - \pi_1 (\hat{\beta}_{23} - \theta_{23}) - a_{22}^{(1)} (\hat{\beta}_{22} - \theta_{22}) \\
& \quad + (-a_{12}^{(1)} + a_{22}^{(1)} \hat{\pi}_1) (\hat{\beta}_{21} - \theta_{21}) & \text{(3.3.47i)} \\
\beta_{20} - \beta_{20} & = \hat{\beta}_{25} - \theta_{25} & \text{(3.3.47j)}
\end{align*}
\]

Equations (3.3.47a) to (3.3.47d) can be written as follows

\[
\begin{align*}
\frac{\rho_0^N}{N^3} (\beta_{11}^{(1)} - \beta_{11}^{(1)}) & = \frac{\rho_0^N}{N^3} (\hat{\beta}_{11} - \theta_{11}) & \text{(3.3.48a)} \\
\frac{\rho_0^N}{N^3} (\beta_{12}^{(1)} - \beta_{12}^{(1)}) & = \frac{\rho_0^N}{N^3} \frac{N^2}{\rho_0^N} \left[ \frac{\rho_0^N}{N^2} (\hat{\beta}_{12} - \theta_{12}) \right] - \hat{\pi}_1 \left[ \frac{\rho_0^N}{N^3} (\hat{\beta}_{11} - \theta_{11}) \right] & \text{(3.3.48b)} \\
\frac{\rho_0^N}{N^3} (\beta_{11}^{(2)} - \beta_{11}^{(2)}) & = \frac{\rho_0^N}{N^3} \frac{N^2}{\rho_0^N} \frac{\rho_0^N}{N} \left( \frac{\rho_0^N}{N^3} (\hat{\beta}_{13} - \theta_{13}) \right) - a_{21}^{(1)} \frac{\rho_0^N}{N^3} \frac{N^2}{\rho_0^N} \left[ \frac{\rho_0^N}{N^2} (\hat{\beta}_{12} - \theta_{12}) \right] \\
& \quad - a_{11}^{(1)} + a_{21}^{(1)} \hat{\pi}_1 \right) \left( \frac{\rho_0^N}{N^3} (\hat{\beta}_{11} - \theta_{11}) \right) & \text{(3.3.48c)}
\end{align*}
\]
\[
\frac{\rho_N}{N^3} \left( \beta_{21} - \beta_{12} \right) = \frac{\rho_N}{N^3} \left( \hat{\beta}_{21} - \hat{\beta}_{12} \right) = \frac{\rho_N}{N^3} \left[ \rho_N \left( \hat{\theta}_{41} - \theta_{14} \right) \right] - \pi_1 \frac{\rho_N}{N^3} \left[ \rho_N \left( \hat{\theta}_{13} - \theta_{13} \right) \right] \\
- a_{12} \frac{\rho_N}{N^3} \frac{N^2}{\rho_0} \left[ \rho_N \left( \hat{\theta}_{12} - \theta_{12} \right) \right] \\
+ \left( -a_{12} + a_{22} \bar{\pi}_1 \right) \frac{\rho_N}{N^3} \left( \hat{\theta}_{11} - \theta_{11} \right) . \tag{3.3.48d}
\]

Similarly we can write the equations (3.3.47f) to (3.3.47i) as follows

\[
\frac{\rho_N}{N^3} \left( \beta_{21} - \beta_{21} \right) = \frac{\rho_N}{N^3} \left( \hat{\theta}_{21} - \theta_{21} \right) \tag{3.3.48e}
\]

\[
\frac{\rho_N}{N^3} \left( \beta_{22} - \beta_{22} \right) = \frac{\rho_N}{N^3} \left[ \rho_N \left( \hat{\theta}_{21} - \theta_{22} \right) \right] - \pi_1 \left[ \rho_N \left( \hat{\theta}_{21} - \theta_{21} \right) \right] \tag{3.3.48f}
\]

\[
\frac{\rho_N}{N^3} \left( \beta_{21} - \beta_{21} \right) = \frac{\rho_N}{N^3} \left[ \rho_N \left( \hat{\theta}_{23} - \theta_{23} \right) \right] - a_{21} \frac{\rho_N}{N^3} \frac{N^2}{\rho_0} \left[ \rho_N \left( \hat{\theta}_{12} - \theta_{12} \right) \right] \\
+ \left( -a_{11} + a_{21} \bar{\pi}_1 \right) \frac{\rho_N}{N^3} \left( \hat{\theta}_{21} - \theta_{21} \right) \tag{3.3.48g}
\]

\[
\frac{\rho_N}{N^3} \left( \beta_{22} - \beta_{22} \right) = \frac{\rho_N}{N^3} \left[ \rho_N \left( \hat{\theta}_{24} - \theta_{24} \right) \right] - \pi_1 \frac{\rho_N}{N^3} \left[ \rho_N \left( \hat{\theta}_{23} - \theta_{23} \right) \right] \\
- a_{22} \frac{\rho_N}{N^3} \frac{N^2}{\rho_0} \left[ \rho_N \left( \hat{\theta}_{22} - \theta_{22} \right) \right] \\
+ \left( -a_{12} + a_{22} \bar{\pi}_1 \right) \frac{\rho_N}{N^3} \left( \hat{\theta}_{21} - \theta_{21} \right) . \tag{3.3.48h}
\]

Invoking theorem (3.3.2) in (3.3.48a), to (3.3.48d), we note that the expressions in the square brackets converge in probability as \( N \to \infty \), to non-degenerate random variables, there by we get

\[
\frac{\rho_N}{N^3} \left( \hat{\beta}_{11} - \beta_{11} \right) \to \xi_{11}(1) = \zeta_{11}(1) \quad \text{(say)} \tag{3.3.49}
\]

\[
\frac{\rho_N}{N^3} \left( \hat{\beta}_{12} - \beta_{12} \right) \to -\pi_1 \xi_{12}(1) = \zeta_{12}(1) \quad \text{(say)} \tag{3.3.49}
\]

\[
\frac{\rho_N}{N^3} \left( \hat{\beta}_{11} - \beta_{11} \right) \to \left( -a_{11} + a_{21} \bar{\pi}_1 \right) \xi_{13}(1) = \zeta_{13}(1) \quad \text{(say)} \tag{3.3.49}
\]

\[
\frac{\rho_N}{N^3} \left( \hat{\beta}_{22} - \beta_{12} \right) \to \left( -a_{12} + a_{22} \bar{\pi}_1 \right) \xi_{14}(1) = \zeta_{14}(1). \quad \text{(say)} \tag{3.3.49}
\]

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in probability as \( N \to \infty \) follows from the fact that the term in the \( d \)-square brackets converges.

\[
\lim_{N \to \infty} \frac{\sum x_i y_i}{\sum x_i y_i} = \frac{\sum x_i y_i}{\sum x_i y_i}
\]

However, the limit of the right side of (3.3.50) follows.

\[
\lim_{N \to \infty} \frac{\sum x_i y_i}{\sum x_i y_i} = \frac{\sum x_i y_i}{\sum x_i y_i}
\]

Now, the limit of the right side of (3.3.50) follows.

\[
\lim_{N \to \infty} \frac{\sum x_i y_i}{\sum x_i y_i} = \frac{\sum x_i y_i}{\sum x_i y_i}
\]

Hence, (4) of the theorem is proved. Statement (q) of the theorem follows as a consequence.

Simultaneously it can be shown that

\[
(\text{see} \) \quad (1)^{\text{ex}} = (1)^{\text{ex}} \left( \frac{\sum x_i y_i}{\sum x_i y_i} \right)
\]

and

\[
(\text{see} \) \quad (1)^{\text{ex}} = (1)^{\text{ex}} \left( \frac{\sum x_i y_i}{\sum x_i y_i} \right)
\]
(3.3.52) together imply, on applying Lemma 6.1 of Bhat (1999),

\[ \hat{\pi}_1 \zeta_{11}(1) + \zeta_{12}(1) = 0, \quad \text{a.s.} \]

which is (i) of (d).

Similarly, (ii) and (iii) of (d) follows from (3.3.50b) and (3.3.50c).

Now using the inverse transformation of (3.3.28), one can prove (iv) to (vi) of (d) on similar lines as above.

Hence the theorem. \( \square \)

**Case 2:** \( \rho_2 = \rho_3 = \rho_4 = \rho_0; \ |\rho_1| > |\rho_0| > 1 \)

**Theorem 3.3.4.** Let \( \{Z(t) = (X(t), Y(t))'\} \) be generated from the model (3.2.1) and 
\[ (\hat{\theta}_1', \hat{\theta}_2') = (\hat{\theta}_{11}, \hat{\theta}_{12}, \hat{\theta}_{13}, \hat{\theta}_{14}, \hat{\theta}_{21}, \hat{\theta}_{22}, \hat{\theta}_{23}, \hat{\theta}_{24}, \hat{\theta}_{25}) \]
be the least squares estimator of the parameter vector \( (\theta_1', \theta_2') \) of the reparametrised model in (3.3.22). Then, under the assumptions of (3.2.2), governed by the model, the following statements hold under the placement \( \rho_2 = \rho_3 = \rho_4 = \rho_0; \ |\rho_1| > |\rho_0| > 1 \) relating to the real roots of \( P(z) = 0. \)

a. \( \left( \frac{\rho_0^N}{N^2} (\hat{\theta}_{11} - \theta_{11}), \frac{\rho_0^N}{N} (\hat{\theta}_{12} - \theta_{12}), \frac{\rho_0^N}{N} (\hat{\theta}_{13} - \theta_{13}), \frac{\rho_0^N}{N} (\hat{\theta}_{14} - \theta_{14}) \right), \)
\( \left( \frac{\rho_0^N}{N^2} (\hat{\theta}_{21} - \theta_{21}), \frac{\rho_0^N}{N} (\hat{\theta}_{22} - \theta_{22}), \frac{\rho_0^N}{N} (\hat{\theta}_{23} - \theta_{23}), \frac{\rho_0^N}{N} (\hat{\theta}_{24} - \theta_{24}) \right) \)
converges in probability, as \( N \to \infty, \) to the random vector \( (\xi_{11}(2), \xi_{12}(2), \xi_{13}(2), \xi_{14}(2), \xi_{21}(2), \xi_{22}(2), \xi_{23}(2), \xi_{24}(2) ) \) (say).

b. Each \( \xi_{ij}(2) \) is distributed like a ratio of two random variables of the generic type 
\[ \sum_{i=1}^4 (c_i \pi_{ij}) / J \] wherein \( \pi_{ij} \) and \( J \) are independent and are distributed like specific linear combinations of \( \epsilon_i(t); \ i = 1, 2; \ t = 1, 2, \ldots. \)

c. \( \left( \sqrt{N} (\hat{\theta}_{15} - \theta_{15}), \sqrt{N} (\hat{\theta}_{25} - \theta_{25}) \right) \)
converges to a bivariate normal random vector with mean vector zero and a diagonal covariance matrix.

**Proof.** The explicit solutions for \( X(t), Y(t) \) and \( X_2(t), Y_2(t) \) as in Theorems 3.2.1-(b) and Theorem 3.2.2-(a) for the Case 2, are used in expanding the elements of the
matrices $M_1^*, M_2^*(l)$ and $M_2^*(l)$; $l = 1, 2, \ldots, 5$. The determinants of these matrices say, $D_2, D_21(l)$ and $D_22(l)$; $l = 1, 2, \ldots, 5$ are computed using MAPLE software. These are substituted in the definition of $\hat{\theta}_{ij} - \theta_{ij}; i = 1, 2; j = 1, 2, \ldots, 5$ (3.3.29) resulting in

$$\rho_1^{N - 2N} \rho_0^{6N} D_2 = \frac{4\rho_1^6 (\rho_0 - \rho_1)^6 (\pi_1 - \pi_2)^2 (\pi_1 - \pi_2)^2 H_{34}^2 H_{33}^2 H_{11}^2 H_{12}^2 H_{13}^2 H_{12}^2}{(\rho_0 - 1)^6 (\rho_1^2 - 1) (\rho_0 \rho_1 - 1)^6 \rho_1^2} + o_P(1)$$

Note that $\rho_1^{2N} \rho_0^{-6N} D_2 \rightarrow \xi_2$ (say), such that $P(\xi_2 = 0) = 0$ a.s.

$$\rho_0^{N-N} (\hat{\theta}_{11} - \theta_{11}) = \frac{1}{(\pi_2 - \pi_1)} \left(\frac{a_{11} l_{10}(1) + a_{12} l_{10}(1) + a_{13} l_{11}(1) + a_{14} l_{12}(1)}{H_{33} H_{34}}\right) + o_P(1)$$

$$\rho_0^{N} (\hat{\theta}_{12} - \theta_{12}) = \frac{1}{(\pi_2 - \pi_1)} \left(\frac{a_{21} l_{10}(1) + a_{22} l_{10}(1) + a_{23} l_{11}(1) + a_{24} l_{12}(1)}{H_{33}}\right) + o_P(1)$$

$$\rho_0^{N} (\hat{\theta}_{13} - \theta_{13}) = \frac{1}{(\pi_2 - \pi_1)} \left(\frac{a_{31} l_{10}(1) + a_{32} l_{10}(1) + a_{33} l_{11}(1) + a_{34} l_{12}(1)}{H_{33} H_{34}}\right) + o_P(1)$$

$$\rho_0^{N} (\hat{\theta}_{14} - \theta_{14}) = \frac{1}{(\pi_2 - \pi_1)} \left(\frac{a_{41} l_{10}(1) + a_{42} l_{10}(1) + a_{43} l_{11}(1) + a_{44} l_{12}(1)}{H_{33}}\right) + o_P(1)$$

$$\rho_0^{N-N} (\hat{\theta}_{21} - \theta_{21}) = \frac{1}{(\pi_2 - \pi_1)} \left(\frac{a_{11} l_{10}(1) + a_{12} l_{20}(1) + a_{13} l_{21}(1) + a_{14} l_{22}(1)}{H_{33} H_{34}}\right) + o_P(1)$$

$$\rho_0^{N-N} (\hat{\theta}_{22} - \theta_{22}) = \frac{1}{(\pi_2 - \pi_1)} \left(\frac{a_{21} l_{20}(1) + a_{22} l_{20}(1) + a_{23} l_{21}(1) + a_{24} l_{22}(1)}{H_{33}}\right) + o_P(1)$$

$$\rho_0^{N} (\hat{\theta}_{23} - \theta_{23}) = \frac{1}{(\pi_2 - \pi_1)} \left(\frac{a_{31} l_{20}(1) + a_{32} l_{20}(1) + a_{33} l_{21}(1) + a_{34} l_{22}(1)}{H_{33} H_{34}}\right) + o_P(1)$$

$$\rho_0^{N} (\hat{\theta}_{24} - \theta_{24}) = \frac{1}{(\pi_2 - \pi_1)} \left(\frac{a_{41} l_{20}(1) + a_{42} l_{20}(1) + a_{43} l_{21}(1) + a_{44} l_{22}(1)}{H_{33}}\right) + o_P(1)$$
\[
\sqrt{N} \left( \hat{\theta}_{15} - \theta_{15} \right) = N^{-1/2} D_2 \sum \epsilon_1(t) + o_P(1)
\]
\[
\sqrt{N} \left( \hat{\theta}_{25} - \theta_{25} \right) = N^{-1/2} D_2 \sum \epsilon_2(t) + o_P(1)
\]

