During earlier stages of study of flow over circular cylinders, the tools of investigation were physical experiments and some mathematical models, the studies were mostly directed towards determination of coefficient of drag, distribution of pressure on the surface and study of various flow phenomena such as separation of flow from surface, recirculation of flow and shedding of vortices.

Attention has been directed in this chapter to a re-examination of considerable amount of experimental and theoretical investigations of laminar, viscous flow over bluff bodies in particular circular cylinders. Besides, presenting a history of past developments, available literature has been examined with a view to make comments on the state of the art and to recognise the scope of further research on the subject.

The work reported has been classified under the heads: 1. Experimental Studies: 2. Analytical studies.

2.1 Experimental Studies:

Earlier works on laminar viscous flow over bluff bodies date back to the turn of 20th century when Wieselberger (1914), Fure (1934) and Homann (1936) conducted experiments to establish the
quantitative relationship between drag coefficient and extent of wake, and the flow Reynolds number when flow occurs past a cylindrical bluff body. Schlichting (1968) reports the work of Wieselberger (1914) who obtained a curve for values of drag at different Reynolds numbers for flow over circular cylinders. This curve shows that coefficient of drag is continuously decreasing with increase in Reynolds number. At Re=1800 a minimum value of \( \text{Cd} = 0.95 \) is attained. With further increase in Reynolds number there is increase in value of \( \text{Cd} \) to 1.2 at \( \text{Re} \approx 2 \times 10^5 \). A marked fall in \( \text{Cd} \) occurs from 1.2 to 0.36 at \( \text{Re} > 2 \times 10^5 \).

The work of Homann (1936) has been reported by Goldstein (1965) and Schlichting (1968). Homann photographed streamlines for flow of oil over a circular cylinder at Reynolds numbers of 3.9, 18.6, 31.6, 33.5, 54.8, 65.2, 73, 101.5, 161, 225 and 281. The state of affairs depicted by these photographs is a representative of the typical characteristics of flow over bluff bodies.

Study of work by Wieselberger (1914), Thom (1933), Fage (1934) reveal that with increase in Reynolds number, there occurs a remarkable change in the nature of flow separation and its subsequent effects. At very low value of \( \text{Re} \), i.e., of order zero, inertial forces are negligible as compared with viscous forces. Pressure gradients which depend upon velocity square are insignificant and obviously the skin friction for accounts for a large part of the total drag. At \( \text{Re} \) ranging from 6 to 40, the laminar flow separates. The separation is, however, symmetrical and is characterised by the formation
of two eddies or vortices which rotate in opposite directions. Beyond the eddies, the streamlines close together and thereby limit the size of wake. With further increase in $Re$, $40 < Re < 70$ there occurs an increase in the intensity of formation of eddies. There is a change in the shape of vortices too; they become longer and longer in the direction of flow. A waviness begins to develop downstream and a periodic oscillation of the wave is discerned. With Reynolds number increasing beyond 70, the vortices detach alternatively from opposite sides at root of the body and form two staggered rows of uniformly spaced vortices, called Karman Vortex trails (Karman-1911) or Karman Vortex Street. Energy of the vortices is consumed by fluid viscosity and consequently their regular pattern disappears beyond a certain distance downstream from the cylinder. The frequency with which the vortices are cast off from the circular cylinder depends uniquely on Reynolds number and is prescribed by the relation

$$\frac{f_d}{U} = 0.198 \left( 1 - \frac{121}{Re^2} \right)$$

$40 < Re < 400$

The Reynolds number at which the vortices become asymmetrical and ready to leave the obstacle is called the critical Reynolds number and its value is dictated by the shape of the obstacle, the degree of turbulence of main stream and on proximity of channel walls.

Fage (1934) as reported by Goldstein (1965), photographed flow of ordinary tap water (without any particles added) past long circular cylinders. Photographs taken within Reynolds
number range 17.7 to 170 indicated elongation of vortices in the direction of flow with increasing Reynolds numbers. Similar conclusions were drawn by Tenada (1956) with respect to appearance of twin vortices behind a circular cylinder, and the effect of Reynolds number on the elongation of vortices and oscillation of wake.

