CHAPTER IV

Pseudovalueation and Pseudonorm

§ 1. Introduction:- A linear space is a vector space over the field of real numbers or over the field of complex numbers. A linear topological space is a linear Hausdorff space in which the operations of addition and scalar multiplication are continuous.

A linear space V is said to be a normed space if to each x in V there is associated a non-negative real number \( \| x \| \) called the norm of x in such a way that

1. \( \| x + y \| \leq \| x \| + \| y \| \) for all x, y in V
2. \( \| \alpha x \| = |\alpha| \| x \| \) for all x in V and all scalars \( \alpha \)
3. \( \| x \| = 0 \) if and only if \( x = 0 \),

where \( |\alpha| \) denotes the ordinary absolute value of real (complex) number \( \alpha \).

Every normed space may be regarded as a metric space in which the distance \( \phi (x, y) \) between x and y is \( \| x - y \| \). The relevant properties of \( \phi \) are

1. \( 0 \leq \phi (x, y) < \infty \) for all x and y
2. \( \phi (x, y) = 0 \) if and only if \( x = y \)
3. \( \phi (x, y) = \phi (y, x) \) for all x and y
4. \( \phi (x, z) \leq \phi (x, y) + \phi (y, z) \) for all x, y, z.
In any metric space, the open ball with centre at $x$ and radius $r$ is the set

$$B_r(x) = \{y : \rho(x, y) < r\}$$

In particular if a metric in a normed space $V$ is derived from a norm as above, then a topology is obtained by declaring a subset of $V$ to be open if it is a (possibly empty) union of open balls. It is easy to verify that the vector space operations (addition and scalar multiplication) are continuous in this topology.

In this chapter we introduce the concept of a

$\omega$-pseudonorm which generalizes the notion of a norm on an arbitrary vector space $V$ over an arbitrary field $K$ with pseudovalue $\omega$ and give a necessary and sufficient condition for a topological vector space $V$ over a topological field $K$ to be $\omega$-pseudonormable, that is to say one can introduce a pseudovalue $\omega$ in $K$ and a $\omega$-pseudonorm $\|\|$ in $V$ such that the topology induced in $K$ by the pseudovalue $\omega$ and the topology induced in $V$ by the $\omega$-pseudonorm $\|\|$ are the same as those given in $K$ and $V$ respectively. (For the definition of $\omega$-pseudonorm $\|\|$ and the topology induced by a $\omega$-pseudonorm $\|\|$ see section 2 of this chapter). For this purpose we extend the concept of gauge set introduced by P. M. Cohn to pair gauge sets and also use the concept of bounded sets in a topological vector spaces.
This result may be viewed as a generalization of the following result of Kolmogrov.

A linear topological space $V$ is normable that is to say one can introduce a norm in $V$ such that the topology in $V$ defined by this norm is the same as the given topology, if and only if, $V$ contains a bounded convex neighbourhood $C$ of zero. Here $C \subseteq V$ is said to be bounded if $\epsilon_n x_n \to 0$ for arbitrary elements $x_n \in C$ and scalars $\epsilon_n \to 0$ and $C$ is called convex if with $x_1, x_2 \in C$,

$$t x_1 + (1 - t) x_2 \in C$$

for all real numbers $t$ with $0 \leq t \leq 1$.

§ 2. **Pseudonormed Vector Spaces**

In this section we shall introduce basic notations and give examples.

**Definition 2.1**: Let $K$ be a field with a pseudovaluation and $V$ be a vector space over $K$. By a $\omega$-pseudonorm on $V$, we mean a real valued function on $V$ such that

1. $\|v\| \geq 0$ for all $v \in V$ and $\|0\| = 0$ where 0 is zero vector of $V$;
2. $\|a v\| \leq \omega(a) \|v\|$ for all $a \in K$ and $v \in V$;
3. $\|v_1 - v_2\| \leq \|v_1\| + \|v_2\|$ for all $v_1, v_2 \in V$;
4. there exists $v \in V$ such that $\|v\| \neq 0$. 
We shall hereafter refer the \( \omega \) - pseudonorm as just pseudonorm. We observe that
\[
\| - v \| = \| v \| \quad \text{for all } v \in V
\]
and
\[
\| v_1 + v_2 \| \leq \| v_1 \| + \| v_2 \| \quad \text{for all } v_1, v_2 \in V.
\]

**Example 2.2:** The field \( \mathbb{R} \) with the ordinary absolute value, is a field with pseudovaluation. Let \( V = \mathbb{R}^n \) be the \( n \)-dimensional vector space over \( \mathbb{R} \). Define for each \( x = (a_1, a_2, \ldots, a_n) \in V \)
\[
\| x \| = \sqrt{a_1^2 + \ldots + a_n^2}
\]
Then \( \| . \| \) is a pseudonorm on \( V \).