(3.3.54)

Where the constants \( a_{ij} \)'s are

\[
a_{11} = -\frac{1}{2} \frac{\left( p_1^{2-1} \right) \left( p_2^{2-1} \right) \left( p_0 p_1 \right)^3}{p_0^5 (p_1 - p_0)^4}
\]
\[
a_{12} = -\frac{1}{2} \frac{\left( p_2^{2-1} \right) \left( p_0 p_1 \right)^3}{p_0^5 (p_1 - p_0)^4}
\]
\[
a_{13} = \frac{1}{4} \frac{\left( p_2^{2-1} \right) \left( p_0 p_1 \right)^3}{p_0^5 (p_1 - p_0)^4}
\]
\[
a_{14} = \frac{1}{4} \frac{\left( p_0 p_1 \right)^3}{p_0^5 (p_1 - p_0)^4}
\]
\[
a_{21} = \frac{1}{4} \frac{\left( p_2^{2-1} \right)^3}{p_0^5 (p_1 - p_0)^4}
\]
\[
a_{22} = \frac{1}{4} \frac{\left( p_0 p_1 \right)^3}{p_0^5 (p_1 - p_0)^4}
\]
\[
a_{23} = \frac{1}{4} \frac{\left( p_2^{2-1} \right)^3}{p_0^5 (p_1 - p_0)^4}
\]
\[
a_{24} = \frac{1}{4} \frac{\left( p_0 p_1 \right)^3}{p_0^5 (p_1 - p_0)^4}
\]
\[
a_{31} = \frac{1}{4} \frac{\left( p_2^{2-1} \right)^3}{p_0^5 (p_1 - p_0)^4}
\]
\[
a_{32} = \frac{1}{4} \frac{\left( p_0 p_1 \right)^3}{p_0^5 (p_1 - p_0)^4}
\]
\[
a_{33} = \frac{1}{4} \frac{\left( p_2^{2-1} \right)^3}{p_0^5 (p_1 - p_0)^4}
\]
\[
a_{34} = \frac{1}{4} \frac{\left( p_0 p_1 \right)^3}{p_0^5 (p_1 - p_0)^4}
\]
\[
a_{41} = \frac{1}{4} \frac{\left( p_0 p_1 \right)^3}{p_0^5 (p_1 - p_0)^4}
\]
\[
a_{42} = \frac{1}{4} \frac{\left( p_2^{2-1} \right)^3}{p_0^5 (p_1 - p_0)^4}
\]
\[
a_{43} = \frac{1}{4} \frac{\left( p_0 p_1 \right)^3}{p_0^5 (p_1 - p_0)^4}
\]
\[
a_{44} = \frac{1}{4} \frac{\left( p_2^{2-1} \right)^3}{p_0^5 (p_1 - p_0)^4}
\]

From the results in (3.3.53), we see that

\[
\left( \frac{\rho_0}{N^2} \theta_{11} - \theta_{11}, \frac{\rho_0}{N^2} \theta_{12} - \theta_{12}, \frac{\rho_0}{N^2} \theta_{13} - \theta_{13}, \frac{\rho_0}{N^2} \theta_{14} - \theta_{14}, \frac{\rho_0}{N^2} \theta_{21} - \theta_{21}, \frac{\rho_0}{N^2} \theta_{22} - \theta_{22}, \frac{\rho_0}{N^2} \theta_{23} - \theta_{23}, \frac{\rho_0}{N^2} \theta_{24} - \theta_{24} \right)
\]

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converges in probability, as \( N \to \infty \), to a random vector \((\xi_{11}(2), \xi_{12}(2), \xi_{13}(2), \xi_{14}(2), \xi_{21}(2), \xi_{22}(2), \xi_{23}(2), \xi_{24}(2))\) (say).

Hence the statement (a) of the theorem is proved.

Proof of (b) goes on exactly similar lines to the proof of Theorem 3.3.2-(b) and using (3.3.54), proof of (c) goes on exactly similar lines to the proof of Theorem 3.3.2-(c).

We now state and prove the main theorem under Case 2 relating to the least squares estimators of the coefficients of the model in (3.2.1).

Theorem 3.3.5. Let \( \hat{\beta}' = (\hat{\beta}'_{11}, \hat{\beta}_{10}, \hat{\beta}_{20}) = (\hat{\beta}_{11}^{(1)}, \hat{\beta}_{12}^{(1)}, \hat{\beta}_{11}^{(2)}, \hat{\beta}_{12}^{(2)}, \hat{\beta}_{21}^{(1)}, \hat{\beta}_{22}^{(1)}, \hat{\beta}_{21}^{(2)}, \hat{\beta}_{22}^{(2)}, \hat{\beta}_{10}, \hat{\beta}_{20}) \) be the least squares estimators of \( \beta' = (\beta'_1, \beta_{10}, \beta_{20}) = (\beta_{11}^{(1)}, \beta_{12}^{(1)}, \beta_{11}^{(2)}, \beta_{12}^{(2)}, \beta_{21}^{(1)}, \beta_{22}^{(1)}, \beta_{21}^{(2)}, \beta_{22}^{(2)}, \beta_{10}, \beta_{20}) \), the vector of structural parameters of the model in (3.2.1) satisfying the assumptions of (3.2.2). Then the following statements hold under the placement \( \rho_2 = \rho_3 = \rho_4 = \rho_0; \ |\rho_1| > |\rho_0| > 1 \) relating to the roots of \( P(z) = 0 \).

a. \( \left( \left( \rho_{ij}/N^2 \right) (\hat{\beta}_i - \beta_i) \right)' \) converges in probability as \( N \to \infty \), to a random vector \((\zeta_{11}(2), \zeta_{12}(2), \zeta_{13}(2), \zeta_{14}(2), \zeta_{21}(2), \zeta_{22}(2), \zeta_{23}(2), \zeta_{24}(2))\) (say).

b. Each \( \zeta_{ij}(2) \) is distributed like a ratio of two random variables of the generic type \( \sum_{i=1}^{d} c_i \pi_{ij}/J \), wherein \( \pi_{ij} \) and \( J \) are independent and are distributed like specific linear combinations of \( \epsilon_i(t) \); \( i = 1, 2; t = 1, 2, \ldots \).

c. \( \left\{ \sqrt{N} \left( \hat{\beta}_{10} - \beta_{10} \right), \sqrt{N} \left( \hat{\beta}_{20} - \beta_{20} \right) \right\} \) converges in distribution as \( N \to \infty \), to a bivariate random vector with mean vector a null vector and a diagonal covariance matrix.

d. (i) \( \pi_1 \zeta_{11}(2) + \zeta_{12}(2) = 0 \) (a.s.)

(ii) \( a_{11}^{(1)} \zeta_{11}(2) + a_{21}^{(1)} \zeta_{12}(2) + \zeta_{13}(2) = 0 \) (a.s.)

(iii) \( \left( a_{12}^{(1)} + \pi_1 a_{11}^{(1)} \right) \zeta_{11}(2) + \left( a_{21}^{(1)} \pi_1 + a_{22}^{(1)} \right) \zeta_{12}(2) + \pi_1 \zeta_{13}(2) + \zeta_{14}(2) = 0 \) (a.s.)
Using the inverse transformation (3.3.27) and (3.3.28), we can express the centered least squares estimation $\hat{\beta}_c - \beta_c$ as

$$
\hat{\beta}^{(1)}_{11} - \beta^{(1)}_{11} = \tilde{\theta}_{11} - \theta_{11}
$$

$$
\hat{\beta}^{(1)}_{12} - \beta^{(1)}_{12} = \left( \tilde{\theta}_{12} - \theta_{12} \right) - \tilde{\pi}_1 \left( \tilde{\theta}_{11} - \theta_{11} \right)
$$

$$
\hat{\beta}^{(2)}_{11} - \beta^{(2)}_{11} = \left( \tilde{\theta}_{13} - \theta_{13} \right) - a^{(1)}_{21} \left( \tilde{\theta}_{12} - \theta_{12} \right) + \left( -a^{(1)}_{11} + \tilde{\pi}_1 a^{(1)}_{21} \right) \left( \tilde{\theta}_{11} - \theta_{11} \right)
$$

$$
\hat{\beta}^{(2)}_{12} - \beta^{(2)}_{12} = \left( \tilde{\theta}_{14} - \theta_{14} \right) - \pi_1 \left( \tilde{\theta}_{13} - \theta_{13} \right) - a^{(1)}_{22} \left( \tilde{\theta}_{12} - \theta_{12} \right) + \left( -a^{(1)}_{12} + \tilde{\pi}_1 a^{(1)}_{22} \right) \left( \tilde{\theta}_{11} - \theta_{11} \right)
$$

Knowing the rate of convergence in probability of the $\hat{\theta}$s in Theorem 3.3.4, we have

$$
\hat{\beta}^{(1)}_{11} - \beta^{(1)}_{11} = N^2 \frac{\rho^N_0}{N^2} \left( \tilde{\theta}_{11} - \theta_{11} \right)
$$

$$
\hat{\beta}^{(1)}_{12} - \beta^{(1)}_{12} = \frac{N}{N^2} \frac{\rho^N_0}{N} \left( \tilde{\theta}_{12} - \theta_{12} \right) - \tilde{\pi}_1 \frac{N^2}{\rho^N_0} \frac{\rho^N_0}{N^2} \left( \tilde{\theta}_{11} - \theta_{11} \right)
$$

$$
\hat{\beta}^{(2)}_{11} - \beta^{(2)}_{11} = \frac{1}{\rho^N_0} \left[ \rho^N_0 \left( \tilde{\theta}_{13} - \theta_{13} \right) \right] - a^{(1)}_{21} \frac{N}{\rho^N_0} \frac{\rho^N_0}{N} \left( \tilde{\theta}_{12} - \theta_{12} \right)
$$

$$
\left( -a^{(1)}_{11} + \tilde{\pi}_1 a^{(1)}_{21} \right) \frac{N^2}{\rho^N_0} \frac{\rho^N_0}{N^2} \left( \tilde{\theta}_{11} - \theta_{11} \right)
$$

$$
\hat{\beta}^{(2)}_{12} - \beta^{(2)}_{12} = \frac{1}{\rho^N_0} \left[ \rho^N_0 \left( \tilde{\theta}_{14} - \theta_{14} \right) \right] - \pi_1 \frac{1}{\rho^N_0} \left[ \rho^N_0 \left( \tilde{\theta}_{13} - \theta_{13} \right) \right] - a^{(1)}_{22} \frac{N}{\rho^N_0} \frac{\rho^N_0}{N} \left( \tilde{\theta}_{13} - \theta_{13} \right)
$$

$$
\left( -a^{(1)}_{12} + \tilde{\pi}_1 a^{(1)}_{22} \right) \frac{N^2}{\rho^N_0} \frac{\rho^N_0}{N^2} \left( \tilde{\theta}_{14} - \theta_{14} \right)
$$

Applying statement (a) of the Theorem 3.3.4 to the quantities in square brackets, we see that the least weight that stabilises each of $\left( \hat{\beta}^{(k)}_{1j} - \beta^{(k)}_{1j} \right)$; $k = 1, 2$; $j = 1, 2$ is
\( \hat{\rho}_{\mathbf{0}}^{N} \), and therefore we get

\[
\begin{align*}
\frac{\hat{\rho}_{\mathbf{0}}^{N}}{N^2} (\hat{\beta}_{11}^{(1)} - \beta_{11}^{(1)}) & \xrightarrow{P} \zeta_{11}(2) \quad \text{ (say)} \\
\frac{\hat{\rho}_{\mathbf{0}}^{N}}{N^2} (\hat{\beta}_{12}^{(2)} - \beta_{12}^{(1)}) & \xrightarrow{P} -\hat{\pi}_{1} \xi_{11}(2) = \zeta_{12}(2) \quad \text{ (say)} \\
\frac{\hat{\rho}_{\mathbf{0}}^{N}}{N^2} (\hat{\beta}_{11}^{(2)} - \beta_{11}^{(2)}) & \xrightarrow{P} (-a_{11}^{(1)} + \hat{\pi}_{1} a_{21}^{(1)}) \xi_{11}(2) = \zeta_{13}(2) \quad \text{ (say)} \\
\frac{\hat{\rho}_{\mathbf{0}}^{N}}{N^2} (\hat{\beta}_{12}^{(2)} - \beta_{12}^{(2)}) & \xrightarrow{P} (-a_{11}^{(1)} + \hat{\pi}_{1} a_{22}^{(1)}) \xi_{11}(2) = \zeta_{14}(2) \quad \text{ (say)}
\end{align*}
\]

Similar results can be obtained for \( (\hat{\beta}_{2j}^{(k)} - \beta_{2j}^{(k)}) \); \( k = 1, 2; \ j = 1, 2 \) using the transformation (3.3.28).

Hence (a) of the theorem is proved. \( \Box \)

Proofs of (b),(c) and (d) are on exactly similar lines of the proofs for (b),(c) and (d) of the Theorem 3.3.3.

