CHAPTER I
INTRODUCTION

The studies of the equilibrium and the oscillations of self-gravitating gaseous masses with a prevalent magnetic field have been of considerable interest in recent years. These investigations have been carried out with a view to understand the role played by the magnetic field in the structure of stellar bodies.

The significance of magnetic fields for the equilibrium and the stability of stellar bodies was first pointed out by Chandrasekhar and Fermi (1953) who showed that a homogeneous self-gravitating sphere with a uniform magnetic field inside and a dipole field outside is not an equilibrium configuration; the sphere tends to become oblate in the direction of the field. Gjellestad (1954) determined the optimum eccentricity of such an oblate spheroid. This configuration, however, is unstable with respect to all deformations except those which transform the spheroid into another spheroid of different eccentricity. Ferraro (1954) discussed the equilibrium of a self-gravitating homogeneous fluid configuration with a weak poloidal magnetic field and a small uniform rotation and calculated the eccentricity of the resulting equilibrium spheroid. Roberts (1955) has extended this work by
obtaining a series expansion for the equation of the surface of a homogeneous fluid body pervaded by a Ferraro magnetic field, but there is some doubt about the convergence of the series.

Prendergast (1956) obtained an exact solution for the equilibrium of a self-gravitating homogeneous fluid sphere with a prevalent axisymmetric magnetic field which vanishes at the surface. Auluck and Nayyer (1960) and Pratap (1961) have also examined magnetic configurations which can keep a homogeneous self-gravitating fluid sphere in equilibrium. Woltjer (1960) and Wentzel (1961) considered a compressible model with a barotropic equation of state and determined the structure of the magnetic field for some simple density distributions. Wentzel (1961) also calculated the first order reaction of the magnetic field on the density-pressure fields showing that the equilibrium surface is a prolate spheroid. Roxburgh (1962, 1963 b) determined the equilibrium structure of a weak toroidal magnetic field in a gaseous polytrope. The reaction of the toroidal magnetic field (considered by Roxburgh) on the equilibrium structure has been examined by Sinha (1968). Monaghan (1965, 1966 a,b) studied the equilibrium of a polytrope with a weak poloidal magnetic field. The equilibrium structure of a weak magnetic field (with both toroidal and poloidal components) in a gaseous polytrope
has been examined by Roxburgh (1966). A systematic treatment of the structure of a gaseous polytrope in the presence of a magnetic field with both a toroidal and a poloidal component has been given by Trehan and Uberoi (1972) employing a method similar to the one used by Chandrasekhar (1933) for rotationally distorted polytropes.

The oscillations and the stability of self-gravitating systems with prevalent magnetic fields have been studied by a number of authors. The radial pulsations of a system containing a general magnetic field which vanishes on the boundary were first examined by Chandrasekhar and Limber (1954) using the virial method (see §1.2). Assuming the pulsations to take place adiabatically (with a constant $\gamma$, the ratio of the specific heats) they found the frequency of oscillation, $\sigma$, of the fundamental mode, to be given by the approximate expression

$$\sigma^2 = (3 \gamma - 4) \frac{W}{I} (1 - \frac{\gamma I}{W}) ,$$  

(1)

where $W$, $I$, and $\gamma I$ denote the gravitational energy, the moment of inertia and the magnetic energy, respectively, of the configuration. The approximate nature of the solution (1) is to be viewed in the following sense. In the absence of a magnetic field the perturbed form of the virial equations can be solved exactly without any assumption regarding the form of the Lagrangian displacement. In obtaining (1) a trial function is chosen which has the form
of the exact Lagrangian displacement in the absence of a
magnetic field. The result (1) obtained under this
assumption will thus be qualitatively reliable for weak
magnetic fields. The effect of a $P_2(\mu)^*$ deformation on an
equilibrium sphere with a magnetic field vanishing at the
boundary was examined by Woltjer (1952). He
found that the frequency of the lowest mode of oscillation
is given by

$$\sigma^2 = \frac{4}{5} \frac{1W1}{I} (1 + 5 \frac{\pi^2}{W}) .$$

(2)

This result showed that a prevailing weak magnetic field,
regardless of its structure, increases the frequency
belonging to the second order harmonic deformation of a
spherically symmetric configuration.

