Appendix: A

Algorithm for the Polarimetric Mueller matrix

This appendix discusses the algorithm used for characterising the Mueller matrix or the response matrix of the polarimeter developed for this thesis. The polarimeter operates in three modes for the measurement of $I \pm Q$, $I \pm U$, and $I \pm V$. The input Stokes vector for all these three modes are taken as $[I, Q, U, V]^T$ where superscript 'T' refers to the transpose operation. The algorithm is derived for these three modes separately. An algorithm is also derived to remove the systematic errors produced by the polarimeter because of the misalignments of the optical components.

A.1 Mode - I

This mode of operation is used to measure the $I \pm Q$. The Glan-Thompson prism (GTP) used as the analyser in the polarimeter is rotated and stops at the reference position sensed by the IR-sensor (refer to Chapter 3 for the details of the polarimeter). This position is set at $45^\circ$ with respect to the slit direction. The optical components involved in this measurement are the GTP and the grating. For ideal measurements of $I \pm Q$, the transmission axis of the GTP will be placed at $45^\circ$ and $-45^\circ$ with respect to the slit direction. Assuming an idealistic nature of the GTP, the Mueller matrix of the GTP can be written as,
where, $c_{21}$ and $s_{21}$ are the cosine and sine of twice the angle $\theta$ made by the transmission axis of the GTP and the slit direction. Ideally this $\theta$ will be $45^\circ$ and $-45^\circ$ for the two orthogonal polarisation measurement (i.e., I±Q). Let us assume a generalised Mueller matrix for the grating as,

$$[M_P] = \frac{1}{2} \begin{pmatrix} 1 & c_{21} & s_{21} & 0 \\ c_{21} & c_{21}^2 & c_{21} s_{21} & 0 \\ s_{21} & c_{21} s_{21} & s_{21}^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

The grating response to the input polarisation has been measured for the grating used for the polarimetric observations (see Chapter 3 for the details). From these measurements, it was found that the grating acts as a partial polariser with $G_{13}$ & $G_{14}$ equals to zero. The response of the grating is maximum along the grating groove direction compared to the perpendicular direction. Taking the reference axis as the grating groove direction, the misalignments of the grating groove direction and the slit direction need to be corrected. If $\theta_2$ is the misalignment angle, then the Mueller matrix for the rotation is given by,

$$[M_R] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{22} & s_{22} & 0 \\ 0 & -s_{22} & c_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$
Where, $c_{22}$ and $s_{22}$ are the cosine and sine of twice the angle $\theta_2$, respectively. Ideally this angle should be $0^\circ$. Multiplying these three Mueller matrices in order, i.e., Grating matrix ($M_G$), rotation matrix ($M_R$), and GTP matrix ($M_P$) (in other words, $M_G M_R M_P$), the output intensity can be written as,

$$I_{out}' = [I + Qc_{21} + Us_{21}][G_{11} + G_{12}\cos(2(\theta_1 - \theta_2)) + G_{13}\sin(2(\theta_1 - \theta_2))]. \quad (A.1)$$

Where $\theta_1$ and $\theta_2$ are the angles made by the transmission axis of the GTP and the groove direction of the grating with the slit direction respectively. The two orthogonal measurements were done with $\theta_1$ at $45^\circ$ and $-45^\circ$ for the measurement of $I \pm Q$. However, in the real situation, there will be a relative error in positioning of the GTP. Assuming $\delta \theta_1$ as the error in the positioning of the GTP, the values for the $\theta_1$ for these two measurements will be, $45^\circ + \delta \theta_1$ and $-45^\circ$. Similarly, assuming an error of $\delta \theta_2$ for the positional accuracy of the grating with respect to the slit direction, then $\theta_2$ will be equal to $\delta \theta_2$. By substituting these values into the Equation A.1, the two output intensity of the two orthogonal position of the GTP can be calculated. By subtracting and adding these two output intensity, the observed $Q/I$ can be calculated and is given by,