Case 3: \( \rho_3 = \rho_4 = \rho_0; \ |\rho_1| > |\rho_2| > |\rho_0| > 1 \)

Theorem 3.3.6. Let \( \{Z(t) = (X(t), Y(t))'\} \) be generated from the model (3.2.1) and \( (\hat{\theta}_1', \hat{\theta}_2') = (\hat{\theta}_{11}, \hat{\theta}_{12}, \hat{\theta}_{13}, \hat{\theta}_{14}, \hat{\theta}_{15}, \hat{\theta}_{21}, \hat{\theta}_{22}, \hat{\theta}_{23}, \hat{\theta}_{24}, \hat{\theta}_{25}) \) be the least squares estimator of the parameter vector \( (\theta_1', \theta_2') \) of the reparametrised model in (3.3.22). Then, under the assumptions of (3.2.2), governed by the model, the following statements hold under the placement \( \rho_3 = \rho_4 = \rho_0; \ |\rho_1| > |\rho_2| > |\rho_0| > 1 \) relating to the real roots of \( P(z) = 0 \).

a. \( \left( \frac{\hat{\rho}_0^{N}}{N} \right) (\hat{\theta}_{11} - \theta_{11}), \frac{\hat{\rho}_0^{N}}{N} (\hat{\theta}_{12} - \theta_{12}), \frac{\hat{\rho}_0^{N}}{N} (\hat{\theta}_{13} - \theta_{13}), \frac{\hat{\rho}_0^{N}}{N} (\hat{\theta}_{14} - \theta_{14}), \frac{\hat{\rho}_0^{N}}{N} (\hat{\theta}_{21} - \theta_{21}), \frac{\hat{\rho}_0^{N}}{N} (\hat{\theta}_{22} - \theta_{22}), \frac{\hat{\rho}_0^{N}}{N} (\hat{\theta}_{23} - \theta_{23}), \frac{\hat{\rho}_0^{N}}{N} (\hat{\theta}_{24} - \theta_{24}) \) converges in probability as \( N \to \infty \) to the random vector \( (\xi_{11}(3), \xi_{12}(3), \xi_{14}(3), \xi_{14}(3), \xi_{21}(3), \xi_{22}(3), \xi_{23}(3), \xi_{24}(3)) \) (say).

b. Each \( \xi_{ij}(3) \) is distributed like a ratio of two random variables of the generic type \( \sum_i c_i \pi_{ij}/J \), wherein \( \pi_{ij} \) and \( J \) are independent and are distributed like specific linear combinations of \( \epsilon_i(t); \ i = 1, 2; \ t = 1, 2, \ldots \).
c. \((\sqrt{N} (\hat{\theta}_{15} - \theta_{15}), \sqrt{N} (\hat{\theta}_{25} - \theta_{25}))\) converges in to a bivariate normal random vector with mean vector zero and a diagonal covariance matrix.

**Proof.** Explicit solutions for \(X(t), Y(t), X_2(t)\) and \(Y_2(t)\) under Case 3 are given in Theorems 3.2.1 and 3.2.2. These solutions are substituted for the elements of the matrices \(M^*_l, M^*_j(l)\); \(j = 1, 2; l = 1, 2, \ldots, 5\) and expanded using the notations in (3.3.36). The determinants, say \(D_3, D_{3j}(l); j = 1, 2; l = 1, 2, \ldots, 5\) of the matrices \(M^*_l\) and \(M^*_j(l)\); \(j = 1, 2; l = 1, 2, \ldots, 5\) respectively, are computed. These are then substituted in the definition of \((\hat{\theta}_i - \theta_i); i = 1, 2; l = 1, 2, \ldots, 5\) given in (3.3.29), to get the following results.

\[
p_0^{-4N} \rho_1^{-2N} \rho_2^{-2N} D_3 = d_0 + o_P(1)
\]

\[
\frac{\rho_0^N}{N} (\hat{\theta}_{11} - \theta_{11}) = \frac{\rho_0^N D^{(1)}_{31}}{N D_3} = \frac{1}{(\bar{\pi}_2 - \bar{\pi}_1)} \frac{(a_{11} l_{10}(1) + a_{12} m_{10}(1) + a_{13} p_{10}(1) + a_{14} p_{11}(1))}{H_{33} H_{34}} + o_P(1)
\]

\[
\rho_0^N (\hat{\theta}_{12} - \theta_{12}) = \frac{\rho_0^N D^{(2)}_{31}}{D_3} = \frac{(a_{21} l_{10}(1) + a_{22} m_{10}(1) + a_{23} p_{10}(1) + a_{24} p_{11}(1))}{H_{33}} + o_P(1)
\]

\[
\rho_0^N (\hat{\theta}_{13} - \theta_{13}) = \frac{\rho_0^N D^{(3)}_{31}}{D_3} = \frac{1}{(\bar{\pi}_2 - \bar{\pi}_1)} \frac{(a_{31} l_{10}(1) + a_{32} m_{10}(1) + a_{33} p_{10}(1) + a_{34} p_{11}(1))}{H_{11} H_{12}} + o_P(1)
\]

\[
\rho_0^N (\hat{\theta}_{14} - \theta_{14}) = \frac{\rho_0^N D^{(4)}_{31}}{D_3} = \frac{(a_{41} l_{10}(1) + a_{42} m_{10}(1) + a_{43} p_{10}(1) + a_{44} p_{11}(1))}{H_{11}} + o_P(1)
\]

\[
\rho_0^N (\hat{\theta}_{21} - \theta_{21}) = \frac{\rho_0^N D^{(1)}_{32}}{N D_3} = \frac{1}{(\bar{\pi}_2 - \bar{\pi}_1)} \frac{(a_{11} l_{20}(1) + a_{12} m_{20}(1) + a_{13} p_{20}(1) + a_{14} p_{21}(1))}{H_{33} H_{34}} + o_P(1)
\]

\[
\rho_0^N (\hat{\theta}_{22} - \theta_{22}) = \frac{\rho_0^N D^{(2)}_{32}}{D_3} = \frac{(a_{21} l_{20}(1) + a_{22} m_{20}(1) + a_{23} p_{20}(1) + a_{24} p_{21}(1))}{H_{33}} + o_P(1)
\]
\[
\rho_2^N (\theta_{23} - \theta_{23}) = \rho_2^N \frac{D_{32}(3)}{D_3} = \frac{1}{(\pi_2 - \pi_1)} \frac{(a_{31} l_{20}(1) + a_{32} m_{20}(1) + a_{33} p_{20}(1) + a_{34} p_{21}(1))}{H_{11} H_{12}} + o_P(1)
\]
\[
\rho_1^N (\theta_{24} - \theta_{24}) = \rho_1^N \frac{D_{32}(4)}{D_3} = \frac{(a_{41} l_{20}(1) + a_{42} m_{20}(1) + a_{43} p_{20}(1) + a_{44} p_{21}(1))}{H_{11}} + o_P(1) \\
(3.3.55)
\]
\[
\hat{\theta}_{15} - \theta_{15} = \frac{D_{31}(5)}{D_3} = N^{-1} \sum \epsilon_1(t) + o_P(1)
\]
\[
\hat{\theta}_{25} - \theta_{25} = \frac{D_{32}(5)}{D_3} = N^{-1} \sum \epsilon_2(t) + o_P(1) \\
(3.3.56)
\]

where

\[
v_0(2) = \sum \rho_1^{-2u} \quad u_0(2) = \sum \rho_2^{-2u} \quad x_p(1) = \sum u^p \rho_0^{-u} \rho_1^{-u} \quad l_{40}(1) = \sum \rho_1^{-u} \epsilon_i(N - u + 1) \\
\]
\[
m_{40}(1) = \sum \rho_2^{-u} \epsilon_i(N - u + 1)
\]

\[
d_3 = \frac{H_{33}^4 H_{34}^2 (p_{12} - p_{11})^4 H_{11}^4 p_{0}^8 (-p_2 + p_1)^2 (p_0 - p_1)^4 (p_0 - p_3)^4 H_{12}^2 (\pi_2 - \pi_1)^2}{\rho_2^2 \rho_1^2 \rho_0^4 (p_0 - p_1)^4 (p_0 - p_2 - 1)^4 (p_0 - p_3 - 1)^4 (p_0 - p_4 - 1)^4 (p_1 - 1) (p_2 - 1) (p_3 - 1) (p_4 - 1)}
\]
\[
a_{11} = -\frac{(p_0 - 1)^2 (p_0 + 1)^2 (p_0 + 1 - 1)^2 (p_0 + 1 - 1) (p_0 - 1) (p_0 - 1) (p_0 - 1)}{(p_2 - p_0) (p_3 - p_0) (p_0 - p_1)^2 \rho_0}
\]
\[
a_{12} = \frac{(p_0 - 1)^2 (p_0 + 1)^2 (p_0 + 1 - 1) (p_0 - 1) (p_0 - 1) (p_0 - 1) (p_0 - 1) (p_0 - 1) (p_0 + 1)}{(p_2 - p_0)^2 (p_2 - p_1) (p_0 - p_1) \rho_0}
\]
\[
a_{13} = -\frac{(p_0 - 2)^2 (p_0 - 1)^2 (p_0 - 1) (p_0 - 1)^2}{(p_2 - p_0)^2 (p_0 - p_1)^2 \rho_0} \cdot \frac{\text{numerator13}}{(p_2 - p_0)^2 (p_0 - p_1)^2 \rho_0}
\]
\[
a_{14} = \frac{(p_0 - 2)^2 (p_0 - 1)^2 (p_0 - 1)^2 (p_0 - 1)^2}{(p_2 - p_0)^2 (p_0 - p_1)^2 \rho_0^2}
\]
\[
a_{21} = -\frac{(p_0 - 1)^2 (p_0 + 1)^2 (p_0 + 1 - 1) (p_0 + 1 - 1) (p_0 - 1) (p_0 - 1) (p_0 - 1) (p_0 - 1) (p_0 - 1) (p_0 - 1) (p_0 - 1)}{(-p_2 - p_0) (p_0 - p_1)^4 (p_0 - p_1)^2}
\]
\[
a_{22} = \frac{(p_0 - 1)^2 (p_0 + 1)^2 (p_0 + 1 - 1) (p_0 + 1 - 1) (p_0 - 1) (p_0 - 1) (p_0 - 1) (p_0 - 1) (p_0 - 1) (p_0 - 1) (p_0 - 1)}{(-p_2 - p_0) (p_0 - p_1)^4 (p_0 - p_1)^2}
\]
\[
a_{23} = \frac{(p_0 - 1)^2 (p_0 + 1)^2 (p_0 - 1)^2 (p_0 - 1)}{(-p_2 - p_0)^2 (p_0 - p_1)^4 \rho_0}
\]
\[
a_{24} = -\frac{(p_0 - 1)^2 (p_0 - 1)^2 (p_0 - 1)^2 (p_0 - 1)^2}{(-p_2 + p_0)^2 (p_0 - p_1)^4 \rho_0^2}
\]
\[
a_{31} = -\frac{\rho_0^4 (p_0 - 2 p_0 + 4 p_0^2) (p_0 + 1)^2 (p_0 - 1)^2 (p_0 + 1) (p_0 - 1) (p_0 - 1)^2 (p_0 + 1)^2}{\rho_0^4 (-p_2 - p_0)^2 (p_0 - p_1)^4 (-p_3 - p_0)^2}
\]
\[
a_{32} = \frac{(p_0 - 1)^2 (p_0 + 1)^2 (p_0 + 1)^2 (p_0 + 1)^2}{(-p_2 + p_0)^2 (p_0 - p_1)^4 \rho_0^2}
\]
\[
a_{33} = -\frac{(p_0 - 2) (p_0 + 1)^2 (p_0 - 1)^2 (p_0 - 1)^2}{\rho_0^4 (-p_2 + p_0)^2 (p_0 - p_1)^4 (-p_3 + p_0)^2}
\]
where

\[
\text{numerator 13} = 2 \rho_2^2 \rho_0 \rho_1^2 + 2 \rho_2^2 \rho_0^4 \rho_1 - \rho_1 \rho_2^2 - \rho_0^3 \rho_2^2 + \rho_1 \rho_0^2 \rho_2^2 - 3 \rho_1^2 \rho_0^3 \rho_2^2 + \rho_2 \rho_0^2 - 2 \rho_2 \rho_0^5 \rho_1 + \rho_1^2 \rho_0^2 \rho_2 - 2 \rho_1 \rho_0^3 \rho_2 + 2 \rho_2 \rho_0^4 \rho_1^2 + \rho_2 - \rho_1^2 \rho_2
\]

\[
\text{numerator 23} = -\rho_2 \rho_0^2 - \rho_1 + 2 \rho_2 \rho_0^5 \rho_1 + \rho_2 \rho_0 \rho_1 - 2 \rho_2 \rho_0^4 \rho_1^2 + 3 \rho_1^2 \rho_0^3 \rho_2^2 + \rho_2 \rho_0^2 + 2 \rho_1 \rho_0^3 \rho_2 - 2 \rho_2 \rho_0^2 \rho_0 \rho_1^2 + 2 \rho_0 + \rho_1 \rho_2
\]

\[
\text{numerator 33} = \rho_0^4 \rho_1^2 \rho_2^3 - \rho_0^4 \rho_1 \rho_2^4 + 2 \rho_0^3 \rho_2^5 \rho_1 - 2 \rho_0^3 \rho_1^3 \rho_2^3 - \rho_0^3 \rho_1 \rho_2^4 - 3 \rho_0^2 \rho_2^5 \rho_1^2 + 2 \rho_0^5 \rho_1^2 \rho_2^3 + 3 \rho_0^3 \rho_1^3 \rho_2^2 - \rho_0^2 \rho_2^5 - 2 \rho_0 \rho_1^2 \rho_2^4 + 2 \rho_0 \rho_2^5 \rho_1 + 2 \rho_0 \rho_2^4 - 2 \rho_0 \rho_1 \rho_2^3 - \rho_2^5 + \rho_2^3 - \rho_1^3 \rho_2^3 - \rho_2^5 \rho_1^2 + 2 \rho_2 \rho_0^2 + \rho_2 - 2 \rho_0^3 \rho_2^2 + \rho_0^2 \rho_1 + \rho_0^4 \rho_2
\]

Statement (a) of the theorem follows from (3.3.55).

Proof for (b) of the theorem follows on exactly similar lines of the proof for (b) of Theorem 3.3.2 on using (3.3.56), the proof for (c) can be given on exactly similar lines as that for (c) of the Theorem 3.3.2.