**Example 2.3:** Let \( K \) be any field with a pseudovaluation and \( V = K[x] \) be the polynomial ring over \( K \). Define for each \( f(x) = a_0 + a_1x + \ldots + a_n x^n \in V \)
\[
\| f(x) \| = \text{Max} \{ \omega(a_0), \omega(a_1), \ldots, \omega(a_n) \}.
\]
Then \( \| . \| \) is a pseudonorm on \( V \).

Let \( K \) be a field with a pseudovaluation \( \omega \) and \( V \) be a vector space over \( K \) with a pseudonorm \( \| . \| \). Then there is a unique topology \( T_\omega \) on \( K \) and \( T_\| \) on \( V \) with respect to which the collection \( \mathcal{U} = \{ U_\| (a) \}_{\| = 1,2,\ldots} \) where
\[
U_\| (a) = \{ b \in K : \omega(b - a) < 1/\| \}
\]
and the collection $\mathcal{N} = \{\mathcal{N}_r(v)\}_{r=1,2,\ldots}$ where

$$\mathcal{N}_r(v) = \{u \in V : \|v - u\| < 1/r\},$$

are basis of neighbourhoods at $a$ and $v$ respectively. It is not hard to verify that under these topologies $K$ becomes a topological field and $V$ a topological vector space.

**Definition 2.4.** A pseudovaluation $\omega$ (and a pseudonorm $\| \cdot \|$) on a field $K$ (on a vector space $V$) is said to be almost trivial if the topology induced by $\omega$ on $K$ (by $\| \cdot \|$ on $V$) is discrete.

Any other pseudovaluation on $K$ (pseudonorm on $V$) is called almost non-trivial.

One can verify easily that if $(V, \| \cdot \|)$ is a pseudonormed vector space over an almost non-trivial pseudovaluated field $(K, \omega)$, then the topology induced by $\| \cdot \|$ on $V$ is non-discrete.

**Definition 2.5.** A subset $A$ in a topological field $K$ is called bounded, if given any neighbourhood $U_1$ of $o$, there exists a neighbourhood $U_2$ of $o$ such that $U_2 A \subseteq U_1$. Here

$$U_2 A = \{x a : x \in U_2, a \in A\}.$$

Analogously, we define a subset $B$ of a topological vector space $V$ over a topological field $K$ bounded if given a neighbourhood $N$ of $o$ in $V$ there can be found a neighbourhood $U$ of $o$ in $K$ such that $UB \subseteq N$. Here

$$UB = \{a v : a \in U, v \in B\}.$$
Remark 2.6: The set \( K \) in the topological field \( K \) can not be bounded unless the topology on \( K \) is discrete. Likewise, the set \( V \) of the topological vector space \( V \) over a topological field \( K \) can not be bounded unless the topology on \( K \) is discrete.

Lemma 2.7: If \( B_1 \) and \( B_2 \) are two bounded subsets of a topological vector space \( V \) over a topological field \( K \) then \( B_1 + B_2 \) and \( a \cdot B_1 \) for all \( a \) in \( K \) are also bounded.

Proof: Since \( B_1 \) and \( B_2 \) are bounded, there exist suitable neighbourhoods \( U_1 \) and \( U_2 \) of \( 0 \) in \( K \) such that

\[
U_1 B_1 \subseteq N', \quad i = 1, 2
\]

for any given neighbourhood \( N' \) of \( 0 \) in \( V \). If \( N \) is any given neighbourhood of \( 0 \) in \( V \), then choosing \( N' \) to be such that \( N' + N' \subseteq N \) and \( U \) to be \( U_1 \cap U_2 \), we find that

\[
U(B_1 + B_2) \subseteq U_1 B_1 + U_2 B_2 \subseteq N' + N' \subseteq N.
\]

From the continuity of scalar multiplication, we get for any given neighbourhood \( N_1 \) of \( 0 \), another neighbourhood \( N_2 \) of \( 0 \) and \( U_1 \) of \( a \) such that \( U_1 N_2 \subseteq N_1 \). As \( B_1 \) is bounded, there exist a \( U_2 \) such that \( U_2 B_1 \subseteq N_2 \). Therefore \( U_1 (U_2 B_1) \subseteq U_1 N_2 \subseteq N_1 \). Thus \( a B_1 \) is bounded.

Corollary 1: If \( A_1 \) and \( A_2 \) are bounded subsets of a topological field \( K \), then so are \( A_1 + A_2 \) and \( a A_1 \) for all \( a \) in \( K \).
Corollary 2: If \( B \) is a bounded subset of a topological vector space \( V \) over a topological field \( K \), then \( \{ v + B \} \) where \( v \) in \( V \) is also a bounded subset of \( V \).

The proof of corollary 1 follows from specializing \( V \) to \( K \) in lemma 2.7 and of corollary 2 from the fact that singleton sets are always bounded.