Tritton’s (1959) work concerns primarily with measurements of drag on circular cylinders in a flow stream with Reynolds number range 0.5 to 100. Drag estimates were made by observing the bending of quartz fibre. The test results advance experimental evidence for there being a transition in the nature of vortex street in the cylinder wake at Reynolds number around 90.

Experimental investigations for the steady separated flow past a circular cylinder have been reported by Grove et al (1964). The wake was artificially stabilised and the experimental results gave a clear insight into the asymptotic character of steady separated flows at high Reynolds numbers. It was found that the friction coefficient at the rear of the cylinder remained unchanged for $25 < Re < 177$; that the circulation velocity within the wake approached a non-zero limit as $Re$ increased; and that the wake length increased in direct proportion to Reynolds number.

Dimopoulas and Hanratty (1968) used an electrochemical technique to study the flow around solid objects by measurements of velocity gradient at the solid boundary.
between Reynolds number of 60 and 360. For Re > 150
the measurements between the front stagnation point and
separation were found to be in conformity with the values
predicted by the boundary layer theory. Further, the wall
velocity gradients in the wake were smaller than those in the
front part of the cylinder and a minimum, quite close to the
separation point.

Acrivos et al (1968) have presented experimental results
for the steady separated flow past a variety of bluff objects.
Included in this work are measurements of rear stagnation point,
pressure coefficients, the pressure distribution along the
base of a backward facing step, velocity profiles within the
circulating wake bubble as well as in the outer flow, lengths-
maximum widths and vortex centre location of the corresponding
wake bubbles, and the local shear stress along the surface
of a circular cylinder. In every case, the experimental data
were found to be consistent with the theoretical model
prepared by Acrivos et al (1965) for steady separated flow
past bluff objects in the limit of large Reynolds number.

In a comparatively recent investigation, Huner and
Hussey (1977) made drag measurements in range of Reynolds
number from 0.23 to 2.6. Empirical corrections were made to
account for the effect of finite boundaries and finite
length. The boundary correction was small ( < 0.1\%) but length
correction ranged up to 6.6\%. The experimental results were
within good agreement with the theory of Kaplan (1957) for
Re < 0.5 and with numerical results of Takami and Keller
(1969) at Re = 1 and 2.
2.2 Analytical Studies:

The analytical studies have been carried out by different approaches. In one approach, models for wake flow are proposed. In the second approach, simplification of basic equations of motions is done and perturbation type of solutions obtained or boundary layer solutions obtained. In the third approach, series truncation method is used. The most popular approach has been the use of finite difference equations of basic Navier Stokes equations to obtain numerical solutions on digital computers.

2.2.1 Wake Models:

Kirchoff (1869) proposed a model for wake flow in case of a circular cylinder with vanishingly small viscosity. The model gives the width, extent and nature of wake region - which is called as 'dead water region' (Fig. 2).

Helmholtz (1868) and Kirchoff (1869) suggested a free streamline model which gives proper solution of the Navier Stokes equation with non-viscous flow conditions ($\nu = 0$). The system, however, fails under the limiting condition of $\nu \to 0$, which describes the steady flow due to the pressure of a bluff body in an otherwise uniform stream. A modification to this model, giving a closed wake, was pronounced and tried by Batchelor (1956). A closed wake contains a standing eddy, or eddies, whose general feature can be inferred from
the results of an earlier investigation of a steady flow in a closed region at large Reynolds number. Although no particular case has been dealt with, the work stipulates that for two dimensional and axisymmetrical bodies, the drag coefficient tends to zero as the Reynolds number approaches infinity.

2.2.2 Approximate Analytical Solutions:

Analytical solutions for the slow viscous flows (creeping flow) were first tried by Stokes who reasoned that the inertia forces represented by the convective terms in the Navier-Stokes equations are ineffective because they are quadratic in the velocity. Hence at low Reynolds number the pressure forces must be nearly balanced by viscous forces alone. In plane flow, the result is the biharmonic equation for the stream function. The boundary conditions are zero velocity at the surface and uniform flow at infinity. For flow over a circular cylinder a symmetry condition must be added to rule out circulation. It is found that solution fails to satisfy condition of uniform flow at infinity. The non existence of a solution of Stokes equation for unbounded plane flow past any body is known as Stoke's paradox. Oseen (1910) showed that paradox of Stokes arose from the singular nature of flow at low Reynolds number. Oseen suggested that rather than neglect the convective terms altogether, the convective terms are replaced by their linearised forms valid far from the body. In practice, however, although
the Oseen equations are linear, their solution is sufficiently complex that no second approximations are known. The solution for the circular cylinder using linearised convective terms as suggested by Oseen was given by Lamb (1945) in 1911.