It is interesting to ask the following question:
"If the magnetic field does not vanish at the surface,
what would be the forms of equations (1) and (2)." In
particular if we write equation (1) as

$$\sigma_R^2 = \sigma_T^2 (1 - \alpha \bar{\nu} / W)$$

and equation (2) as

$$\sigma_S^2 = \sigma_S^2 (1 + \beta \bar{\nu} / W), \quad \sigma_T^2$$

and $\sigma_S^2$ being the frequencies of the radial pulsation and
the Kelvin mode of oscillation of a homogeneous sphere,
the precise question is: what is the sign and the magnitude
of the constants $\alpha$ and $\beta$ when the magnetic field does not
vanish at the surface, or when one considers a more general

$^*P_l(\mu)$ is a Legendre polynomial of order $l$. 
configuration e.g. a polytropic configuration. It should be noted that the presence of a magnetic field which is non-vanishing at the surface will not, in general, allow spherical symmetry. Therefore the answer to the above question will depend, to some extent, on the equilibrium configuration considered. For the present purpose we have chosen the equilibrium state discussed by Ferraro (1954) in which there is a poloidal magnetic field inside and a dipole type field outside the configuration. The oscillations of this system have been examined and it is found that the general nature of the results (1) and (2) remains the same.

The results (1) and (2) for the effect of a magnetic field on the radial mode and the non-radial mode belonging to the \( P_2(\mu) \) deformation were derived on the assumption of a uniform density distribution. It is of interest to see how these results are modified when the equilibrium configuration is non-homogeneous. To this end we study the oscillations of magnetically distorted polytropes. Roxburgh and Durney (1967) were the first to examine the oscillations of a polytrope with a prevalent toroidal magnetic field; their studies, however, were confined to the radial oscillations of a polytrope of index \( n = 3 \). Anand (1969) has examined the oscillations of the same equilibrium configuration using second order virial equations.
However, the analysis given by An and suffers from some mathematical inconsistencies (see Chapter IV), which require a re-examination of the problem. Trehan and Billings (1971) have examined the oscillations of a polytrope with a poloidal magnetic field using second order virial equations. Fahliman (1971) has examined the radial and the non-radial (belonging to the harmonic $\ell = 2$) modes of oscillation of the same equilibrium model using a variational principle. The radial modes of oscillation of a polytrope with a magnetic field containing both a toroidal and a poloidal component have been investigated by Trehan and Uberoi (1972) using a variational principle.

In the present study we have made a systematic examination of the radial and the non-radial modes belonging to the harmonics $P_{\ell}^{(\mu)}$ ($\ell = 1, 2, 3$) of a polytrope with a toroidal magnetic field using a variational technique. The radial and the non-radial modes of oscillation of a polytrope with a magnetic field having both toroidal and poloidal components have also been investigated using second order virial equations. These studies have brought out some interesting results. We find that the distortion of the structure due to the magnetic field plays a significant role in modifying the frequencies of the non-radial modes viz. the 'Kelvin' modes of oscillation (cf. equation [2]). The effect of the magnetic field on the oscillation frequencies can be split up into two parts viz.
the direct effect of the magnetic field (excluding the
distortion of the structure) and the effect of the
reaction of the magnetic field on the structure. These
two separate contributions can either be of the same sign
or of opposite signs. In the latter case, their relative
strengths determine the net effect of the magnetic field
on the oscillation frequencies.

It is found that in the "first approximation" the
effect of a toroidal magnetic field alone (excluding the
effect of the distortion of the structure) on the 'Kelvin'
modes belonging to the spherical harmonics $\ell = 2$ and $\ell = 3$
is to increase their frequencies of oscillation, whereas,
in the "second approximation" the effect is to decrease
the frequencies of oscillation for polytropic indices
$n \geq 1.5$. Here the term "first approximation" refers to
the use of a trial function which has the form of the
exact Lagrangian displacement in the absence of a magnetic
field. The term "second approximation" refers to the use
of a one parameter trial function in the variational
principle. For a polytrope with a general magnetic field
containing both a toroidal and a poloidal component, it is
found that the frequencies of the $\ell = 2$ 'Kelvin' mode of
oscillation are increased by the presence of the magnetic
field.

The convective modes of oscillation of a magnetically
distorted polytrope with a purely toroidal field have been examined for the first time. It is found that the presence of a toroidal magnetic field has a de-stabilizing influence on the convective modes belonging to the spherical harmonic $\ell = 1$, whereas it has a stabilizing influence on the convective modes belonging to the spherical harmonics $\ell = 2$ and $\ell = 3$. Since the convective instability first manifests itself through the mode belonging to the lowest spherical harmonic, it is clear that the presence of a toroidal magnetic field has a de-stabilizing influence on the configuration, in so far as the convective stability is concerned.

A comparative study of the various results obtained for magnetically distorted polytropes has been made in Chapter VII. The correspondence of these results with the known results for homogeneous configurations has also been studied.