We now state and prove the main theorem under Case 3 relating to least squares estimators of the coefficients of the model on (3.2.1).
Theorem 3.3.7. Let \( \hat{\beta}' = (\hat{\beta}'_c, \hat{\beta}_10, \hat{\beta}_20) \) be the least squares estimators of \( \beta' = (\beta'_c, \beta_10, \beta_20) = (\beta^{(1)}_{11}, \beta^{(1)}_{12}, \beta^{(2)}_{11}, \beta^{(2)}_{12}, \beta^{(1)}_{21}, \beta^{(1)}_{22}, \beta^{(2)}_{21}, \beta^{(2)}_{22}, \beta_10, \beta_20) \), the vector of structural parameters of the model in (3.2.1) satisfying the assumptions of (3.2.2). Then the following statements hold under the placement \( \rho_1 = \rho_2 = \rho_0; |\rho_0| > |\rho_3| > |\rho_4| > 1 \) relating to the roots of \( P(z) = 0 \).

a. \( \left\{ \left( \rho_0'/N^2 \right) (\hat{\beta}_c - \beta_c) \right\} \) converges in probability as \( N \to \infty, \) to a random vector \((\zeta_{11}(3), \zeta_{12}(3), \zeta_{13}(3), \zeta_{14}(3), \zeta_{21}(3), \zeta_{22}(3), \zeta_{23}(3), \zeta_{24}(3)) \) (say).

b. Each \( \zeta_{ij}(3) \) is distributed like a ratio of two random variables of the generic type
\[
\sum_{i=1}^4 c_i \pi_{ij}/J ,
\]
wherein \( \pi_{ij} \) and \( J \) are independent and are distributed like specific linear combinations of \( \epsilon_i(t); i = 1, 2; t = 1, 2, \ldots \)

c. \( \left\{ \sqrt{N} \left( \hat{\beta}_{10} - \beta_{10} \right), \sqrt{N} \left( \hat{\beta}_{20} - \beta_{20} \right) \right\} \) converges in distribution as \( N \to \infty, \) to a bivariate random vector with mean vector a null vector and a diagonal covariance matrix.

d. (i) \( \bar{\pi}_1 \zeta_{11}(3) + \zeta_{12}(3) = 0 \) (a.s.)
(ii) \( a_{11}^{(1)} \zeta_{11}(3) + a_{21}^{(1)} \zeta_{12}(3) + \zeta_{13}(3) = 0 \) (a.s.)
(iii) \( \left( a_{12}^{(1)} + \pi_1 a_{11}^{(1)} \right) \zeta_{11}(3) + \left( a_{21}^{(1)} \pi_1 + a_{22}^{(1)} \right) \zeta_{12}(3) + \pi_1 \zeta_{13}(3) + \zeta_{14}(3) = 0 \) (a.s.)
(iv) \( \tilde{\pi}_1 \zeta_{21}(3) + \zeta_{22}(3) = 0 \) (a.s.)
(v) \( a_{11}^{(1)} \zeta_{21}(3) + a_{21}^{(1)} \zeta_{22}(3) + \zeta_{23}(3) = 0 \) (a.s.)
(vi) \( \left( a_{12}^{(1)} + \pi_1 a_{11}^{(1)} \right) \zeta_{21}(3) + \left( a_{21}^{(1)} \pi_1 + a_{22}^{(1)} \right) \zeta_{22}(3) + \pi_1 \zeta_{23}(3) + \zeta_{24}(3) = 0 \) (a.s.)

Proof. Using the inverse transformation (3.3.27), we can express the centered least squares estimators \( \hat{\beta}_{1c} - \beta_{1c} \) as
\[
\hat{\beta}^{(1)}_{11} - \beta^{(1)}_{11} = \hat{\theta}_{11} - \theta_{11}
\]
\[
\hat{\beta}^{(1)}_{12} - \beta^{(1)}_{12} = (\hat{\theta}_{12} - \theta_{12}) - \tilde{\pi}_1 \left( \hat{\theta}_{11} - \theta_{11} \right)
\]
\[ \beta_{11}^{(2)} - \beta_{11}^{(1)} = (\hat{\theta}_{13} - \theta_{13}) - a_{21}^{(1)} (\hat{\theta}_{12} - \theta_{12}) + (-a_{11}^{(1)} + \pi_1 a_{21}^{(1)}) (\hat{\theta}_{11} - \theta_{11}) \]

\[ \beta_{12}^{(2)} - \beta_{12}^{(1)} = (\hat{\theta}_{14} - \theta_{14}) - \pi_1 (\hat{\theta}_{13} - \theta_{13}) - a_{22}^{(1)} (\hat{\theta}_{12} - \theta_{12}) \]

\[ + (-a_{12}^{(1)} + \pi_1 a_{22}^{(1)}) (\hat{\theta}_{11} - \theta_{11}) \]

Knowing the rate of convergence in probability of \( \hat{\theta} \)s in Theorem 3.3.6, we have

\[ \hat{\beta}_{11}^{(1)} - \beta_{11}^{(1)} = \frac{N}{\rho_0} \left[ \frac{\rho_0^N}{N} (\hat{\theta}_{11} - \theta_{11}) \right] \]

\[ \hat{\beta}_{11}^{(2)} - \beta_{11}^{(1)} = \frac{1}{\rho_0} \left[ \frac{\rho_0^N}{1} (\hat{\theta}_{12} - \theta_{12}) \right] - \pi_1 \frac{N}{\rho_0} \left[ \frac{\rho_0^N}{N} (\hat{\theta}_{11} - \theta_{11}) \right] \]

\[ \hat{\beta}_{12}^{(2)} - \beta_{12}^{(1)} = \frac{1}{\rho_2} \left[ \frac{\rho_2^N}{N} (\hat{\theta}_{13} - \theta_{13}) \right] - a_{21}^{(1)} \frac{1}{\rho_0} \left[ \frac{\rho_0^N}{1} (\hat{\theta}_{12} - \theta_{12}) \right] \]

\[ + (-a_{11}^{(1)} + \pi_1 a_{21}^{(1)}) \frac{N}{\rho_0} \left[ \frac{\rho_0^N}{N} (\hat{\theta}_{11} - \theta_{11}) \right] \]

\[ \hat{\beta}_{12}^{(2)} - \beta_{12}^{(1)} = \frac{1}{\rho_1} \left[ \frac{\rho_1^N}{N} (\hat{\theta}_{14} - \theta_{14}) \right] - \pi_1 \frac{1}{\rho_2} \left[ \frac{\rho_2^N}{1} (\hat{\theta}_{13} - \theta_{13}) \right] - a_{22}^{(1)} \frac{1}{\rho_0} \left[ \frac{\rho_0^N}{1} (\hat{\theta}_{13} - \theta_{13}) \right] \]

\[ + (-a_{12}^{(1)} + \pi_1 a_{22}^{(1)}) \frac{N}{\rho_0} \left[ \frac{\rho_0^N}{N} (\hat{\theta}_{14} - \theta_{14}) \right] \]

Similarly, we can express \( (\hat{\beta}_{2j}^{(k)} - \beta_{2j}^{(k)}) \) in terms of \( (\hat{\theta}_{2l} - \theta_{2l}) \); \( l = 1, 2, \ldots, 4 \) using (3.3.28). Invoking the rate of convergence of \( \hat{\theta} \)s as obtained in the Theorem 3.3.6 under Case 3, it can be shown that the least weight that stabilises each of \( (\hat{\beta}_{ij}^{(k)} - \beta_{ij}^{(k)}) \); \( k = 1, 2; i, j = 1, 2 \) is \( \frac{\rho_0^N}{N} \). Therefore the statement (a) of this theorem follows.

\[ \square \]

Proof for (b), (c) and (d) of the theorem are on exactly similar lines of the proof of (ii), (iii) and (iv) of Theorem 3.3.3.

Case 4: \( \rho_1 = \rho_2 = \rho_0; \ |\rho_0| > |\rho_3| > |\rho_4| > 1 \).

**Theorem 3.3.8.** Let \( \{ Z(t) = (X(t), Y(t))' \} \) be generated from the model (3.2.1) and \( (\hat{\theta}_1, \hat{\theta}_2) = (\hat{\theta}_{11}, \hat{\theta}_{12}, \hat{\theta}_{13}, \hat{\theta}_{14}, \hat{\theta}_{15}, \hat{\theta}_{21}, \hat{\theta}_{22}, \hat{\theta}_{23}, \hat{\theta}_{24}, \hat{\theta}_{25}) \) be the least squares estimator of the parameter vector \( (\Theta_1', \Theta_2') \) of the reparametrised model in (3.3.22). Then, under the assumptions of (3.2.2), governed by the model, the following statements hold under the placement \( \rho_1 = \rho_2 = \rho_0; \ |\rho_0| > |\rho_3| > |\rho_4| > 1 \) relating to the roots of \( P(z) = 0 \).
a. \( \left( \rho_4^N (\hat{\theta}_{11} - \theta_{11}), \rho_3^N (\hat{\theta}_{12} - \theta_{12}), \frac{\rho_0^N}{N} (\hat{\theta}_{13} - \theta_{13}), \rho_6^N (\hat{\theta}_{14} - \theta_{14}), \rho_4^N (\hat{\theta}_{21} - \theta_{21}), \rho_3^N (\hat{\theta}_{22} - \theta_{22}), \frac{\rho_0^N}{N} (\hat{\theta}_{23} - \theta_{23}), \rho_6^N (\hat{\theta}_{24} - \theta_{24}) \right) \) converges in probability as \( N \to \infty \), to the random vector \((\xi_{11}(4), \xi_{12}(4), \xi_{13}(4), \xi_{14}(4), \xi_{21}(4), \xi_{22}(4), \xi_{23}(4), \xi_{24}(4))\) (say).

b. Each \( \xi_{ij}(4) \) is distributed like a ratio of two random variables of the generic type \( \sum_{i=1}^{4} (c_i \pi_{ij}) / J \) wherein \( \pi_{ij} \) and \( J \) are independent and are distributed like specific linear combinations of \( \epsilon_i(t); \ i = 1, 2; \ t = 1, 2, \ldots \)

c. \( \left( \sqrt{N} (\hat{\theta}_{15} - \theta_{15}), \sqrt{N} (\hat{\theta}_{25} - \theta_{25}) \right) \) converges to a bivariate normal random vector with mean vector zero and a diagonal covariance matrix.

Proof. The explicit solutions \( \mathbf{X}(t), \mathbf{Y}(t), \mathbf{X}_2(t) \) and \( \mathbf{Y}_2(t) \) as given in Theorems 3.2.1 and 3.2.2 under Case 4 are used in expanding the elements of the matrices \( M^*, M_1(l) \) and \( M_2(l) \); \( l = 1, 2, \ldots, 5 \). The determinants of these matrices say, \( D_4, D_{41}(l) \) and \( D_{42}(l) \); \( l = 1, 2, \ldots, 5 \) are computed. These are then substituted in the definition of \( (\hat{\theta}_{ij} - \theta_{ij}); \ i = 1, 2; \ j = 1, 2, \ldots, 5 \) in (3.3.29), resulting in the following.

\[
\rho_0^{-4N} \rho_3^{-2N} \rho_4^{-2N} D_4 = d_0 + o_P(1)
\]
\[
\rho_4^N (\hat{\theta}_{11} - \theta_{11}) = \left( \frac{1}{\pi_2 - \pi_2} \right) \frac{a_{11} p_{10}(1) + a_{12} p_{11}(1) + a_{13} d_{10}(1) + a_{14} e_{10}(1)}{H_{33} H_{34}} + o_P(1)
\]
\[
\rho_3^N (\hat{\theta}_{12} - \theta_{12}) = \left( \frac{1}{\pi_2 - \pi_2} \right) \frac{a_{21} p_{10}(1) + a_{22} p_{11}(1) + a_{23} d_{10}(1) + a_{24} e_{10}(1)}{H_{33}} + o_P(1)
\]
\[
\frac{\rho_0^N}{N} (\hat{\theta}_{13} - \theta_{13}) = \left( \frac{1}{\pi_2 - \pi_1} \right) \frac{a_{31} p_{10}(1) + a_{32} p_{11}(1) + a_{33} d_{10}(1) + a_{34} e_{10}(1)}{H_{11} H_{12}} + o_P(1)
\]
\[
\rho_6^N (\hat{\theta}_{14} - \theta_{14}) = \left( \frac{1}{\pi_2 - \pi_1} \right) \frac{a_{41} p_{10}(1) + a_{42} p_{11}(1) + a_{43} d_{10}(1) + a_{44} e_{10}(1)}{H_{11}} + o_P(1)
\]
\[ \rho_4^N (\hat{\theta}_{21} - \theta_{21}) = \frac{1}{(\pi_2 - \pi_2)} \frac{a_{11}p_{20}(1) + a_{12}p_{21}(1) + a_{13}d_{20}(1) + a_{14}e_{20}(1)}{H_{33}H_{34}} + o_P(1) \]

\[ \rho_3^N (\hat{\theta}_{22} - \theta_{22}) = \frac{a_{21}p_{20}(1) + a_{22}p_{21}(1) + a_{23}d_{20}(1) + a_{24}e_{20}(1)}{H_{33}} + o_P(1) \]

\[ \frac{\rho_0^N}{N} (\hat{\theta}_{23} - \theta_{23}) = \frac{1}{(\pi_2 - \pi_1)} \frac{a_{31}p_{20}(1) + a_{32}p_{21}(1) + a_{33}d_{20}(1) + a_{34}e_{20}(1)}{H_{11}H_{12}} + o_P(1) \]

\[ \frac{\rho_0^N}{N} (\hat{\theta}_{24} - \theta_{24}) = \frac{a_{41}p_{20}(1) + a_{42}p_{21}(1) + a_{43}d_{20}(1) + a_{44}e_{20}(1)}{H_{11}} + o_P(1) \] (3.3.57)

\[ \sqrt{N} (\hat{\theta}_{15} - \theta_{15}) = N^{-1/2} \sum_{t=1}^{N-2} \epsilon_1(t) + o_P(1) \]

\[ \sqrt{N} (\hat{\theta}_{25} - \theta_{25}) = N^{-1/2} \sum_{t=1}^{N-2} \epsilon_2(t) + o_P(1) \] (3.3.58)

where \(a_{ij}s\) and \(d_0\) are

\[ d_4 = \frac{(-H_{33}p_{34}+p_{33}H_{44})p_{33}H_{11}^2H_{55}^2(p_3-p_0)^2(p_4-p_0)^2(p_5-p_0)^2(-H_{11}p_{34}+p_{11}H_{44})^2}{p_0^2(p_0-p_1)^2(p_0-p_2)^2(p_0-p_3)^2} \]