$$\frac{Q_{obs}'}{I_{obs}'} = -\frac{A + B}{1 + A.B}. \quad (A.2)$$

Where,

$$A = \frac{Q}{I} \cos(2\delta \theta_1) + \frac{U}{I} \sin(2\delta \theta_1),$$

$$B = \frac{G_{12}}{G_{11}} \sin(2(\delta \theta_1 - \delta \theta_2)).$$

Note that while deriving the above equations, the value of $G_{13}$ and $G_{14}$ are taken as zero. Equation A.2 gives the observed linear polarisation $Q$ in terms of the input Stokes vector with the errors in the alignment of the polarimetric optics. It can be seen that the circular polarisation does not come into this measurement and hence there will not be any circular to linear cross-talk because of the polarimeter.
A.2 Mode - II

The mode - II measurement is for the other linear polarisation component known as 'U' (i.e., I±U). The GTP is fixed at a position -45° from the reference position marked by the IR sensor (i.e., along the slit direction). The angle θ₁ now for this measurement will be 0° and 90° for the two orthogonal polarisation measurement (i.e., I±U). Since, the response of the grating along the groove direction and perpendicular to it is different, a polaroid called as compensating polaroid, is inserted behind the polarimeter (close to the slit). The transmission axis of this compensating polaroid is set at 45° so that this does not introduce any response difference between the two orthogonal positions with the GTP. Let us assume that θ₂ be the angle of the transmission axis of the polaroid with respect to the slit direction (ideally this angle should be 45°). Now, the combined Mueller matrix for this measurement can be obtained by multiplying the individual Mueller matrices in the following order, the Grating matrix (M_G), the rotation matrix to match the axis with the groove direction (M_R), the compensating polaroid matrix (M_P, similar to the GTP matrix except for the angle difference), and the GTP matrix (M_P) (i.e., M_GM_RM_PM_P^{GTP}). Assuming δθ₁ and δθ₂ to be the errors in positioning the GTP at 0° and the compensating polaroid at 45°, then the calculation of the output intensity at two orthogonal positions of the GTP will give the value of the observed U. The calculated value of U is given by,

\[
\frac{U'_{\text{obs}}}{I'_{\text{obs}}} = \frac{A + B}{1 + A.B}. \tag{A.3}
\]

Where,

\[
A = \frac{Q}{I} \sin(2δθ₁) - \frac{U}{I} \cos(2δθ₁),
\]

\[
B = \sin(2(δθ₁ - δθ₂)).
\]

Note that the grating response does not come into the final equation as the output for this configuration is a linearly polarised with constant polarisation direction because of the fixed compensating polaroid. However, because of this extra polaroid, the
exposure time has been increased by five times in order to get similar signal-to-noise ratio compared to the first one. Equation A.3 gives the output linear polarisation $U$ in terms of the input Stokes vector. In this case also, the circular to linear cross-talk because of the polarimeter is zero.

## A.3 Mode - III

The mode - III measurement is for the circular polarisation, $V$. In this case, a quarter-wave plate (QWP) is inserted in front of the polarimeter (close to the source) to convert the circular polarisation input to a linear polarisation and measure the linear polarisation as measured for $Q$ or $U$. In our case, it was measured like $Q$. The compensating polaroid is removed for this measurement. The Mueller matrix of a linear retarder with its fast axis at $\theta_1$ is given by,

$$[M_{LR}] = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & c_{21}^2 + s_{21}^2 \beta & c_{21} s_{21} (1 - \beta) & -s_{21} \mu \\
0 & c_{21} s_{21} (1 - \beta) & s_{21}^2 + c_{21}^2 \beta & c_{21} \mu \\
0 & s_{21} \mu & -c_{21} \mu & \beta
\end{pmatrix}$$