Corollary 3: If \( A \) is a bounded subset of a topological field \( K \) and \( a \) is any element of \( K \), then \( \{ a + A \} \) is also bounded.

Lemma 2.8: Suppose that \((K, \omega)\) is a field with an almost non-trivial pseudovaluation \( \omega \) and \((V, \|\cdot\|)\) is a pseudonormed vector space over \( K \). Then

1. \( A \subseteq K \) is bounded if and only if there exists a real number \( r \) such that \( \omega(a) \leq r \) for all \( a \) in \( K \).

2. \( B \subseteq V \) is bounded if and only if there exists a real number \( s \) such that \( \| v \| \leq s \) for all \( v \) in \( B \).

Proof: It is sufficient to prove (ii) as (i) can be deduced by taking \( V = K \) and \( \|\cdot\| = \omega \). If \( B \) is bounded, then for any given \( \varepsilon > 0 \), we can find a \( \delta > 0 \) such that if \( \omega(a) < \delta \), for some \( a \) in \( K \), then \( \| a \| \varepsilon \) for all \( v \) in \( B \). Taking \( \varepsilon = 1 \) and a corresponding \( \delta \) from the almost non-trivial nature of \( \omega \), we can find an element \( a \) in \( K \) with \( \omega(a) < \delta \) such that

\[
\| v \| = \| a^{-1} v \| \leq \omega(a^{-1}) \| a \| < \omega(a^{-1}) \text{ for all } v \text{ in } B.
\]

If we take \( s = \omega(a^{-1}) \), we get the desired result.
Conversely, suppose there exists an $s$ with $\|v\| \leq s$ for all $v$ in $B$. Then given any positive integer $t$ if we choose a positive integer $r$ such that $r \geq (s + 1)^t$, then whenever $\omega(a) < 1/r$ for some $a$ in $K$, then

$$\|a v\| \leq \omega(a) \|v\| < \frac{1}{r} s \leq \frac{1}{(s + 1)^t} s < 1/t$$

for all $v$ in $B$. Thus $U_{\omega}(a) B \subseteq K_{\omega}(a)$.

§ 3. Pair gauge sets and pair gauge functions.

P. M. Cohn introduced in [5] the concept of gauge set and gauge function to study pseudovaluation on a field $K$. Here we extend these notions to study pseudonormed vector spaces over pseudovaluated fields.

Definition 3.1. Let $V$ be a vector space over a field $K$. An ordered pair $(H, G)$ of sets where $H \subseteq V$ and $G \subseteq K$ will be called a pair gauge set if the following five conditions hold.

(i) $H = -H$ and $-1 \in G$.

(ii) $GH \subseteq H$ and $GG \subseteq G$ where $GH = \{a v : a \in G, v \in H\}$ and $GG = \{a b : a, b \in G\}$.

(iii) There exists an element $c$ in $K$ with $H + H \subseteq c H$ and $G + G \subseteq c G$. Here $H + H = \{v + w : v, w \in H\}$ and $G + G = \{a + b : a, b \in G\}$.

(iv) $H$ is a proper subset of $V$ and $G$ is a proper subset of $K$.

(v) There exists a non-zero element $d$ in $G$ such that

$$V = \bigcup_{n=1}^{\infty} d^n H \text{ and } K = \bigcup_{n=1}^{\infty} d^n G.$$
Any element \(d\) satisfying condition (v) of the above definition will be called a \textit{gauge element} of the pair gauge set \((H, G)\).

**Example 3.2.** Let \((K, \omega)\) be a almost non-trivial pseudovaluated field with \(\omega(1) = 1\) and \((V, \| \cdot \|)\) be a pseudonormed vector space over \((K, \omega)\). If we take

\[
H = \{ v \in V : \| v \| \leq 1 \}
\]

and

\[
G = \{ a \in K : \omega(a) \leq 1 \}
\]

then \([H, G]\) gives rise to a pair gauge set and any non-zero element \(d\) in \(K\) with \(\omega(d) < 1\) is a gauge element.

**Lemma 3.3.** If \((H, G)\) is a pair gauge set of a vector space \(V\) over a field \(K\) and \(d\) is a gauge element of \((H, G)\), then

\[(i) \ldots \subseteq d^2 H \subseteq dH \subseteq H \subseteq d^{-1} H \subseteq d^{-2} H \subseteq \ldots \quad \text{and}
\]

\[(ii) \ldots \subseteq d^2 G \subseteq dG \subseteq G \subseteq d^{-1} G \subseteq d^{-2} G \subseteq \ldots \]

are strict ascending sequences and \(\bigcap_{n=1}^{\infty} d^n G = \{0\}\).