\[ a_{11} = \frac{(p_0-p_1)^2(p_0-p_1)(p_0-p_1)(p_0-p_1)(p_0-p_1)(p_0-p_1)}{p_0^4(p_3-p_0)^2(p_4-p_0)^2(p_5-p_0)^2} \cdot \frac{\text{numerator 11}}{p_0^4(p_3-p_0)^2(p_4-p_0)^2(p_5-p_0)^2} \]

\[ a_{12} = \frac{(p_0-p_1)^2(p_0-p_1)(p_0-p_1)(p_0-p_1)(p_0-p_1)(p_0-p_1)}{p_0^4(p_3-p_0)^2(p_4-p_0)^2(p_5-p_0)^2} \cdot \frac{\text{numerator 21}}{p_0^4(p_3-p_0)^2(p_4-p_0)^2(p_5-p_0)^2} \]

\[ a_{13} = \frac{(p_0^2+p_2-p_0)p_0(p_0-p_1)(p_0-p_1)(p_0-p_1)(p_0-p_1)(p_0-p_1)}{p_0^4(p_3-p_0)^2(p_4-p_0)^2(p_5-p_0)^2} \]

\[ a_{14} = \frac{(p_0-p_1)^2(p_0-p_1)(p_0-p_1)}{p_0^4(p_3-p_0)^2(p_4-p_0)^2(p_5-p_0)^2} \]

\[ a_{21} = \frac{(p_0-p_1)(p_0-p_1)(p_0-p_1)(p_0-p_1)(p_0-p_1)(p_0-p_1)}{(-p_0+p_3)(-p_0+p_3)(-p_0+p_3)(-p_0+p_3)(-p_0+p_3)(-p_0+p_3)} \cdot \frac{\text{numerator 21}}{(-p_0+p_3)(-p_0+p_3)(-p_0+p_3)(-p_0+p_3)(-p_0+p_3)(-p_0+p_3)} \]

\[ a_{22} = \frac{(p_0-p_1)^2(p_0-p_1)(p_0-p_1)(p_0-p_1)(p_0-p_1)(p_0-p_1)}{p_0(-p_0+p_3)(-p_0+p_3)(-p_0+p_3)(-p_0+p_3)(-p_0+p_3)(-p_0+p_3)} \]

\[ a_{23} = \frac{(p_0-p_1)^2(p_0-p_1)(p_0-p_1)}{p_0^2(p_3-p_0)^2(p_4-p_0)^2(p_5-p_0)^2} \]

\[ a_{24} = \frac{(p_0-p_1)^2(p_0-p_1)(p_0-p_1)(p_0-p_1)(p_0-p_1)(p_0-p_1)}{(-p_0+p_3)(-p_0+p_3)(-p_0+p_3)(-p_0+p_3)(-p_0+p_3)(-p_0+p_3)} \cdot \frac{\text{numerator 31}}{(-p_0+p_3)(-p_0+p_3)(-p_0+p_3)(-p_0+p_3)(-p_0+p_3)(-p_0+p_3)} \]

\[ a_{31} = \frac{(p_0^2-p_0)(p_0-p_1)(p_0-p_1)(p_0-p_1)(p_0-p_1)(p_0-p_1)}{p_0^2(p_3-p_0)^2(p_4-p_0)^2(p_5-p_0)^2} \]

\[ a_{32} = \frac{(p_0-p_1)^2(p_0-p_1)(p_0-p_1)(p_0-p_1)(p_0-p_1)(p_0-p_1)}{p_0(-p_0+p_3)(-p_0+p_3)(-p_0+p_3)(-p_0+p_3)(-p_0+p_3)(-p_0+p_3)} \]
and for \( i = 1, 2 \),

\[
d_{i0}(1) = \sum \rho^u_i \varepsilon_i (N - u + 1) \quad \text{and} \quad \varepsilon_{i0}(1) = \sum \rho^u_i \varepsilon_i (N - u + 1),
\]

where

\[
\text{numerator11} = 2 \rho^3 \rho^3 \rho^4 - 3 \rho^3 \rho^4 \rho^2 + \rho^3 \rho^4 + 2 \rho^5 \rho^4 - \rho^4 \rho^4 \rho^4
\]

\[
+ 3 \rho^2 \rho^4 \rho^2 - \rho^2 \rho^4 - 2 \rho^2 \rho^2 \rho^4 + \rho^5 \rho^4 \rho^2 + \rho^5 \rho^2 \rho^4
\]

\[
- 2 \rho \rho \rho^4 \rho^2 - 2 \rho \rho \rho^4 \rho^2 + 2 \rho \rho \rho^4 \rho^2 + \rho \rho \rho^2 \rho^2 + \rho \rho \rho^2 \rho^2
\]

\[
\text{numerator21} = 3 \rho^3 \rho^2 \rho^3 - \rho^2 \rho^3 - 2 \rho \rho \rho^2 + \rho \rho \rho^2 - 2 \rho \rho \rho^3
\]

\[
+ \rho - 2 \rho + 2 \rho \rho \rho^2 + 2 \rho \rho \rho^2 - 2 \rho \rho \rho^3 + \rho
\]

\[
\text{numerator31} = -\rho^3 \rho^2 \rho^2 - \rho^4 \rho^2 \rho^2 + \rho^3 \rho^2 \rho^2 - 2 \rho \rho \rho^4 \rho^2 + \rho \rho \rho^3 \rho^2
\]

\[
+ \rho^3 \rho \rho^2 + \rho^3 \rho^2 \rho^2 - \rho^3 \rho^2 \rho^2 + 2 \rho \rho \rho \rho^3 \rho^2 + \rho \rho \rho^3 \rho^2
\]

\[
+ \rho \rho \rho^3 + 3 \rho \rho \rho^3 \rho^2 - 2 \rho \rho \rho^2 \rho^2 - 2 \rho \rho \rho^2 \rho^2 - \rho \rho \rho^2 \rho^2 + \rho \rho \rho^2 \rho^2
\]

\[
\text{numerator41} = -\rho^3 \rho^2 - \rho \rho \rho^2 - \rho \rho \rho^4 \rho^2 + \rho \rho \rho \rho^4 \rho^2 - 2 \rho \rho \rho \rho^4 \rho^2 + \rho \rho \rho \rho^4 \rho^2
\]

\[
+ \rho^3 \rho \rho^2 + \rho \rho \rho^2 - \rho \rho \rho^4 \rho^2 + 2 \rho \rho \rho \rho^3 \rho^2 + \rho \rho \rho^3 \rho^2
\]

\[
+ \rho \rho \rho^3 + 3 \rho \rho \rho^3 \rho^2 - 2 \rho \rho \rho^2 \rho^2 - 2 \rho \rho \rho^2 \rho^2 - \rho \rho \rho^2 \rho^2 + \rho \rho \rho^2 \rho^2
\]

From the results in (3.3.57), we see that statement (a) of the theorem holds.

Proof of (b) goes on exactly similar lines to the proof of Theorem 3.3.2-(b).

From results in (3.3.58), proof of statement (c) goes on exactly similar lines to the proof of Theorem 3.3.2-(c).

Below is the main theorem under Case 4.

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Theorem 3.3.9. Let $\hat{\beta}' = (\hat{\beta}'_1, \hat{\beta}'_2, \hat{\beta}'_3)$ be the least squares estimators of $\beta' = (\beta'_1, \beta'_2, \beta'_3)$, the vector of structural parameters of the model in (3.2.1) satisfying the assumptions of (3.2.2). Then the following statements hold under the placement $\rho_1 = \rho_2 = \rho_0; |\rho_0| > |\rho_3| > |\rho_4| > 1$ relating to the roots of $P(z) = 0$.

a. $\left(\rho_4^N \left(\tilde{\beta}_e - \beta_e\right)\right)'$ converges in probability, as $N \to \infty$, to a random vector $(\zeta_{11}(4), \zeta_{12}(4), \zeta_{13}(4), \zeta_{14}(4), \zeta_{21}(4), \zeta_{22}(4), \zeta_{23}(4), \zeta_{24}(4))$ (say).

b. Each of $\zeta_{ij}(4)$ is distributed like a ratio of two random variables of the type $\sum_{i=1}^{4} (c_i \pi_{ij}) / J$, wherein $\pi_{ij}$ and $J$ are independent and are distributed like specific linear combinations of $\varepsilon_i(t); i = 1,2; t = 1,2, \ldots$.

c. $\sqrt{N} \left(\beta_{10} - \beta_{10}\right), \sqrt{N} \left(\beta_{20} - \beta_{20}\right)$ converges in distribution, as $N \to \infty$, to a bivariate random vector with mean vector a null vector and a diagonal covariance matrix.

d. (i) $\bar{\pi}_1 \zeta_{11}(4) + \zeta_{12}(4) = 0$ a.s.

(ii) $a_{11}^{(1)} \zeta_{11}(4) + a_{21}^{(1)} \zeta_{12}(4) + \zeta_{13}(4) = 0$ a.s.

(iii) $(a_{12}^{(1)} + \pi_1 a_{11}^{(1)}) \zeta_{11}(4) + (a_{21}^{(1)} \pi_1 + a_{22}^{(1)}) \zeta_{12}(4) + \pi_1 \zeta_{13}(4) + \zeta_{14}(4) = 0$ a.s.

(iv) $\bar{\pi}_1 \zeta_{21}(4) + \zeta_{22}(4) = 0$ a.s.

(v) $a_{11}^{(1)} \zeta_{21}(4) + a_{21}^{(1)} \zeta_{22}(4) + \zeta_{23}(4) = 0$ a.s.

(vi) $(a_{12}^{(1)} + \pi_1 a_{11}^{(1)}) \zeta_{21}(4) + (a_{21}^{(1)} \pi_1 + a_{22}^{(1)}) \zeta_{22}(4) + \pi_1 \zeta_{23}(4) + \zeta_{24}(4) = 0$ a.s.

Proof. Using the inverse transformation (3.3.27), we can express the centered least squares estimators $(\hat{\beta}_{1c} - \beta_{1c})$ as

$$\hat{\beta}_{11}^{(1)} - \beta_{11}^{(1)} = \frac{1}{\rho_4^N} \left[\rho_4^N \left(\hat{\theta}_{11} - \theta_{11}\right)\right]$$

$$\hat{\beta}_{12}^{(1)} - \beta_{12}^{(1)} = \frac{1}{\rho_3^N} \left[\rho_3^N \left(\hat{\theta}_{12} - \theta_{12}\right)\right] - \pi_2 \frac{1}{\rho_4^N} \left[\rho_4^N \left(\hat{\theta}_{11} - \theta_{11}\right)\right]$$

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Applying statement (a) of the Theorem 3.3.8, the expressions in square brackets of (3.3.59) are bounded in probability and we see that the least weight that stabilises each of \((\hat{\beta}^{(k)}_{1j} - \beta^{(k)}_{1j})\); \(j = 1, 2\); \(k = 1, 2\) is \(\rho^N_k\). Similar results can be obtained for \((\hat{\beta}^{(k)}_{2j} - \beta^{(k)}_{2j})\); \(j = 1, 2\); \(k = 1, 2\). Hence (a) of the theorem is proved.

Proofs for statements (b), (c) and (d) are on exactly similar lines of the proofs of (b), (c) and (d) of the Theorem 3.3.3.

Case 5: \(\rho_1 = \rho_2 = \rho_3 = \rho_0\); \(|\rho_0| > |\rho_4| > 1\)

Theorem 3.3.10. Let \(\{Z(t) = (X(t), Y(t))^T\}\) be generated from the model (3.2.1) and \((\hat{\theta}', \hat{\theta}') = (\hat{\theta}_{11}, \hat{\theta}_{12}, \hat{\theta}_{13}, \hat{\theta}_{14}, \hat{\theta}_{15}, \hat{\theta}_{21}, \hat{\theta}_{22}, \hat{\theta}_{23}, \hat{\theta}_{24}, \hat{\theta}_{25})\) be the least squares estimator of the parameter vector \((\theta_1', \theta_2')\) of the reparametrised model in (3.3.22). Then, under the assumptions of (3.2.2), governed by the model, the following statements hold under the placement \(\rho_1 = \rho_2 = \rho_0\); \(|\rho_0| > |\rho_3| > |\rho_4| > 1\) relating to the roots of \(P(z) = 0\).

a. \(\left(\rho^N_3 (\hat{\theta}_{11} - \theta_{11}), \rho^N_3 (\hat{\theta}_{12} - \theta_{12}), \frac{\rho^N_2}{N} (\hat{\theta}_{13} - \theta_{13}), \rho^N_0 (\hat{\theta}_{14} - \theta_{14}), \rho^N_4 (\hat{\theta}_{21} - \theta_{21}), \frac{\rho^N_2}{N} (\hat{\theta}_{22} - \theta_{22}), \frac{\rho^N_2}{N} (\hat{\theta}_{23} - \theta_{23}), \rho^N_0 (\hat{\theta}_{24} - \theta_{24})\right)\) converges in probability as \(N \to \infty\), to the random vector \((\xi_{11}(5), \xi_{12}(5), \xi_{13}(5), \xi_{14}(5), \xi_{21}(5), \xi_{22}(5), \xi_{23}(5), \xi_{24}(5))\) (say).

b. Each of \(\xi_{ij}(5)\) is distributed like a ratio of two random variables of the generic type \(\sum_{i=1}^{4} (c_i \pi_{ij}) / J\) wherein \(\pi_{ij}\) and \(J\) are independent and are distributed like
specific linear combinations of $\varepsilon_i(t); \ i = 1, 2; \ t = 1, 2, \ldots$

c. $\left(\sqrt{N} \left(\hat{\theta}_{15} - \theta_{15}\right), \sqrt{N} \left(\hat{\theta}_{25} - \theta_{25}\right)\right)$ converges in to a bivariate normal random vector with mean vector zero and a diagonal covariance matrix.