Solutions for arbitrary Reynolds number were carried out by Goldstein (1929) and Tomotika and Aoi (1950). These more complicated results are of limited value because, contrary to Oseen's own views, the approximation is qualitatively as well as quantitatively invalid at high Reynolds number. The detailed flow patterns calculated by Tomotika and Aoi would be of some interest had not Yamada (1954) pointed out that numerical inaccuracy invalidates their qualitative nature even at low Reynolds number. For example, Tomotika and Aoi predict the standing eddies at arbitrarily low Reynolds number whereas Yamada showed that they first appear behind the circular cylinder at \( \text{Re} = 1.51 \) in the Oseen approximation.

Kaplun (1957) has carried the process through one more cycle to find the coefficient of the third term in the Stokes expansion. Also, improved approximate solutions were sought by Imai (1951) who obtained matched asymptotic solution for the governing equations by considering higher order convective terms. The velocity field was modified by considering the influence of disturbance velocity components; the squares, products and derivatives of the disturbance velocity components were however ignored. Imai's approach could, therefore, be used to discover the nature of the flow at small and moderate values of Reynolds numbers for laminar flow.
Solutions obtained by Stokes, Lamb (Lamb-1945) and Imai (1951) are all applicable to flow of viscous incompressible fluids over circular cylinders at low values of Reynolds numbers.

For high values of Reynolds numbers, Prandtl's boundary layer equations have been used for solving the viscous incompressible flow over circular and elliptical cylinders. In the first work of this type undertaken by Blasius (1908) and reported in Schlichting (1968) the equation for the potential velocity field depends upon the shape of the body, and for a particular flow situation it consists of certain terms with known coefficients and distances raised to odd powers e.g., \[ U(x) = u_1 x + u_3 x^3 + u_5 x^5 + \ldots \]

From the prescribed series, the pressure distribution is worked out and the angle of separation is estimated. Prediction for improved results were subsequently obtained by Pohlhausen (1921) who employed an empirical relation for pressure distribution suggested by Hiemenz (1911). Schlichting and Ulrich (1942) obtained solutions for steady laminar boundary layers for flow over elliptic cylinders up to point of separation.

The boundary layer equations are essentially valid only up to the point of separation. With the development of VSTOL aerodynamics where suction has to be applied or tripping of boundary layer is done to augment lift on airfoils, the usefulness and validity of boundary layer approach becomes doubtful. Obviously a need was felt and rightly so for the complete solution of Navier Stokes equations.
Underwood (1969) calculated the steady two dimensional incompressible flow past a circular cylinder for Reynolds number up to ten. A reasonably accurate description of the flow field was found by employing semi analytical method of series truncation to reduce the governing partial differential equations of motion to a system of ordinary differential equations which can be integrated numerically. Results were presented for Reynolds numbers between 0.4 and 10.0. The Reynolds number at which separation first occurs behind the cylinder was found to be 5.75. Over the entire Reynolds number range investigated, characteristics of flow parameters such as drag coefficient, pressure coefficient, standing eddy length, and streamline pattern compared favourably with available experimental results and numerical solutions.

2.2.3 Numerical Solutions:

Thom (1933) used a numerical 'difference' approach for solving the exact equations of motion of a viscous fluid. The system is built up rather laboriously from the network of potential and streamline curves appropriate to the boundaries considered; these curves being obtained from ide and fluid theory. Figure 3 gives a representative picture of the streamline pattern about a circular cylinder as reported by Thom. Vortex layers leaving the surface at the point of separation tends to separate the fluid in the wake from that in the main stream flow region. Thom's investigations further indicate that the critical Reynolds number at which eddies come off the surface depends upon the ratio
of the channel width to cylinder diameter; the critical Reynolds number increases with the diminished ratio. Thom's method can be applied over a wide range of Reynolds numbers but the amount of computations involved is very large.