§ 1.1 Basic Equations

The basic equations governing the motion of a self-gravitating, inviscid and perfectly conducting fluid are:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (3)$$

$$\frac{d}{dt} \int \rho \mathbf{v} \, dV = \int (\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}) \, dV = - \nabla p + \int \mathbf{v} \, d\mathbf{v} + \frac{1}{\sigma} \mathbf{J} \times \mathbf{H}, \quad (4)$$
\[ \nabla \times \mathbf{H} = -\frac{4\pi}{c} \mathbf{J} , \quad \nabla \cdot \mathbf{H} = 0 , \quad (5) \]

\[ \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} \quad \text{and} \quad \mathbf{E} + \frac{1}{c}(\mathbf{v} \times \mathbf{H}) = 0 , \quad (6) \]

where \( \rho \) is the density, \( p \) is the pressure, \( \mathbf{v} \) is the velocity, \( \mathbf{J} \) is the electric current density, \( \mathbf{E} \) and \( \mathbf{H} \) are the electric and magnetic fields respectively and \( c \) is the speed of light. By eliminating \( \mathbf{E} \) from equations (6), we obtain the induction equation

\[ \frac{\partial \mathbf{H}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{H}) . \quad (6-a) \]

The gravitational potential \( V \) is given by the Poisson equation

\[ \nabla^2 V = -4\pi G \rho . \quad (7) \]

The above equations have to be supplemented by an equation of state of the fluid. In the adiabatic approximation, we may assume

\[ \frac{d}{dt} (p \rho - 2^2) = 0 , \quad (8) \]

where \( \gamma \) is the ratio of specific heats. This is a convenient assumption to make when the interest is in dynamical instabilities as opposed to thermal instabilities.

The boundary conditions are

\[ n \times \langle \mathbf{E} \rangle = \frac{1}{2} (n \cdot \mathbf{v}) \langle \mathbf{H} \rangle , \]

\[ \langle p + \frac{\mathbf{H}^2}{8\pi} \rangle = 0 , \quad (9) \]

\[ n \cdot \langle \mathbf{H} \rangle = 0 \quad \text{and} \quad n \times \langle \mathbf{H} \rangle = \mathbf{K} , \]
where $\mathbf{n}$ denotes the unit outward normal to the boundary, and $K$ is the surface current density. The angular brackets denote the jump in a quantity across the boundary of the configuration. In the present studies we will assume that the configuration has axial symmetry about the $z$-axis.

§1.2 The Oscillations and the Stability of the Configuration

The usual method for the study of the oscillations and the stability of an equilibrium configuration is to consider a small (infinitesimal) perturbation to the system and study its behaviour with time. Of the techniques available for such a study, the variational method and the tensor virial method have been found very useful. If we let $\xi(x,t)$ denote the Lagrangian displacement of an element of the fluid, we may assume a time dependence of the form $\xi(x,t) = \xi(x) e^{i\sigma t}$. The linearized equations of motion can then be written in the form

$$\sigma^{-2} \partial^2 \xi = T(\xi), \quad (10)$$

where $T(\xi)$ is an operator linear in $\xi$. Equation (10) together with the linearized form of the boundary conditions (9) defines a characteristic value problem for $\sigma^{-2}$. In the 'normal mode analysis' $\xi$ is expressed in terms of a suitable complete set of orthonormal functions. We solve the differential equation corresponding to each normal mode subject to the appropriate boundary conditions to obtain the characteristic values $\sigma^{-2}$. Let $\sigma_n^2 (n = 1, 2 \ldots \infty)$
be the characteristic frequencies of the various modes of oscillation. If $\sigma_n$ are all real, $|\xi|$ is bounded and the system is stable, whereas if there exists a particular $\sigma_n$ which is complex, $(\sigma_n = \sigma_R + i\sigma_I)$, the system will be unstable if $\sigma_I < 0$.

If the equilibrium configuration is a static one, it can be shown that the operator $T(\xi)$ is self-adjoint (cf. Bernstein et al. 1958, Chandrasekhar 1961, Kovetz 1966). The characteristic values of $\sigma^{-2}$ will then be real and this rules out overstability. In this case equation (10) admits a variational formulation and we can write

$$\sigma^{-2} \int_V \frac{\partial}{\partial \tau} |\xi|^2 \, d\tau = \int_V \xi \cdot T(\xi) \, d\tau, \quad (11)$$

where the integration is over the volume of the configuration. In order to obtain the characteristic values, we substitute in equation (11) a suitable trial function for $\xi$ satisfying the appropriate boundary conditions. The particular form used is suggested by some 'a priori' guess about the nature of the exact solution and may contain a number of variational parameters. Let $\alpha_i (i = 1, \ldots, n)$ be the linear variational parameters. On requiring that equation (11) be stationary, with respect to arbitrary variations of the variational parameters, we obtain a set of $n$ linear homogeneous equations involving $\alpha_i$ and $\sigma^{-2}$. We then solve the secular equation of this set of equations to obtain the characteristic
frequencies. We will use this method in Chapters II, IV and V.