Proof. The explicit solutions $X(t), Y(t), X_2(t)$ and $Y_2(t)$ as given in Theorems 3.2.1 and 3.2.2 under Case 4 are used in expanding the elements of the matrices $M^*, M_1(l)$ and $M_2(l); \ l = 1, 2, \ldots, 5$. The determinants of these matrices say, $D_5, D_{51}(l)$ and $D_{52}(l); \ l = 1, 2, \ldots, 5$ are computed. These are then substituted in the definition of $\left(\hat{\theta}_{ij} - \theta_{ij}\right); \ i = 1, 2; \ j = 1, 2, \ldots, 5$ given in (3.3.29), resulting in the following.

$$\rho_0^{-6N} \rho_4^{-2N} D_5 = d (\bar{\pi}_2 - \bar{\pi}_1)^2 (\pi_2 - \pi_1)^2 H_{33} H_{34} H_{11} H_{12} + o_P(1)$$

$$\rho_4^N \left(\hat{\theta}_{11} - \theta_{11}\right) = \rho_4^N \frac{D_{51}(1)}{D_5} = \frac{1}{(\bar{\pi}_2 - \bar{\pi}_1)} \frac{a_{11}p_{10}(1) + a_{12}p_{11}(1) + a_{13}p_{12}(1) + a_{14}e_{10}(1)}{H_{33} H_{34}} + o_P(1)$$

$$\frac{\rho_4^N}{N^2} \left(\hat{\theta}_{12} - \theta_{12}\right) = \rho_0^N N^{-2} \frac{D_{51}(2)}{D_5} = \frac{a_{21}p_{10}(1) + a_{22}p_{11}(1) + a_{33}p_{12}(1) + a_{44}e_{10}(1)}{H_{33}} + o_P(1)$$

$$\frac{\rho_4^N}{N} \left(\hat{\theta}_{13} - \theta_{13}\right) = \rho_0^N \frac{D_{51}(3)}{N} \frac{D_{51}(3)}{D_5} = \frac{1}{\pi_2 - \pi_1} \frac{a_{31}p_{10}(1) + a_{32}p_{11}(1) + a_{33}p_{12}(1) + a_{34}e_{10}(1)}{H_{11} H_{12}} + o_P(1)$$

$$\rho_4^N \left(\hat{\theta}_{14} - \theta_{14}\right) = \rho_4^N \frac{D_{51}(4)}{D_5} = \frac{a_{41}p_{10}(1) + a_{42}p_{11}(1) + a_{43}p_{12}(1) + a_{44}e_{10}(1)}{H_{11}} + o_P(1)$$

$$\rho_4^N \left(\hat{\theta}_{21} - \theta_{21}\right) = \rho_4^N \frac{D_{52}(1)}{D_5} = \frac{1}{(\bar{\pi}_2 - \bar{\pi}_1)} \frac{a_{11}p_{20}(1) + a_{12}p_{21}(1) + a_{13}p_{22}(1) + a_{14}p_{20}(1)}{H_{33} H_{34}} + o_P(1)$$

$$\frac{\rho_4^N}{N^2} \left(\hat{\theta}_{22} - \theta_{22}\right) = \rho_0^N \frac{D_{52}(2)}{N^2} \frac{D_{52}(2)}{D_5} = \frac{a_{21}p_{20}(1) + a_{22}p_{21}(1) + a_{23}p_{22}(1) + a_{24}e_{20}(1)}{H_{33}} + o_P(1)$$

$$\frac{\rho_4^N}{N} \left(\hat{\theta}_{23} - \theta_{23}\right) = \rho_0^N \frac{D_{52}(3)}{N} \frac{D_{52}(3)}{D_5} = \frac{1}{\pi_2 - \pi_1} \frac{a_{31}p_{20}(1) + a_{32}p_{21}(1) + a_{33}p_{22}(1) + a_{34}e_{20}(1)}{H_{11} H_{12}} + o_P(1)$$

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\[ \rho_0^N (\hat{\theta}_{24} - \theta_{24}) = \rho_0^N \frac{D_{22}(4)}{D_5} = a_{41} p_{20}(1) + a_{42} p_{21}(1) + a_{43} p_{22}(1) + a_{44} e_{20}(1) \]
\[ + o_P(1) \quad (3.3.60) \]
\[ \sqrt{N} (\hat{\theta}_{15} - \theta_{15}) = N^{-1/2} \sum_{t=1}^{N-2} e_1(t) + o_P(1) \]
\[ \sqrt{N} (\hat{\theta}_{26} - \theta_{25}) = N^{-1/2} \sum_{t=1}^{N-2} e_2(t) + o_P(1) \quad (3.3.61) \]

where \( a_{ij} \)'s and \( d \) are constants given by

\[ d_5 = \frac{4 \rho_0^6 (\rho_4^3 - \rho_0^3) (\rho_4^4 - \rho_0^4)^6}{(\rho_0 \rho_4^4 - \rho_0^4)^6 (\rho_0^4 + 1)^3} \]
\[ a_{11} = - \frac{(\rho_4^2 - 1)^2 (\rho_0^2 - 1)^2 (\rho_0 \rho_4^3 - \rho_0^3)^3 (\rho_4^2 - 3 \rho_0 \rho_4^4 + 3 \rho_0^3 \rho_4^2 + \rho_0^3 \rho_4^2 - 3 \rho_0^3 \rho_4)}{\rho_4^3 \rho_0^3 (\rho_4^4 - \rho_0^4)^3} \]
\[ a_{12} = \frac{1}{2} \frac{3 \rho_0^3 \rho_4^4 + 3 \rho_0^3 - 3 \rho_0 - \rho_4^3 (\rho_0 \rho_4^3 - \rho_0^3)^3 (\rho_4^2 - 1)}{\rho_4^3 (\rho_4^4 - \rho_0^4)^3 \rho_0^3} \]
\[ a_{13} = - \frac{1}{2} \frac{(\rho_0 \rho_4^3 - \rho_0^3)^3 (\rho_4^2 - 1)}{\rho_4^3 (\rho_4^4 - \rho_0^4)^3 \rho_0^3} \]
\[ a_{14} = \frac{(\rho_0 \rho_4^3 - \rho_0^3)^3 (\rho_4^2 - 1)}{\rho_4^3 (\rho_4^4 - \rho_0^4)^3 \rho_0^3} \]
\[ a_{21} = \frac{1}{2} \frac{(\rho_0^2 - 1)^3 (\rho_0 \rho_4^3 - \rho_0^3)^3 (\rho_4^2 - 3 \rho_0 \rho_4^4 + 3 \rho_0^3 \rho_4^2 + \rho_0^3 \rho_4^2 - 3 \rho_0^3 \rho_4)}{\rho_0^6 (\rho_4^4 - \rho_0^4)^3} \]
\[ a_{22} = - \frac{1}{4} \frac{(\rho_0 \rho_4^3 - \rho_0^3)^3 (\rho_4^2 - 3 \rho_0 \rho_4^4 + 3 \rho_0^3 \rho_4^2 - 3 \rho_0^3 \rho_4)}{\rho_0^6 (\rho_4^4 - \rho_0^4)^3} \]
\[ a_{23} = \frac{1}{4} \frac{(\rho_0 \rho_4^3 - \rho_0^3)^3 (\rho_0^2 - 1)^5}{\rho_0^6 (\rho_4^4 - \rho_0^4)^3} \]
\[ a_{24} = - \frac{1}{2} \frac{(\rho_0 \rho_4^3 - \rho_0^3)^3 (\rho_4^2 - 1)}{\rho_0^6 (\rho_4^4 - \rho_0^4)^3 \rho_4^3} \]
\[ a_{31} = \frac{1}{2} \frac{(\rho_0^2 - 1)^4 (\rho_0 \rho_4^3 - \rho_0^3)^3 (\rho_4^2 - 3 \rho_0 \rho_4^4 + 3 \rho_0^3 \rho_4^2 - 3 \rho_0 \rho_4^4 + \rho_0^3 \rho_4^2 + 4 \rho_4^3 \rho_0^3 - \rho_0^3 \rho_4 - 5 \rho_0 + 3 \rho_4)}{\rho_0^6 (\rho_4^4 - \rho_0^4)^3} \]
\[ a_{32} = \frac{1}{2} \frac{(\rho_0 \rho_4^3 - \rho_0^3)^3 (\rho_4^2 - 1)}{\rho_0^6 (\rho_4^4 - \rho_0^4)^3 \rho_4^3} \]
\[ a_{33} = - \frac{1}{2} \frac{(\rho_0^2 - 1)^5 (\rho_0 \rho_4^3 - \rho_0^3)^3}{\rho_0^6 (\rho_4^4 - \rho_0^4)^3 \rho_4^3} \]
\[ a_{34} = \frac{(\rho_0 \rho_4^3 - \rho_0^3)^3 (\rho_4^2 - 1)}{\rho_0^6 (\rho_4^4 - \rho_0^4)^3 \rho_4^3} \]
\[ a_{41} = \frac{1}{2} \frac{(\rho_0^2 - 1)^4 (\rho_0 \rho_4^3 - \rho_0^3)^3 (\rho_4^2 - 3 \rho_0 \rho_4^4 + 3 \rho_0^3 \rho_4^2 - 3 \rho_0 \rho_4^4 + \rho_0^3 \rho_4^2 + 4 \rho_4^3 \rho_0^3 - \rho_0^3 \rho_4 - 5 \rho_0 + 3 \rho_4)}{\rho_0^6 (\rho_4^4 - \rho_0^4)^3 \rho_4^3} \]
\[ a_{42} = - \frac{1}{4} \frac{(\rho_0 \rho_4^3 - \rho_0^3)^3 (\rho_4^2 - 1)}{\rho_0^6 (\rho_4^4 - \rho_0^4)^3 \rho_4^3} \]
\[ a_{43} = \frac{1}{4} \frac{(\rho_0 \rho_4^3 - \rho_0^3)^3 (\rho_4^2 - 1)}{\rho_0^6 (\rho_4^4 - \rho_0^4)^3 \rho_4^3} \]
\[ a_{44} = - \frac{1}{2} \frac{(\rho_0 \rho_4^3 - \rho_0^3)^3 (\rho_4^2 - 1)}{\rho_0^6 (\rho_4^4 - \rho_0^4)^3 \rho_4^3} \]
From the results in (3.3.60), we see that statement (a) of the theorem holds.

Proof of (b) goes on exactly similar lines to the proof of Theorem 3.3.2-(b).

From result (3.3.61), proof of statement (c) goes on exactly similar lines to the proof of Theorem 3.3.2-(c).

Now, we state and prove the main theorem under Case 5.

**Theorem 3.3.11.** Let \( \hat{\beta}' = (\hat{\beta}_c', \hat{\beta}_{10}, \hat{\beta}_{20}) = (\hat{\beta}^{(1)}_{11}, \hat{\beta}^{(1)}_{12}, \hat{\beta}^{(2)}_{11}, \hat{\beta}^{(2)}_{12}, \hat{\beta}^{(1)}_{21}, \hat{\beta}^{(2)}_{21}, \hat{\beta}^{(1)}_{22}, \hat{\beta}^{(2)}_{22}, \hat{\beta}_{10}, \hat{\beta}_{20}) \) be the least squares estimators of \( \beta' = (\beta_c', \beta_{10}, \beta_{20}) = (\beta^{(1)}_{11}, \beta^{(1)}_{12}, \beta^{(2)}_{11}, \beta^{(2)}_{12}, \beta^{(1)}_{21}, \beta^{(2)}_{21}, \beta^{(1)}_{22}, \beta^{(2)}_{22}, \beta_{10}, \beta_{20}) \), the vector of structural parameters of the model in (3.2.1) satisfying the assumptions of (3.2.2). Then the following statements hold under the placement \( \rho_1 = \rho_2 = \rho_0; \quad |\rho_0| > |\rho_3| > |\rho_4| > 1 \) relating to the roots of \( P(z) = 0 \).

a. \( \left\{ \rho_N^N \left( \hat{\beta}' - \beta' \right)' \right\} \) converges in probability, as \( N \to \infty \), to a random vector \( \{\zeta_{11}(5), \zeta_{12}(5), \zeta_{13}(5), \zeta_{14}(5), \zeta_{21}(5), \zeta_{22}(5), \zeta_{23}(5), \zeta_{24}(5)\} \) (say).

b. Each of \( \zeta_{ij}(5) \) is distributed like a ratio of two random variables of the type \( \sum_{i=1}^{4} (c_i \xi_{ij}) / J \), wherein \( \xi_{ij} \) and \( J \) are independent and are distributed like specific linear combinations of \( e_i(t) \); \( i = 1, 2; \quad t = 1, 2, \ldots \).

c. \( \left( \sqrt{N} \left( \hat{\beta}_{10} - \beta_{10} \right), \sqrt{N} \left( \hat{\beta}_{20} - \beta_{20} \right) \right) \) converges in distribution, as \( N \to \infty \), to a bivariate random vector with mean vector a null vector and a diagonal covariance matrix.

(i) \( \bar{\pi}_1 \zeta_{11}(5) + \zeta_{12}(5) = 0 \) a.s.

(ii) \( a_{11}^{(1)} \zeta_{11}(5) + a_{21}^{(1)} \zeta_{12}(5) + \zeta_{13}(5) = 0 \) a.s.

(iii) \( \left( a_{12}^{(1)} + \pi_1 a_{11}^{(1)} \right) \zeta_{11}(5) + \left( a_{21}^{(1)} \pi_1 + a_{22}^{(1)} \right) \zeta_{12}(5) + \pi_1 \zeta_{13}(5) + \zeta_{14}(5) = 0 \) a.s.

(iv) \( \bar{\pi}_1 \zeta_{21}(5) + \zeta_{22}(5) = 0 \) a.s.

(v) \( a_{11}^{(1)} \zeta_{21}(5) + a_{21}^{(1)} \zeta_{22}(5) + \zeta_{23}(5) = 0 \) a.s.