Where, $c_{21}$ and $s_{21}$ are the cosine and sine of twice the angle made by the fast axis of the linear retarder with the slit direction respectively. $\beta$ and $\mu$ are the cosine and sine of the retardance of the linear retarder. For ideal configuration, $\theta_1$ should be zero or $90^\circ$ and the retardance of the QWP should be $90^\circ$. The combined Mueller matrix for this configuration can be obtained by multiplying the individual Mueller matrices in the following order, grating response matrix ($M_G$), rotation matrix for the grating ($M_R$), matrix for the GTP ($M_P$), and the matrix for the QWP ($M_{LR}$) (i.e., $M_G M_R M_P M_{LR}$). From the calculation of the output intensity from this combined Mueller matrix for the two orthogonal polarisation position, the value for the observed
$V$ can be calculated and is given by,

$$\frac{V_{\text{obs}}'}{I_{\text{obs}}'} = -\frac{A + B}{1 + A.B}. \quad (A.4)$$

Where,

$$A = Q_p \cos(2\delta \theta_1) - U_p \sin(2\delta \theta_1),$$
$$B = \sin(2(\delta \theta_1 - \delta \theta_2)).$$

$\delta \theta_1$ and $\delta \theta_2$ are the misalignment of the QWP fast axis and the GTP axis. $Q_p$ and $U_p$ are related to the input Stokes vector and defined as,

$$Q_p = [c_{21}^2 - s_{21}^2 \sin(\delta \Delta)]Q + [c_{21}s_{21}(1 + \sin(\delta \Delta))]U - s_{21} \cos(\delta \Delta)V,$$
$$U_p = [c_{21}s_{21}(1 + \sin(\delta \Delta))]Q + [s_{21}^2 - c_{21}^2 \sin(\delta \Delta)]U + c_{21} \cos(\delta \Delta)V.$$

$\delta \Delta$ is the error in the retardance and $c_{21}, s_{21}$ are the cosine and sine of twice the angle made by the GTP with the slit direction. Equation A.4 gives the output observed circular polarisation in terms of the input Stokes vector with the optical alignment errors in the polarimeter. The Equations A.2, A.3, and A.4 are used to get the polarimeter response to an input Stokes vector. However, in order to eliminate the spurious polarisation produced because of the mis-alignments, the corrected Stokes profiles need to be calculated from the observed Stokes profiles. In order to generalise the above three modes, let us assume the following,

- $\delta \theta_1$ as the error in the positioning of the QWP.
- $\delta \theta_2$ as the error in the initial position of the GTP.
- $\delta \theta_3$ as the error in the positioning of the compensating polaroid.
- $\delta \theta_4$ as the error in the positioning of the grating groove.
- $\delta \Delta$ as the error in the retardance of the QWP.
With the above assumptions, the generalised equations for the three different modes of operation can be derived as,

\[
\frac{Q_{obs}'}{I_{obs}'} = -\frac{A_1 + A_2}{1 + A_1 A_2},
\]

(A.5)

where,

\[
A_1 = \frac{Q}{I} \cos(2\delta \theta_2) + \frac{U}{I} \sin(2\delta \theta_2),
\]

\[
A_2 = \frac{G_{12}}{G_{11}} \sin(2(\delta \theta_2 - \delta \theta_4)).
\]

For the U-measurement,

\[
\frac{U_{obs}'}{I_{obs}'} = \frac{B_1 + B_2}{1 + B_1 B_2},
\]

(A.6)

where,

\[
B_1 = \frac{Q}{I} \sin(2\delta \theta_2) - \frac{U}{I} \cos(2\delta \theta_2),
\]

\[
B_2 = \sin(2(\delta \theta_2 - \delta \theta_3)).
\]

For the V-measurement,

\[
\frac{V_{obs}'}{I_{obs}'} = -\frac{C_1 + C_2}{1 + C_1 C_2},
\]