Kawaguti (1953) solved the finite difference form of steady state Navier Stokes equation for flow over a circular cylinder at Reynolds number of 40. Apelt (1958) used a different approach to obtain numerical solutions at Re=40 and 44. Apelt used different boundary conditions and transformations.

Allen and Southwell (1955) applied relaxation technique to the general case of steady laminar motion of an incompressible viscous fluid past a stationary cylinder; that is to motion at speeds such that neither inertia nor viscosity can be neglected. The numerical computations relate to a circular cylinder, but the method is claimed to be applicable to any shape (an initial conformal transformation changes independent variable x and y to irrotational velocity potential $-\alpha$ and stream function $-\beta$). The flow patterns, contours of the irrotational velocity potential and the stream function change with variation in the flow Reynolds number. Introduction of variables involving Reynolds numbers, however, makes the change relatively slow and thus the accepted solution for Re = 10 could be made a good starting assumption for Re = 100.
In the numerical technique adopted by Payne (1958), the velocity equation representing the unsteady flow of viscous flow past a circular cylinder was solved by the step by step integration procedure. Computations showed that for Reynolds number 40 and 100, the calculated values of drag diminish with time to a value very near to that for the steady flows. At Reynolds number of 40, the value of drag coefficient is in reasonable agreement with Kawaguti's (1953). Further it is observed that at Reynolds number of 100 the mesh size is too coarse. Ingham (1968) improved the method developed by Payne to cover larger time intervals. At Reynolds number of 40 Ingham's results are in agreement with Apelt's (1958), and at Reynolds number of 100 improvement is made over results obtained by Payne. Though these results are not accurate predictions.

Fromm and Harlow (1963) developed a method for solution of time dependent problems concerning the flow of viscous incompressible fluids in two space dimensions. The approach was applied to study the development of a vortex street behind a plate which was impulsively accelerated to constant speed in the channel of finite width; the Reynolds number range investigated was $15 \leq Re \leq 6000$. Computations were made for critical Reynolds numbers for vortex shedding, drag coefficient, Strouhal number, vortex configuration, and channel wall effects; the nature of early stages of flow pattern development was also investigated. In this confined flow problem stream function and vorticity transport finite difference equations were solved. The numerical technique however necessitated a high speed computer.
Imai's (1951) analytical results on the asymptotic behaviour of viscous fluid at a great distance from a cylindrical body were used as the boundary conditions in the numerical scheme suggested by Keller and Takami (1966). The authors claimed successful calculations for Re = 2, 4, 10 and 15 for steady viscous flow past circular cylinders. This method was improved by Takami and Keller (1969) and the range of Reynolds number extended from 1 to 60. In these methods the Navier Stokes equations were replaced by a set of finite difference equations and the numerical solution was obtained by means of iterative method. The results were compared with existing analytical, numerical and experimental results.

Kawaguti and Jain (1966) carried out systematic numerical study of the unsteady viscous incompressible fluid flow past circular cylinder. The range of Reynolds number considered was 1 to 100. The steady state solutions could be obtained at Reynolds number Re = 10, 20, 30, 40, 50 as the limit of unsteady state solutions. The results were in agreement with existing results. Plots were obtained for flow pattern, vorticity distribution and pressure distribution at different Reynolds numbers, their variation in time was also plotted.

Jain and Rao (1969) extended the work of Kawaguti and Jain (1966). They obtained limiting steady state of unsteady viscous incompressible fluid flow past a circular
cylinder at Reynolds number upto 50. After removing condition of symmetry, a study was made to investigate existence of the limiting steady state or Karman Vortex Street at Reynolds number of 40, 60, 100 and 200. Results were compared with existing ones.