(vi) \( a_{12}^{(1)} + \pi_1 a_{11}^{(1)} \zeta_{21}(5) + \left( a_{21}^{(1)} \pi_1 + a_{22}^{(1)} \right) \zeta_{22}(5) + \pi_1 \zeta_{23}(5) + \zeta_{24}(5) = 0 \) a.s.
Proof. Using the relationship between \( \left( \hat{\beta}_c - \beta_c \right) \) and \( \left( \hat{\theta}_c - \theta_c \right) \), and rate of convergence of \( \left( \hat{\beta}_c - \theta_c \right) \) under Case 5 studied in Theorem 3.3.10, we have

\[
\begin{align*}
\hat{\beta}_{11}^{(1)} - \beta_{11}^{(1)} &= \frac{1}{\rho_4^N} \left[ \rho_4^N \left( \hat{\beta}_{11} - \beta_{11} \right) \right] \\
\hat{\beta}_{12}^{(1)} - \beta_{12}^{(1)} &= \frac{N^2}{\rho_0^N} \left[ \rho_0^N \left( \hat{\beta}_{12} - \beta_{12} \right) \right] - \pi_1 \frac{1}{\rho_4^N} \left[ \rho_4^N \left( \hat{\beta}_{11} - \beta_{11} \right) \right] \\
\hat{\beta}_{11}^{(2)} - \beta_{11}^{(2)} &= \frac{N}{\rho_0^N} \left[ \rho_0^N \left( \hat{\beta}_{11} - \beta_{11} \right) \right] - \pi_1 \frac{N^2}{\rho_0^N} \left[ \rho_0^N \left( \hat{\beta}_{12} - \beta_{12} \right) \right] \\
\quad + \left( -a_{11}^{(1)} + \pi_1 a_{21}^{(1)} \right) \frac{1}{\rho_4^N} \left[ \rho_4^N \left( \hat{\beta}_{11} - \beta_{11} \right) \right] \\
\hat{\beta}_{12}^{(2)} - \beta_{12}^{(2)} &= \frac{1}{\rho_0^N} \left[ \rho_0^N \left( \hat{\beta}_{12} - \beta_{12} \right) \right] - \pi_1 \frac{N}{\rho_0^N} \left[ \rho_0^N \left( \hat{\beta}_{11} - \beta_{11} \right) \right] \\
\quad - a_{22}^{(1)} \frac{N^2}{\rho_0^N} \left[ \rho_0^N \left( \hat{\beta}_{12} - \beta_{12} \right) \right] + \left( -a_{12}^{(1)} + \pi_1 a_{22}^{(1)} \right) \frac{1}{\rho_4^N} \left[ \rho_4^N \left( \hat{\beta}_{11} - \beta_{11} \right) \right] \\
&= (3.3.62)
\end{align*}
\]

Applying statement (a) of the Theorem 3.3.10, the expressions in square brackets are bounded in probability and we see that the least weight that stabilises each of \( \left( \beta_{1j}^{(k)} - \beta_{1j}^{(k)} \right); \ j = 1, 2; \ k = 1, 2 \) is \( \rho_4^N \).

Similar results can be proved for \( \left( \beta_{2j}^{(k)} - \beta_{2j}^{(k)} \right); \ j = 1, 2; \ k = 1, 2 \). Hence (a) of the theorem is proved.

Proofs for statements (b), (c) and (d) are on exactly similar lines of the proofs of (b), (c) and (d) of the Theorem 3.3.3.

Case 6: \( \rho_1 = \rho_2 = \rho_0; \rho_3 = \rho_4 = \rho_0; \ |\rho_0| > |\rho_0| > 1 \)

Theorem 3.3.12. Let \( \{ Z(t) = (X(t), Y(t))' \} \) be generated from the model (3.2.1) and \( \left( \hat{\theta}_1', \hat{\theta}_2' \right) = \left( \hat{\theta}_{11}, \hat{\theta}_{12}, \hat{\theta}_{13}, \hat{\theta}_{14}, \hat{\theta}_{15}, \hat{\theta}_{21}, \hat{\theta}_{22}, \hat{\theta}_{23}, \hat{\theta}_{24}, \hat{\theta}_{25} \right) \) be the least squares estimator of the parameter vector \( (\theta_1', \theta_2') \) of the reparametrised model in (3.3.22). Then, under the assumptions of (3.2.2), governed by the model, the following statements hold under the placement \( \rho_1 = \rho_2 = \rho_0 = \rho_4 = \rho_0; \ |\rho_0| > |\rho_0| > 1 \) relating to the roots of \( P(z) = 0 \).
a. \( \left( \frac{\rho_0^N}{N} (\hat{\theta}_{11} - \theta_{11}), \frac{\rho_0^N}{N} (\hat{\theta}_{12} - \theta_{12}), \frac{\rho_0^N}{N} (\hat{\theta}_{13} - \theta_{13}), \frac{\rho_0^N}{N} (\hat{\theta}_{14} - \theta_{14}) \right) \), \( \left( \frac{\rho_0^N}{N} (\hat{\theta}_{21} - \theta_{21}), \frac{\rho_0^N}{N} (\hat{\theta}_{22} - \theta_{22}), \frac{\rho_0^N}{N} (\hat{\theta}_{23} - \theta_{23}), \frac{\rho_0^N}{N} (\hat{\theta}_{24} - \theta_{24}) \right) \) converges in probability as \( N \to \infty \) to the random vector \( \{\xi_{11}(6), \xi_{12}(6), \xi_{13}(6), \xi_{14}(6), \xi_{21}(6), \xi_{22}(6), \xi_{23}(6), \xi_{24}(6)\} \) (say).

b. Each of \( \xi_{ij}(6) \) is distributed like a ratio of two random variables of the generic type \( \sum_{i=1}^{4} (c_i \pi_{ij}) / J \) wherein \( \pi_{ij} \) and \( J \) are independent and are distributed like specific linear combinations of \( \epsilon_i(t); \ i = 1, 2; \ t = 1, 2, \ldots \).

c. \( \left( \sqrt{N} (\hat{\theta}_{15} - \theta_{15}), \sqrt{N} (\hat{\theta}_{26} - \theta_{26}) \right) \) converges to a bivariate normal random vector with mean vector zero and a diagonal covariance matrix.

Proof. The explicit solutions \( X(t), Y(t), X_2(t) \) and \( Y_2(t) \) as given in Theorems 3.2.1 and 3.2.2 under Case 6 are used in expanding the elements of the matrices \( M^*, M_1(l) \) and \( M_2(l); \ l = 1, 2, \ldots, 5 \). The determinants of these matrices say, \( D_6, D_{61}(l) \) and \( D_{62}(l); \ l = 1, 2, \ldots, 5 \) are computed. These are then substituted in the definition of \( \left( \hat{\theta}_{ij} - \theta_{ij} \right); \ i = 1, 2; \ j = 1, 2, \ldots, 5 \), resulting in the following.

\[
\rho_0^{-4N} \rho_1^{-4N} D_6 = d (\bar{\pi}_2 - \bar{\pi}_1)^2 (\pi_2 - \pi_1)^2 H_{11}^4 H_{33}^4 H_{12}^2 H_{34}^2 + o_P(1)
\]

where

\[
d = \frac{\rho_0^4 (\rho_0 - \rho_1)^8}{\rho_1^4 (\rho_1^2 - 1)^4 (\rho_0 \rho_1 - 1)^8 (\rho_0^2 - 1)^4}
\]

\[
\rho_0^N (\hat{\theta}_{11} - \theta_{11}) = \frac{\rho_0^N D_{61}(1)}{N D_6} = \frac{1}{(\bar{\pi}_2 - \bar{\pi}_1)} \frac{a_{11}m_{10}(1) + a_{12}m_{11}(1) + a_{13}p_{10}(1) + a_{14}p_{11}(1)}{H_{33} H_{34}} + o_P(1)
\]

\[
\rho_0^N (\hat{\theta}_{12} - \theta_{12}) = \frac{\rho_0^N D_{61}(1)}{N D_6} = \frac{a_{21}p_{10}(1) + a_{22}p_{11}(1) + a_{23}p_{10}(1) + a_{24}p_{11}(1)}{H_{33}} + o_P(1)
\]

\[
\rho_0^N (\hat{\theta}_{13} - \theta_{13}) = \frac{\rho_0^N D_{61}(1)}{N D_6} = \frac{1}{(\bar{\pi}_2 - \bar{\pi}_1)} \frac{a_{31}m_{10}(1) + a_{32}m_{11}(1) + a_{33}p_{10}(1) + a_{34}p_{11}(1)}{H_{11} H_{12}} + o_P(1)
\]

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\[ \rho_1^N (\theta_{14} - \theta_{14}) = \rho_1^N \frac{D_{54}(1)}{D_6} = \frac{a_{41}m_{10}(1) + a_{42}m_{11}(1) + a_{43}P_{10}(1) + a_{44}P_{11}(1)}{H_{11}} + o_P(1) \]

Similarly we get

\[ \rho_0^N \left( \hat{\theta}_{21} - \theta_{21} \right) = \frac{1}{(\pi_2 - \pi_2)} \frac{a_{11}m_{20}(1) + a_{12}m_{21}(1) + a_{13}P_{20}(1) + a_{14}P_{21}(1)}{H_{33}H_{34}} + o_P(1) \]

\[ \rho_0^N \left( \hat{\theta}_{22} - \theta_{22} \right) = \frac{a_{21}m_{20}(1) + a_{22}m_{21}(1) + a_{23}P_{20}(1) + a_{24}P_{21}(1)}{H_{33}} + o_P(1) \]

\[ \rho_1^N \left( \hat{\theta}_{23} - \theta_{23} \right) = \frac{1}{(\pi_2 - \pi_1)} \frac{a_{31}m_{20}(1) + a_{32}m_{21}(1) + a_{33}P_{20}(1) + a_{34}P_{21}(1)}{H_{11}H_{12}} + o_P(1) \]

\[ \rho_0^N \left( \hat{\theta}_{24} - \theta_{24} \right) = \frac{a_{41}m_{20}(1) + a_{42}m_{21}(1) + a_{43}P_{20}(1) + a_{44}P_{21}(1)}{H_{11}} + o_P(1) \] (3.3.63)

\[ \sqrt{N} (\hat{\theta}_{15} - \theta_{15}) = N^{-1/2} \sum_{t=1}^{N-2} e_1(t) + o_P(1) \]

\[ \sqrt{N} (\hat{\theta}_{25} - \theta_{25}) = N^{-1/2} \sum_{t=1}^{N-2} e_2(t) + o_P(1) \] (3.3.64)

\[ a_{11} = -\frac{(p_1^2 p_0^2 - 3 p_0^2 p_1^2 + 3 p_0 p_1 - 2 + p_0^3)(p_0^2 - 1)^2 (p_0 p_1 - 1)^2 (p_1^2 - 1)}{p_0^3 (-p_0 + p_1)^3} \]

\[ a_{12} = \frac{(p_0^2 - 1)^2 (p_0 p_1 - 1)^3 (p_1^2 - 1)^2}{p_0^2 (-p_0 + p_1)^3 p_1} \]

\[ a_{13} = \frac{3 p_0^2 p_1^2 - 2 p_1^2 - p_0^3 p_0 - 3 p_0 p_1 + 2 - p_0^3}{p_0^3 (-p_0 + p_1)^3} \]

\[ a_{14} = \frac{(p_0 p_1 - 1)^4 (p_0^2 - 1)^3}{p_1^4 p_0^3 (-p_1 + p_0)^3} \]

\[ a_{21} = \frac{3 p_0^2 p_1^3 - 2 p_1^3 - 3 p_0 p_1 + 2}{(p_0 p_1 - 1)^3 (p_1^2 - 1) (p_0^2 - 1)} \]

\[ a_{22} = \frac{-p_1^5 (p_0 p_1 - 1)^3 (p_1^2 - 1)^2 (p_0^2 - 1)^2}{(-p_1 + p_0)^3 p_0^3} \]

\[ a_{23} = \frac{p_0^2 p_1^3 - 3 p_0^2 p_1^2 + p_0^2 + p_0^2 - 2 + 2 p_1^2}{(p_0 p_1 - 1)^3 (p_1^2 - 1)^2 (p_0^2 - 1)^2} \]

\[ a_{24} = \frac{-p_1^4 (p_0 p_1 - 1)^4 (p_0^2 - 1)^3}{(-p_1 + p_0)^3 p_0^3} \]

\[ a_{31} = \frac{-2 p_0^3 + 3 p_0^3 p_1^2 - p_0^3 p_0 + 2 - p_1^2}{(p_0 p_1 - 1)^3 (p_1^2 - 1)^2 (p_0^2 - 1)^2} \]

\[ a_{32} = \frac{(p_0 p_1 - 1)^4 (p_1^2 - 1)^3}{p_1 (-p_1 + p_0)^3} \]
From the results in (3.3.63), statement (i) of the theorem holds. □

Proof of (b) goes on exactly similar lines to the proof of Theorem 3.3.2-(b).

From (3.3.64), proof of statement (c) goes on exactly similar lines to the proof of Theorem 3.3.2-(c). The following is the main theorem related to the asymptotic distribution of least squares estimators under Case 6.

**Theorem 3.3.13.** Let \( \hat{\beta}' = (\hat{\beta}_e, \beta_{10}, \beta_{20}) = (\hat{\beta}_{11}^{(1)}, \hat{\beta}_{12}^{(1)}, \hat{\beta}_{12}^{(2)}, \hat{\beta}_{21}^{(1)}, \hat{\beta}_{21}^{(2)}, \hat{\beta}_{22}^{(1)), \hat{\beta}_{22}^{(2)}, \beta_{10}, \beta_{20}) \) be the least squares estimators of \( \beta' = (\beta_e, \beta_{10}, \beta_{20}) = (\beta_{11}^{(1)}, \beta_{12}^{(1)}, \beta_{12}^{(2)}, \beta_{21}^{(1)}, \beta_{21}^{(2)}, \beta_{22}^{(1)), \beta_{22}^{(2)}, \beta_{10}, \beta_{20}) \), the vector of structural parameters of the model in (3.2.1) satisfying the assumptions of (3.2.2). Then the following statements hold under the placement \( \rho_1 = \rho_2 = \rho_0 = \rho_3 = \rho_4 = \rho_0; |\rho_0| > |\rho_0| > 1 \) relating to the roots of \( P(z) = 0 \).

a. \( \frac{N}{N} (\hat{\beta}' - \beta_e)' \) converges in probability, as \( N \to \infty \), to a random vector 
\( (\xi_{11}(6), \xi_{12}(6), \xi_{13}(6), \xi_{14}(6), \xi_{21}(6), \xi_{22}(6), \xi_{23}(6), \xi_{24}(6)) \) (say).

b. Each \( \xi_{ij}(1) \) is distributed like a ratio of two random variables of the type \( \sum_{i=1}^{4} (c_{1i} \pi_{ij}) / J \), wherein \( \pi_{ij} \) and \( J \) are independent and are distributed like specific linear combinations of \( \epsilon_i(t); i = 1, 2; t = 1, 2, \ldots \)

c. \( N^{1/2} (\hat{\beta}_{10} - \beta_{10}), N^{1/2} (\hat{\beta}_{20} - \beta_{20}) \) converges in distribution, as \( N \to \infty \), to a bivariate random vector with mean vector a null vector and a diagonal
A combined reading of Theorems 3.3.3, 3.3.7, 3.3.9, 3.3.11, and 3.3.13 reveals some important observations following from the main theorems.