In the virial method, we take moments of the equations of motion. We obtain the virial equation of various orders by taking the scalar product of the equation of motion with $x$, $xx$, $...$ and integrating over the volume of the configuration. An advantage in considering these moment equations is that the equations of lowest orders have simple physical interpretation. We can study the oscillations of a system by examining the linearized versions of these moment equations. In the present study we will be interested in virial equations of the second order which can be written in the form (see Chandrasekhar 1961)

$$\frac{d}{dt} \int_V \xi v_i x_k d\gamma = 2 T_{ik} + \delta_{ik} \left[ (\gamma - 1) U + \gamma \widetilde{\rho} \right] M_{ik} - 2M_{ik} + \delta_{ik},$$

(12)

where

$$T_{ik} = \frac{1}{2} \int_V \xi v_i v_k d\gamma , \quad T = T_{ii},$$

$$U = \frac{1}{(\gamma - 1)} \int_V p d\gamma ,$$

$$M_{ik} = \frac{1}{8\pi} \int_V H_i H_k d\gamma , \quad \gamma \omega_l = M_{ii},$$

(13)

$$W_{ik} = - \frac{1}{2} \int_V \xi v_{ik} d\gamma , \quad W = W_{ii} ,$$

...
\[ V_{ik}(x) = G \int_{V'} \phi'(x') \frac{(x_i - x'_i)(x_k - x'_k)}{|x - x'|^3} \, dV', \quad V = V_{ii}, \]
\[ S_{ik} = \frac{1}{8\pi} \int_S x_k (2H_1 H_j - |H|^2 \delta_{ij}) \, dS, \]

and the various integrations extend over the volume \( V \) bounded by the surface \( S \) of the equilibrium configuration.

The quantities \( T_{ik}, M_{ik}, W_{ik} \) and \( V_{ik} \) are the tensor generalizations of the kinetic energy \( T \), the magnetic energy \( W \), the gravitational energy \( M \) and the gravitational potential \( V \) respectively and \( U \) is the internal energy of the system. In order to investigate the oscillations of the system, we obtain the linearized version of equation (12). For the Lagrangian displacement \( \xi \), we take a trial function of the form (cf. Chandrasekhar 1961)
\[ \xi_{ij} = \chi_{ik} x_k e^{i\sigma t}, \tag{14} \]

where \( \chi_{ik} \) are nine constants. The justification for this assumed form for \( \xi \) lies in the fact this form leads to the exact characteristic frequencies of the radial pulsation mode and the Kelvin mode of oscillation of a homogeneous sphere in the absence of rotation and magnetic fields. Therefore, the results to be derived on the basis of equation (14) should be qualitatively reliable when the magnetic field and/or rotation are present. On substituting equation (14) in the linearized version of equation (12),
we obtain a system of linear homogeneous equations for the nine coefficients $X_{ik}$. The secular determinant of this system of equations is set equal to zero and the resulting characteristic equation determines the spectrum of nine values of $\alpha^{-2}$. This method has been very successfully used for configurations in which rotation and magnetic field are present. We will use this method in chapters III and VI.

It may be noted that the virial method is essentially equivalent to a variational technique (cf. Chandrasekhar and Lebovitz 1964). This has been explicitly demonstrated for second order virial equations by Clement (1964).

1.3 Outline Of The Thesis

In chapter II we examine the 'Kelvin' modes of oscillation of the Ferraro model (without rotation) using a variational principle. The second order virial equations are used to examine the transverse shear, the toroidal and the pulsation modes of the Ferraro model in Chapter III. Chapters IV and V are devoted to a study of the various modes of oscillation of a gaseous polytrope with a toroidal magnetic field using a variational principle. In chapter IV we examine the radial modes of oscillation, while in chapter V we develop the form of the variational principle for deformations belonging to a spherical harmonic of order
i.e. $P_\ell (\mu)$ [\(\ell \neq 0\)] and then examine the non-radial modes belonging to $\ell = 1, 2 \text{ and } 3$. Finally in chapter VI the second order virial equations are used to study the various modes of oscillation of a polytrope with a magnetic field containing both a toroidal and a poloidal component. In chapters IV to VI, the calculations have been carried out for polytropic indices $n = 1, 1.5, 2, 3 \text{ and } 3.5 \text{ and for values of } \gamma = 1.55, 1.60, 1.65 \text{ and } 5/3$.

References to equations in other chapters will be given by adding the chapter number (in Roman numeral) to the equation number; thus (IV-47) refers to equation (47) in chapter IV.