Hamielec and Raal (1969) have obtained numerical solutions for steady, incompressible Newtonian flow around a circular cylinder for Reynolds number 1, 2, 4, 10, 15, 30, 50, 100 and 500. The solutions were obtained for steady state Navier Stokes equations. The results have been presented in the form of vorticity and streamfunction distributions. Drag coefficients, pressure distribution and vortex dimensions are compared with experimental data and with available theoretical predictions for Reynolds numbers upto 50. In this method, the influence of incorrect boundary conditions is decreased by extrapolating the value of parameters when the boundary is shifted to infinity.

Son and Hanratty (1969) obtained numerical solutions for flow around a cylinder at Reynolds number of 40, 200 and 500. Finite difference solution for the time dependent equations of motion were carried out in order to extend the range of available data on steady flow around a cylinder at large Reynolds numbers. At termination of iteration procedure, the vorticity distribution around the surface of the cylinder was very close to their steady state values. For Re = 500 the separation angle and the drag coefficient were very close to their steady state values but the pressure
distribution and vorticity distribution at the rear of the cylinder were still changing slightly. The results at Re=500 were found to be quite different from those at Re=200 so the results for Re→∞ could not be predicted. The forces on the cylinder due to viscous drag and due to pressure drag were found to be smaller for steady flow than for laboratory experiments where the wake is unsteady.

Computations for time dependant viscous flow over a circular cylinder were made by Thoman and Szewezyk (1969). A finite difference method was extended to high Reynolds number with particular emphasis given to the quantitative description of fine flow features. The method is of the explicit type and includes a directional difference scheme for non-linear terms which enhances calculational stability at high Reynolds numbers. A cell structure is chosen which provides local cell dimension consistent with the structure of solutions expected. Solutions were obtained for a range of Reynolds numbers from $1 \text{ to } 3 \times 10^5$ in which the flow was started impulsively from rest, and the development is studied up to the approach of steady state or the limit cycle condition, whichever, is appropriate to the particular Reynolds numbers.

The latest work of this kind reported in the literature is that of Dey (1970) who made numerical studies of the Navier Stokes equations for two dimensional, time dependant, laminar, incompressible fluid motion. With the streamfunction and vorticity as the dependent variables, the equations together with the boundary conditions, were written in a
two dimensional orthogonal curvilinear coordinate system. As an application for numerical studies, polar coordinates 
\((r, \theta)\) were used. As a result, this would correspond to the study of viscous flow past a circular cylinder. To form the difference equations, central difference expressions for the space derivatives and backward difference equations were put into matrix forms for the analysis of stability and convergence. A transformation \(r = e^\phi\) was used to simplify the problem. At first the vorticity and the streamfunction equations were solved simultaneously by an ordinary Gauss-Seidel type iterative scheme. Afterwards, relaxation factors were introduced and the same technique was accelerated to produce faster convergence. Computer run time was considerably reduced. An unconditional convergence criterion was obtained in the accelerated technique. Computations were made for different flow situations with Reynolds number attaining the value as high as \(4 \times 10^5\). A re-examination, made in the preceding paragraphs, of the laminar viscous flow would reveal that majority of the experimental and theoretical investigations have been made for the flow past a circular cylinder. Further the perturbation type analytical solutions are applicable only for small values of Reynolds numbers and they fail to satisfy the boundary conditions. Again, the boundary layer theory fails to give satisfactory results if the flow Reynolds number is less than 150. Their usefulness and validity is doubtful for low speed viscous flows. Analytical approaches involving iteration and successive integration procedures, and the finite difference solution for the time
dependent equations necessitate the use of high speed computers. •

There exist numerous practical applications in the sphere of hydrodynamics, aerodynamics, oil and chemical technology where the scientists and engineers have to deal with low speed viscous flows past bluff bodies with shapes different from circular cylinders. Again need is felt for the evolution of far away boundary conditions which would conform to the physics of fluid flow and develop a numerical technique that would give good distribution of grid points without involving much transformations, i.e., less computer time. Attention has been focussed on these aspects in the present research programme, with a view to develop a new technique by which the Navier Stokes equations can be solved for separated flows over elliptical cylinders, where the application of boundary layer equations becomes doubtful.

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FIG 2 - WAKE REGION FOR VANISHING VISCOSITY
FIG 3 - STREAMLINES RECIRCULATORY REGION (THOM)