**Proof:** From \(d\) in \(G\) and \(dH \subseteq H\), we derive that

\[
d^n H = d^{n-1}(d H) \subseteq d^{n-1} H \quad \text{for all } n.
\]

If \(d^n H = d^{n-1} H\) for some \(n\), then \(d^{-1} H = H\). Now using induction on \(n\) we can prove that \(d^{-n} H = H\) for all integers \(n \geq 1\).
Therefore $V = \bigcup_{n=1}^{\infty} d^{-n} H = H$ which is a contradiction, because $H \neq V$.

Similarly we can show that

$$
... \subseteq d^2 G \subseteq d G \subseteq G \subseteq d^{-1} G \subseteq d^{-2} G \subseteq ...
$$

is a strictly ascending sequence.

We now prove that $\bigcap_{n=1}^{\infty} d^n G = \{e\}$. Suppose $a \neq e$ is an element of $\bigcap_{n=1}^{\infty} d^n G$. As $a^{-1} \in K = \bigcup_{n=1}^{\infty} d^n G$, we can find an integer $n$ such that $a^{-1} \in d^n G$. Therefore $a^{-1} d^n \in G$.

Now

$$
d^{-1} = (a^{-1} d^n) (a d^{-(n+1)}) \in G G \subseteq G
$$

which is not true. For otherwise $d^{-1} G \subseteq G G \subseteq G$, and we get a contradiction.

**Remark 3.4:** If $d$ is a gauge element of a pair gauge set $(H, G)$ then so is $d^t (t > 0)$. Let $c$ be the element occurring in (iii) in the definition of pair gauge set. As $c \in K = \bigcup_{n=1}^{\infty} d^n G$ implies that there exists an integer $t > 0$ such that $c \in d^{-t} G$. Hence $G + G \subseteq c G \subseteq (d^{-t} G) G \subseteq d^{-t} G$ and $H + H \subseteq c H \subseteq (d^{-t} G) H \subseteq d^{-t} H$.

Therefore, when the gauge element is not fixed in advance, we may choose $d$ such that $G + G \subseteq d^{-1} G$ and $H + H \subseteq d^{-1} H$.

With each pair gauge set $(H, G)$ and gauge element $d$, we associate a pair of extended integer valued functions $\varphi$ and $f$ on $V$ and $K$ respectively, as follows.
(i) If \( v \in V \) and \( v \in \bigcap_{n=1}^{\infty} d^n \), then \( \varphi(v) = \infty \).

If \( v \in d^n \) and \( v \notin d^{n+1} \), then \( \varphi(v) = n \).

(ii) Suppose \( a \in K \) and \( a \neq 0 \). Then we set \( f(a) = 1 \) if \( a \in d^n \) and \( a \notin d^{n+1} \). We also set \( f(0) = \infty \).

\( (\varphi, f) \) is called a pair gauge function associated with the pair gauge set \((H, G)\) and gauge element \(d\).

Let \( V \) be a vector space over \( K \) and \((H, G)\) be a pair gauge set with \(d\) as a gauge element. If \((\varphi, f)\) is the pair gauge function associated with \((H, G)\) and \(d\), then we have:

\( (i) \ \varphi(v) = \varphi(-v) \) for all \( v \in V \) and \( f(-1) = 0 \)

\( (ii) \ \varphi(av) \geq f(a) + \varphi(v) \) for all \( a \in K \) and \( v \in V \),

and \( f(ab) \geq f(a) + f(b) \) where \( a \) and \( b \) vary over \( K \).

(iii) There exists a positive integer \( t \) such that

\[ \varphi(v_1 + v_2) \geq \min \{ \varphi(v_1), \varphi(v_2) \} - t \] for all \( v_1, v_2 \in V \)

and

\[ f(a + b) \geq \min \{ f(a), f(b) \} - t \] for all \( a, b \in K \).

(iv) \( f(a) = f(d^{-1}) = 1 \)

(v) \( \varphi(0) = \infty, f(0) = \infty \).
Given any pair of extended integer valued functions \((\varphi, \psi)\) on \(V\) and \(K\) satisfying the above properties with some non-zero \(d\) in \(K\), we get a pair of sets \(H\) and \(G\) given by

\[
H = \{v \in V : \varphi(v) \geq d\} \quad \text{and} \quad G = \{a \in K : \psi(a) \geq d\}.
\]

One can see without difficulty that \((H, G)\) is a pair gauge set with \(d\) as a gauge element and \((\varphi, \psi)\) is the pair gauge function associated with \((H, G)\) and \(d\).