(A.7)

where,

\[
C_1 = Q_p \cos(2\delta \theta_2) - U_p \sin(2\delta \theta_2),
\]

\[
C_2 = \frac{G_{12}}{G_{11}} \sin(2(\delta \theta_2 - \delta \theta_4)).
\]

with

\[
Q_p = \left(\sin^2(2\delta \theta_1) - \cos^2(2\delta \theta_1) \sin(\delta \Delta)\right) \frac{Q}{I} - \cos(2\delta \theta_1) \sin(2\delta \theta_1) (1 + \sin(\delta \Delta)) \frac{U}{I} - \cos(2\delta \theta_1) \cos(\delta \Delta) \frac{V}{I},
\]

\[
U_p = -\cos(2\delta \theta_1) \sin(2\delta \theta_1) (1 + \sin(\delta \Delta)) \frac{Q}{I} + \cos^2(2\delta \theta_1) - \sin^2(2\delta \theta_1) \sin(\delta \Delta) \frac{U}{I} - \sin(2\delta \theta_1) \cos(\delta \Delta) \frac{V}{I}.
\]
For the idealistic case of $\delta \theta_1 = \delta \theta_2 = \delta \theta_3 = \delta \theta_4 = \delta \Delta = 0$, Equations A.5, A.6, and A.7 reduces to,

\[
\begin{align*}
\frac{I'_{\text{obs}}}{I_{\text{obs}}} &= -\frac{Q}{I}, \\
\frac{U'_{\text{obs}}}{I_{\text{obs}}} &= \frac{U}{I}, \\
\frac{V'_{\text{obs}}}{I_{\text{obs}}} &= \frac{V}{I}.
\end{align*}
\]

Equations A.5, A.6, and A.7 are used to calculate the output Stokes vector for an input Stokes vector with the polarimeter as the instrument. However, in the actual situation the observed Stokes vector are available and the input Stokes vector has to be inferred from this observed Stokes vector. These can be found out by solving the above three equations (i.e., Equations A.5, A.6, and A.7). The solution is written as,

\[
\frac{Q}{I} = A_1' + A_2',
\]

where,

\[
\begin{align*}
A_1' &= -\frac{(Q'_{\text{obs}} + A) \cos(2\delta \theta_2)}{A (Q'_{\text{obs}} + 1)}, \\
A_2' &= -\frac{(U'_{\text{obs}} - B) \sin(2\delta \theta_2)}{B (U'_{\text{obs}} - 1)}.
\end{align*}
\]

For input U,

\[
\frac{U}{I} = B_1' + B_2',
\]

where,

\[
\begin{align*}
B_1' &= -\frac{(Q'_{\text{obs}} + A) \sin(2\delta \theta_2)}{A (Q'_{\text{obs}} + 1)}, \\
B_2' &= \frac{(U'_{\text{obs}} - B) \cos(2\delta \theta_2)}{B (U'_{\text{obs}} - 1)}.
\end{align*}
\]

For input V,

\[
\frac{V}{I} = C_1' + C_2' + C_3',
\]

(A.10)
where,

\[ C_1' = \frac{(V'_{\text{disk}} + C)}{C'_{\text{disk}}}, \]
\[ C_2' = \frac{Q}{I}(D + E), \]
\[ C_3' = \frac{U}{I}(F + G). \]

The values of A, B, C, D, E, F, and G are given as,

\[ A = \frac{G_{12}}{G_{11}} \sin(2(\delta \theta_2 - \delta \theta_4)), \]
\[ B = \sin(2(\delta \theta_2 - \delta \theta_3)), \]
\[ C = \frac{G_{12}}{G_{11}} \sin(2(\delta \theta_2 - \delta \theta_4)), \]
\[ D = \cos(2\delta \theta_2)(\sin^2(2\delta \theta_4) - \cos^2(2\delta \theta_1) \sin(\delta \Delta)), \]
\[ E = -\sin(2\delta \theta_2) \cos(2\delta \theta_1) \sin(2\delta \theta_4)(1 + \sin(\delta \Delta)), \]
\[ F = \cos(2\delta \theta_2) \cos(2\delta \theta_1) \sin(2\delta \theta_4)(1 + \sin(\delta \Delta)), \]
\[ G = -\sin(2\delta \theta_2)\cos^2(2\delta \theta_2) - \sin^2(2\delta \theta_2) \sin(\delta \Delta)]. \]

Equations A.8, A.9, and A.10 gives the corrected Stokes profile from the observed Stokes profiles. This algorithm is needed and used to remove the polarimetric responses from the observed data.