The following interesting observations are (c) (q) and (p) are exactly similar. The proofs of the theorems can be justified. The statement (q) of the theorem can be justified. The proof follows the transformation in (3.3.27) and the use of the covariance matrix.
1. It can be seen that the rate of convergence of the centered least squares estimators of $\beta_c$ is the $N^{th}$ power of the smallest root of $P(z) = 0$ whenever the smallest root is a single root. If the smallest root say $\rho_0$, is a multiple root with multiplicity $r$, then the rate of convergence of least squares estimator $\hat{\beta}_c$ is $\rho_0^N/N^{(r-1)}$. Thus the rate of convergence of least squares estimators comes down as the multiplicity of the smallest root of $P(z) = 0$ increases.

2. The linear restrictions among the components of the limiting random vector relating to the least squares estimator of $\beta_c$ are identical in all the seven cases in (3.2.8). These restrictions, consequently reduce the rank of the variance-covariance matrix of the limiting random vector to 2, in all cases.

3. The reparametrisation indicates that the specific linear combinations of centered least squares estimators of $\beta_c$ have higher rates of convergence (to zero), than the centered least squares estimators of $\beta_c$. This will be exploited in the next chapter to establish a series of limit theorems on the least squares residuals.

4. One can always derive the limit distribution properties of least squares of $\beta$ first, and use them to derive the limit distribution properties of least squares estimator of the parameter vector $\theta$ of the reparametrised model. This was the approach made by Venkataraman and his school, primarily for models generating scalar time series and by Karthikeyan (1997) for the second order stochastic difference equation of dimension two with distinct roots. This approach when adopted first in the present context led to highly cumbersome algebra and in many situations even the MAPLE software brokedown. However, our new approach of deriving asymptotic properties of least squares estimator of $\beta$ from those of $\theta$ resolves this problem effectively. It appears that this approach would have reduced the algebraic manipulation considerably, had it been applied in the earlier works.
3.3.6 A unified theorem on purely explosive vector time series

In this section a unified limit theorem on the asymptotic properties of least squares estimator of $\beta$ is enunciated consolidating the new results contained in this thesis and those in the work of Karthikeyan (1997) pertaining to purely explosive vector time series.

Let us now introduce, for the sake of unifying the results in different cases, the following notations

i.

$$W_1(N) = \begin{cases} 
N^N & \text{when } \rho_1 = \rho_2 = \rho_3 = \rho_4 = \rho_0; |\rho_0| > 1 \\
\frac{\rho_0^N}{N} & \text{when } \rho_2 = \rho_3 = \rho_4 = \rho_0; |\rho_1| > |\rho_0| > 1 \\
\frac{\rho_0^N}{N} & \text{when } \rho_3 = \rho_4 = \rho_0; |\rho_1| > |\rho_2| > |\rho_0| > 1 \\
\rho_4^N & \text{when } \rho_1 = \rho_2 = \rho_0; |\rho_0| > |\rho_4| > 1 \\
\rho_4^N & \text{when } \rho_1 = \rho_2 = \rho_3 = \rho_0; |\rho_0| > |\rho_4| > 1 \\
\rho_4^N & \text{when } \rho_1 = \rho_2 = \rho_0; |\rho_0| > |\rho_4| > 1 \\
\rho_4^N & \text{when } |\rho_1| > |\rho_2| > |\rho_3| > |\rho_4| > 1 \
\end{cases}$$

ii.

$$W_2(N) = \begin{cases} 
\rho_0^N & \text{when } \rho_1 = \rho_2 = \rho_3 = \rho_4 = \rho_0; |\rho_0| > 1 \\
\rho_0^N & \text{when } \rho_2 = \rho_3 = \rho_4 = \rho_0; |\rho_1| > |\rho_0| > 1 \\
\rho_0^N & \text{when } \rho_3 = \rho_4 = \rho_0; |\rho_1| > |\rho_2| > |\rho_0| > 1 \\
\rho_3^N & \text{when } \rho_1 = \rho_2 = \rho_0; |\rho_0| > |\rho_3| > |\rho_4| > 1 \\
\frac{\rho_0^N}{N} & \text{when } \rho_1 = \rho_2 = \rho_3 = \rho_0; |\rho_0| > |\rho_4| > 1 \\
\rho_0^N & \text{when } \rho_1 = \rho_2 = \rho_0; |\rho_0| > |\rho_4| > 1 \\
\rho_3^N & \text{when } |\rho_1| > |\rho_2| > |\rho_3| > |\rho_4| > 1 \
\end{cases}$$
when $\rho_1 = \rho_2 = \rho_3 = \rho_4 = \rho_0; |\rho_0| > 1$

when $\rho_2 = \rho_3 = \rho_4 = \rho_0; |\rho_1| > |\rho_0| > 1$

when $\rho_3 = \rho_4 = \rho_0; |\rho_1| > |\rho_2| > |\rho_0| > 1$

when $\rho_1 = \rho_3 = \rho_4 = \rho_0; |\rho_0| > |\rho_4| > 1$

when $\rho_1 = \rho_2 = \rho_3 = \rho_0; |\rho_0| > |\rho_4| > 1$

when $\rho_4 = \rho_0; |\rho_0| > |\rho_0| > 1$

when $|\rho_1| > |\rho_2| > |\rho_3| > |\rho_4| > 1$

$\rho_0^N$ when $\rho_1 = \rho_2 = \rho_3 = \rho_4 = \rho_0; |\rho_0| > 1$

$\rho_1^N$ when $\rho_2 = \rho_3 = \rho_4 = \rho_0; |\rho_1| > |\rho_0| > 1$

$\rho_2^N$ when $\rho_3 = \rho_4 = \rho_0; |\rho_1| > |\rho_2| > |\rho_0| > 1$

$\rho_3^N$ when $\rho_1 = \rho_2 = \rho_3 = \rho_4 = \rho_0; |\rho_0| > |\rho_4| > 1$

$\rho_4^N$ when $\rho_1 = \rho_2 = \rho_3 = \rho_0; |\rho_0| > |\rho_4| > 1$

$\rho_0^N$ when $|\rho_1| > |\rho_2| > |\rho_3| > |\rho_4| > 1$

$\rho_1^N$ when $|\rho_1| > |\rho_2| > |\rho_3| > |\rho_4| > 1$

Note that $W_1(N), W_2(N), W_3(N)$ and $W_4(N)$ are the rates of $(\hat{\theta}_{i1} - \theta_{i1}), (\hat{\theta}_{i2} - \theta_{i2}), (\hat{\theta}_{i3} - \theta_{i3})$ and $(\hat{\theta}_{i4} - \theta_{i4}); i = 1, 2$, respectively, in the six cases discussed in Theorem 3.3.2, 3.3.4, 3.3.6, 3.3.8, 3.3.10 and (3.3.12).

Unifying all these results if it is possible to restate Theorems 3.3.2 to 3.3.13 and Theorem 4.3.1 of Karthikeyan (1997) as follows

**Theorem 3.3.14. A unified limit theorem.**

Let $\hat{\beta}' = (\hat{\beta}'_e, \hat{\beta}_{10}, \hat{\beta}_{20}) = (\hat{\beta}_{11}^{(1)}, \hat{\beta}_{12}^{(1)}, \hat{\beta}_{11}^{(2)}, \hat{\beta}_{12}^{(2)}, \hat{\beta}_{21}^{(1)}, \hat{\beta}_{22}^{(1)}, \hat{\beta}_{21}^{(2)}, \hat{\beta}_{22}^{(2)}, \hat{\beta}_{10}, \hat{\beta}_{20})$ be the least
squares estimator of \( \beta' = (\beta'_e, \beta_{10}, \beta_{20}) = (\beta_{11}^{(1)}, \beta_{12}^{(1)}, \beta_{12}^{(2)}, \beta_{21}^{(1)}, \beta_{22}^{(1)}, \beta_{22}^{(2)}, \beta_{10}, \beta_{20}) \), the vector of structural parameters of the model in (3.2.1), generating a purely explosive bivariate time series governed by the assumptions in (3.3.2).

Let \( \hat{\beta}' = (\hat{\beta}_e, \hat{\beta}_{15}, \hat{\theta}) = (\hat{\beta}_{11}, \hat{\beta}_{12}, \hat{\beta}_{13}, \hat{\beta}_{14}, \hat{\beta}_{21}, \hat{\beta}_{22}, \hat{\beta}_{23}, \hat{\beta}_{24}, \hat{\theta}_{15}, \hat{\theta}_{25}) \) be the least squares estimators of the reparametrised model in (3.3.22). Then under the basic assumptions of the model, in (3.2.2), the following statements hold under all placements of the roots of \( P(z) = 0 \) as considered in (3.2.8).

a. \( (W_1(N) (\hat{\beta}_e - \beta_e)^\prime) \) converges in probability as \( N \to \infty \) to a non-degenerate random vector \( \eta \) (say). And hence each of its components are bounded in probability.

b. Each component, \( \eta_i; i = 1, 2, \ldots, 8 \), of \( \eta \) is distributed like a ratio of two random variables of the type \( \sum_{i=1}^{4} (c_i \pi_{ij}) \) / \( J \), wherein \( \pi_{ij} \) and \( J \) are independent and are distributed like specific linear combinations of \( \epsilon_i(t); i = 1, 2; t = 1, 2 \ldots \) and \( c_i \)s are constants.

c. (i) \( \pi_1 \eta_1 + \eta_2 = 0 \) a.s.

(ii) \( a_{11}^{(1)} \eta_1 + a_{21}^{(1)} \eta_2 + \eta_3 = 0 \) a.s.

(iii) \( (a_{12}^{(1)} + \pi_1 a_{11}^{(1)}) \eta_1 + (a_{21}^{(1)} \pi_1 + a_{22}^{(1)}) \eta_2 + \pi_1 \eta_3 + \eta_4 = 0 \) a.s.

(iv) \( \pi_1 \eta_5 + \eta_6 = 0 \) a.s.

(v) \( a_{11}^{(1)} \eta_5 + a_{21}^{(1)} \eta_6 + \eta_7 = 0 \) a.s.

(vi) \( (a_{12}^{(1)} + \pi_1 a_{11}^{(1)}) \eta_5 + (a_{21}^{(1)} \pi_1 + a_{22}^{(1)}) \eta_6 + \pi_1 \eta_7 + \eta_8 = 0 \) a.s.

d. The covariance matrix of the asymptotic random vector \( \eta \) is singular and is of rank 2.

e. Each of \( \{W_1(N) (\hat{\theta}_{11} - \theta_{11})\}, \{W_2(N) (\hat{\theta}_{12} - \theta_{12})\}, \{W_3(N) (\hat{\theta}_{13} - \theta_{13})\}, \{W_4(N) (\hat{\theta}_{14} - \theta_{14})\}, \{W_1(N) (\hat{\theta}_{21} - \theta_{21})\}, \{W_2(N) (\hat{\theta}_{22} - \theta_{22})\} \),

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\[ \{ W_3(N) \left( \hat{\theta}_{23} - \theta_{23} \right) \}, \{ W_4(N) \left( \hat{\theta}_{24} - \theta_{24} \right) \} \text{ is bounded in probability, where} \]

the weight functions \( W_j(N); j = 1, 2, \ldots, 4 \) are case specific.

### 3.4 Rate of convergence of any linear combination of centered least squares estimators

In this section, we study the rate of convergence in probability to zero, as \( N \to \infty \), of any linear combination of the structural parameters of the model (3.2.2).

In section 3 of this chapter, we have studied, rates at which specific linear combinations of centered least squares estimators converge to zero in probability as \( N \to \infty \). These are statement (d) of the Theorems 3.3.3, 3.3.5, 3.3.7, 3.3.9, 3.3.11, 3.3.13. We now study such a result for any linear combination of centered least squares estimators. These results are multivariate analogue of those obtained for scalar time series by Suresh Chandra et al. (1999).

The inverse transformations in (3.3.27) and (3.3.28) together with the Theorems 3.3.2, 3.3.4, 3.3.6, 3.3.8, 3.3.10, 3.3.12 of section 3 of this chapter related to \( \hat{\theta}_s \), gives the multivariate analogue of the Theorem 2.7.1 (Theorem 3.1 of Suresh Chandra et al. (1999)).

The result can be written as, for \( i = 1, 2 \)

\[ \hat{\beta}_i - \beta_i = T \left( \hat{\theta}_i - \theta_i \right) \]

and \( \{ W_j(N) \left( \hat{\theta}_{ij} - \theta_{ij} \right) \} \) is bounded in probability as \( N \to \infty \). Here \( W_j(N); j = 1, 2, \ldots, 4 \) are case specific.

The matrix \( T \) of constants and the random vector \( \left( \hat{\theta} - \theta_1 \right) \) are similar to the matrix \( A = (a_{ij}) \) (say) and the random vector \( (T_1(N), T_2(N), \ldots, T_k(N))^\prime \) respectively of the Theorem 2.7.1.

With the above understanding, we can now express any linear combination of
\[(\hat{\beta}_{ic} - \beta_{ic}) \text{ say } l' \left( \hat{\beta}_{ic} - \beta_{ic} \right) \text{ as} \]

\[l' \left( \hat{\beta}_{ic} - \beta_{ic} \right) = l'T \left( \hat{\theta}_{ic} - \theta_{ic} \right) \]

\[= m' \left( \hat{\theta}_{ic} - \theta_{ic} \right) ; \quad m' = l'T \]

\[= m_1 \frac{1}{W_1(N)} \left[ W_1(N) \left( \hat{\theta}_{i1} - \theta_{i1} \right) \right] + m_2 \frac{1}{W_2(N)} \left[ W_2(N) \left( \hat{\theta}_{i2} - \theta_{i2} \right) \right] \]

\[+ m_3 \frac{1}{W_3(N)} \left[ W_3(N) \left( \hat{\theta}_{i3} - \theta_{i3} \right) \right] + m_4 \frac{1}{W_4(N)} \left[ W_4(N) \left( \hat{\theta}_{i4} - \theta_{i4} \right) \right] \]

It may be noted that if \( l' \left( \hat{\beta}_{ic} - \beta_{ic} \right) \) eliminates any of \( \left( \hat{\theta}_{ij} - \theta_{ij} \right) ; \quad j = 1, 2, \ldots, 4 \) with lower rate of convergence, then its rate of convergence in probability will be the smallest rate associated with the remaining terms. Otherwise, such linear combinations will have \( W_1(N) \) as the rate of convergence in probability to zero. Hence the required result.