**Remark 3.5.** Let \(t\) be as in (iii) above, if we take \(d^t\) as gauge element instead of \(d\), then the pair gauge function \((\varphi', \psi')\) associated with \((H, G)\) and \(d^t\) satisfy the inequalities of (iii) with \(t\) replaced by 1, that is to say

\[
\varphi'(v_1 + v_2) \geq \min \{\varphi'(v_1), \varphi'(v_2)\} - 1 \quad \text{and} \quad \psi'(a + b) \geq \min \{\psi'(a), \psi'(b)\} - 1.
\]

**4. The Pseudoevaluation and Pseudonorm Defined by a Pair Gauge Set.**

Let \(V\) be a vector space over a field \(K\), \((H, G)\) be any pair gauge set with gauge element \(d\) and \((\varphi, \psi)\) the corresponding gauge function. By remark 3.5 above we may choose a \(d\) so that \(t = 1\) (Here \(t\) is as in the properties of the pair gauge function).

Now let \(e\) be a real number such that \(1 < e \leq 2\). We set
\[ \psi(v) = \begin{cases} e^{-\varphi(v)} & \text{if } \varphi(v) \neq \infty \\ 0 & \text{if } \varphi(v) = \infty \end{cases} \]

and
\[ g(a) = \begin{cases} e^{-f(a)} & \text{if } f(a) \neq \infty \\ 0 & \text{if } f(a) = \infty \end{cases} \]

Obviously, \( \psi \) and \( g \) are real valued functions on \( V \) and \( K \) respectively with values \( e^a \) and 0 such that

(i) \( g(-1) = 1 \) and \( \psi(-v) = \psi(v) \) for all \( v \) in \( V \),

(ii) \( g(a + b) \leq g(a) g(b) \) and \( \psi(a + v) \leq \psi(a) \psi(v) \) for all \( a, b \) in \( K \) and \( v \) in \( V \),

(iii) \( g(a + b) \leq e \max \{ g(a), g(b) \} \) for all \( a, b \) in \( K \) and

\[ \psi(u + v) \leq e \max \{ \psi(u), \psi(v) \} \) for all \( u, v \) in \( V \),

(iv) \( g(c) = 0 \) and \( \psi(c) = 0 \).

Now we derive a pseudovaluation from \( g \) and a pseudonorm from \( \psi \). For this we need the following lemmas.

**Lemma 4.1.** (i) If \( a_1, a_2 \) in \( K \) are such that \( g(a_1) \leq g(a_2) \) and \( g(a_1 + a_2) > g(a_1) + g(a_2) \) then \( e \cdot g(a_1) \leq g(a_2) \) where \( e \) and \( g \) are as above.

(ii) If \( v_1, v_2 \) in \( V \) are such that \( \psi(v_1) \leq \psi(v_2) \) and \( \psi(v_1 + v_2) > \psi(v_1) + \psi(v_2) \) then \( e \cdot \psi(v_1) \leq \psi(v_2) \) where \( e \) and \( \psi \) are as above.
Proof. We prove (11) as (1) can also be proved in a similar way. Suppose $\psi(v_1) = \psi(v_2)$. Then

$$\psi(v_1 + v_2) \leq \max \{ \psi(v_1), \psi(v_2) \} \leq 2 \psi(v_1) = \psi(v_1) + \psi(v_2).$$

This contradicts the hypothesis. Hence $\psi(v_1) < \psi(v_2)$.

In case $\psi(v_1) = 0$, then $\psi(v_1) = 0 < \psi(v_2)$. If not, then from the definition of $\psi$ we get $\varphi(v_2) < \varphi(v_1)$ so that

$$1 + \varphi(v_2) \leq \varphi(v_1)$$

as $\varphi$ is an integer valued function.

Consequently

$$e_1 - \varphi(v_1) \leq e_2 \varphi(v_2)$$

which gives $e \psi(v_1) \leq \psi(v_2)$.

Lemma 4.2 (1) Given any decomposition $a = a_1 + a_2 + \ldots + a_n$ of $a$ in $K$, there exist a second decomposition $a = b_1 + \ldots + b_m$ of $a$ in $K$ such that

$$\gamma(b_1) + \ldots + \gamma(b_m) \leq \gamma(a_1) + \ldots + \gamma(a_n)$$

and

$$\gamma(a) \leq e \gamma(b_m)$$

where $\gamma$ and $e$ as above.

(ii) Given any decomposition $v = v_1 + v_2 + \ldots + v_n$ of $v$ in $V$, we can find another decomposition of $v$ as $v = w_1 + \ldots + w_m$ such that

$$\psi(v_1) + \ldots + \psi(v_n) \leq \psi(v_1) + \ldots + \psi(v_n).$$

and

$$\psi(v) \leq e \psi(v)$$

where $\psi$ and $e$ are as above.
Proof: We prove (1i) as proof of (1) is also exactly the same. In case
\[ \psi(v) \leq \psi(v_1) + \ldots + \psi(v_n), \]
then take \( v = v_1 \) and the result follows. Otherwise we claim that there is a decomposition of \( v \) as \( v = v_1 + \ldots + v_m \) such that
\[
\begin{align*}
\psi(v_1) + \ldots + \psi(v_m) &\leq \psi(v_1) + \ldots + \psi(v_n) \\
\text{and} \\
\psi(v_i + v_j) &> \psi(v_i) + \psi(v_j) \quad \text{for all } i, j = 1, 2, \ldots, m \quad i \neq j
\end{align*}
\]
If \( v = v_1 + v_2 + \ldots + v_n \) itself has the property (*) we keep it.
Otherwise, there exist \( i, j \) such that \( \psi(v_i + v_j) \leq \psi(v_i) + \psi(v_j) \).
Then
\[
v = (v_i + v_j) + v_1 + \ldots + v_{i-1} + v_{i+1} + \ldots + v_{j-1} + v_{j+1} + \ldots + v_n
\]
\[
= v_1' + v_2' + \ldots + v_{n-1}' \quad \text{Here } v_1' = v_1 + v_j, \ v_2' = v_1', \ldots \text{ etc}
\]
In this decomposition \( \psi(v_1') + \ldots + \psi(v_{n-1}') \leq \psi(v_1) + \ldots + \psi(v_n). \)
If this decomposition of \( v \) as a sum of \( v_1' \) satisfies (*) we are done.
In the contrary case we shall continue with this process till we get the desired type of decomposition. Note that \( m \geq 2 \). Arrange the \( w_j \) such that \( \psi(w_1) \leq \psi(w_2) \leq \ldots \leq \psi(w_m) \). Then by lemma 4.1 (ii), \( \psi(w_j) \leq \psi(w_{j+1}) \) for \( j = 1, 2, \ldots, m-1 \). Now we claim that \( \psi(v_1 + \ldots + v_t) \leq \psi(w_t) \) for all \( t = 1, 2, \ldots, m \).
This follows by induction and we get \( \psi(v) \leq \psi(\Sigma v_j) \leq \psi(v_m) \).

**Theorem 4.3** Let \( V \) be a vector space over \( K \), \((H,G)\) pair gauge set on \( V \) and \( \psi \), \( \gamma \) be real valued functions defined as above. We set

\[
\omega(a) = \inf \sum_j \gamma_j(a_j) \quad \sum_j a_j = a, \quad a_j \in K
\]

and

\[
\|v\| = \inf \sum_j \psi(v_j) \quad \sum_j v_j = v, \quad v_j \in V
\]

where the infima are taken over all possible finite decompositions of \( a = \sum a_j \) and \( v = \sum v_j \). Then \( \omega \) is a pseudovaluation on \( K \) and \( \| \cdot \| \) is a pseudonorm on \( V \) satisfying

\[
\omega(a) \leq \gamma(a) \leq \varepsilon \omega(a)
\]
and

\[
\|v\| \leq \psi(v) \leq \|v\| \quad \text{for all } v \text{ in } K
\]

**Proof:** We prove only the pseudonorm part, the pseudovaluation part can also be proved in similar way. It is clear that for all \( v \) in \( V \), \( \|v\| \leq \psi(v) \). For the other inequality, take any decomposition of \( v \) as \( v = v_1 + \ldots + v_n \) and apply lemma 4.2, to get another decomposition of \( v \) as \( v = v_1 + \ldots + v_m \) such that

\[
\sum \psi(v_j) \leq \sum \psi(v_k) \quad \text{and} \quad \psi(v) \leq \varepsilon \psi(v_m).
\]

Now

\[
\psi(v) \leq \varepsilon \psi(v_m) \leq \varepsilon \sum_{j=1}^m \psi(v_j) \leq \varepsilon \sum_{j=1}^n \psi(v_j).
\]

Taking lower bounds over all finite decompositions of \( v \), we get \( \psi(v) \leq \varepsilon \|v\| \).
We now verify that \( \| \cdot \| \) is a \( (\omega) \)-pseudonorm on \( V \).

That \( \| v \| > 0 \) and \( \| 0 \| = 0 \) are obvious from the definition.

Since \( \psi \) is not identically zero on \( V \), we see from the
inequality \( \psi(v) \leq e \| v \| \) and the fact that \( e > 1 \), that \( \| v \| \) is
not identically zero on \( V \). To show that \( \| av \| < (\omega(a)) \| v \| \)
for all \( a \in K \) and \( v \in V \), we take for any \( \varepsilon > 0 \), a decomposition of
\( a = a_1 + \ldots + a_n, a_i \in K \) and a decomposition of \( v = v_1 + \ldots + v_m \)
with \( v_i \in V \) such that
\[
\sum_{i,j} \psi(a_j) < (\omega(a)) + \frac{\varepsilon}{2(\omega(a)+1)}
\]
and
\[
\sum_{i,j} \psi(v_j) < \| v \| \frac{\varepsilon}{2(\omega(a)+1)}.
\]

Then \( \psi(v_j) < \psi(v_j) + \psi(a_j) \psi(v_j) < (\omega(a)) \| v \| + \varepsilon + \frac{\varepsilon^2}{4v} \)
As \( v = \sum_{i,j} a_j v_j \), we get \( \| av \| \leq (\omega(a)) \| v \| \).

In a similar fashion we can show that
\[
\| v - w \| \leq \| v \| + \| w \|.
\]

5. The topology associated with "pair gauge set."

In section 4 we have seen that to each pair gauge set
\((\mathcal{H}, \mathcal{G})\) of a vector space \( V \) over a field \( K \), we can associate a
pseudo-valuation \( \omega \) on \( K \) and a \( (\omega) \)-pseudo-norm \( \| \cdot \| \) on \( V \). Note
that \( \omega \) and \( \| \cdot \| \) still depend upon the choice of gauge element \( d \).
We will now show that \( \omega \) and \( \| \cdot \| \) are independent of the choice
of gauge element \( d \) up to equivalence. (Two pseudonorms on a vector
space $V$ are said to be equivalent if they define the same topology on $V$).

**Theorem 5.1.** Let $(\mathcal{H}, G)$ be any pair gauge set of a vector space $V$ over a field $K$ with $\| \cdot \|$ and $(\omega)$ associated pseudonorm and pseudovaluation on $V$ and $K$ respectively. If $d$ is any gauge element of $(\mathcal{H}, G)$ then $\{d^n H\}_{n=0,1,\ldots}$ and $\{d^n G\}_{n=0,1,\ldots}$ form a neighbourhood base at $0$ and $0$ respectively in the topology defined by $\| \cdot \|$ and $(\omega)$ on $V$ and $K$.

**Proof:** Theorem 7.2 of Chapter I shows that for each gauge element $d$ of $(\mathcal{H}, G)$, $\{d^n H\}_{n=0,1,\ldots}$ is a neighbourhood base at $0$ in the topology defined by $(\omega)$ on $K$. (Because $G$ is a gauge set of $K$ with gauge element $d$).

We now prove that $\{d^n H\}_{n=0,1,\ldots}$ is a neighbourhood base at $0$ in the topology defined by $\| \cdot \|$ on $V$.

Take $U = \{v \in V : \|v\| < 1/e\}$ (where $1 < e \leq 2$ is as in section 4).

Now $U$ is a neighbourhood of $0$ in the topology defined by $\| \cdot \|$ on $V$. Since $\psi(v) \leq e \|v\|$ for all $v$ in $V$, if we take $u \in U$ then

$$\psi(u) \leq e \|u\| < 1.$$

This means $u \in H$. Therefore $H$ itself is a neighbourhood of $0$ in the topology defined by $\| \cdot \|$. Continuity of scalar multiplication shows that $\{a H\}$ for all non-zero $a$ in $K$ are neighbourhoods of $0$. 


In particular for any choice of gauge element \( d \) of \((H, G)\)
\[
\{d^n H\}_n = 0, 1, \ldots
\]
are neighbourhoods of \( \theta \) in the topology defined by \( \| \cdot \| \) on \( V \).

Let now \( d_j \) be gauge element of \((H, G)\) used to define \( \| \cdot \| \) on \( V \).
Then \( d_j^{-1} \subseteq d^{-t} G \) for some integer \( t \geq 0 \). Therefore \( d^t \subseteq d_j G \).
If \( N \) is any neighbourhood of \( \theta \) in the topology defined by \( \| \cdot \| \) on \( V \),
then there exists a real number \( \varepsilon > 0 \) such that all elements with
\[
\| v \| < \varepsilon
\]
belong to \( N \). Now choose a positive integer \( n \) such that \( \varepsilon^{-n} \subseteq \varepsilon \).
If \( v \in d_j^n H \) then \( \| v \| \leq \psi(v) \leq \varepsilon^{-n} \subseteq \varepsilon \) implies \( v \in N \).
Therefore \( d_j^n H \subseteq N \) and hence \( d_j^m H \subseteq N \). Thus \( \{d^n H\}_n = \emptyset, 1, \ldots \)
form a neighbourhood base at \( \theta \) to the topology defined by \( \| \cdot \| \) on \( V \).

\section{Characterization of Pseudonorm Topology on a given Vector Space.}

In this section we will find the necessary and sufficient conditions for a topological vector space \((V, J_v)\) over a topological field \((K, J_K)\) to have their topologies \( J_v \) and \( J_K \) induced by a
\( \omega \)-pseudonorm on \( V \) and a pseudovaluation \( \omega \) on \( K \) respectively.
We need the concept of topologically nilpotent element in a topological field.

\textbf{Definition 6.1.} An element \( a \) of a topological field \( K \) is said to be topologically nilpotent if \( a^n \rightarrow 0 \) as \( n \rightarrow \infty \) in the topology of \( K \).
Theorem 6.2. Let \((K, \mathcal{T}_1)\) be a non-discrete topological field and \((V, \mathcal{T}_2)\) be a topological vector space over \((K, \mathcal{T}_1)\). Then \(\mathcal{T}_1\) can be obtained by almost non-trivial pseudovaluation \(\omega\) on \(K\) and \(\mathcal{T}_2\) by almost non-trivial \(\omega\)-pseudonorm \(\| \cdot \|\) on \(V\) if and only if

(i) \(K\) contains a non-empty open bounded set

(ii) There exists in \(K\) a non-zero topologically nilpotent element

(iii) \(V\) contains a non-empty open bounded set.

Proof: Let \(\mathcal{T}_1\) be defined by some almost non-trivial pseudovaluation \(\omega\) on \(K\) and \(\mathcal{T}_2\) by some almost non-trivial \(\omega\)-pseudonorm \(\| \cdot \|\) on \(V\). Then take

\[ A = \{ a \in K : \omega(a) < 1 \} \quad \text{and} \quad B = \{ v \in V : \| v \| < 1 \}. \]

clearly \(A \neq \emptyset\) and \(B \neq \emptyset\) (because \(0 \in A\) and \(0 \in B\)) and are open. By lemma (2.8) \(A\) is bounded in \(K\) and \(B\) is bounded in \(V\). Since \(\omega\) is almost non-trivial there exists a non-zero element \(a\) in \(K\) such that \(\omega(a) < 1\) and this \(a\) is a nilpotent element in \(K\).

For establishing the converse, we adopt the following strategy. We construct from the given conditions a pair gauge set and show that the pseudonorm and pseudovaluation associated with this pair gauge set give rise to the topology \(\mathcal{T}_1\) on \(V\) and \(\mathcal{T}_2\) on \(K\).

Let \(A\) be a non-empty bounded open set in \(K\). If \(0 \notin A\), we choose some element \(a\) in \(A\) and replace \(A\) by \(A' = A - a\). The set \(A'\) is again open. By lemma 2.7 corollary 3, it is also bounded
and it contains $a = a - a$. Replace $A'$ by $A'' = (A') \cap (-A')$.
Then $A''$ is a symmetric, open bounded set containing $o$.

Thus we can assume without loss of generality that $A$ is a symmetric, open and bounded subset of $K$ containing $o$. Because $A$ is bounded, there exists for each neighbourhood $N$ of $o$, a neighbourhood $P$ of $o$, such that $P \subseteq N$. Since $K$ is non-discrete, $P$ contains a non-zero element $a$ and therefore $a \cdot A \subseteq K$. Clearly $a \cdot A$ is again a neighbourhood of $o$ and since $N$ is arbitrary, the collection $\{a \cdot N\}$ where $a$ varies over the non-zero elements of $K$ form a neighbourhood base of $o$.

By continuity of multiplication, there is a non-zero element $b$ in $K$ such that

$$b \cdot A \subseteq A.$$  

Put $b^2 \cdot A = A_2$, then $A_2$ is again a symmetric, bounded neighbourhood of $o$. In addition

$$A_2 \cdot A_2 = b^4 \cdot A \subseteq b^2 \cdot A = A_2$$

and the sets $\{a \cdot A_2\}$ where $a$ varies over the non-zero elements of $K$ form again a neighbourhood base of $o$. Now define

$$G = \{a \in K : a \cdot A_2 \subseteq A_2\}.$$  

Since $A_2 \subseteq G$, $G$ is neighbourhood of $o$, and by definition

$$A_2 \cdot G \subseteq A_2.$$
Let $c$ be a non-zero topologically nilpotent element of $K$. Since $c^n \to 0$ and $G$ is a neighbourhood of $0$, $c^k \in G$ for some $k \geq 1$. Write $d = c^k$, then $d$ is again a topologically nilpotent element of $K$ and $d^n G$ is a neighbourhood of $0$ for each $n > 0$.

Let $N$ be any neighbourhood of $0$ then there is an $a \neq 0$ in $K$ such that $a A_2 \subseteq N$.

Now $d^n \to 0$, hence $d^n a^{-1} \to 0$ and so for some $n > 0$, $d^n a^{-1} \in A_2$.

Now $d^n \to 0$, hence $d^n a^{-1} \to 0$ and so for some $n > 0$, $d^n a^{-1} \in A_2$.

Therefore $\{d^n G\}_n = 0, 1, \ldots$ is a neighbourhood base of $0$ in $(K, J)$.

Next we shall construct $B$. Let $B$ be the given open bounded set in $V$. Then as before we may assume without loss of generality that $B$ is a symmetric open bounded set containing $0$.

Thus $B$ is a neighbourhood of $0$. The collection $\{b B\}$ as $b$ varies over the non-zero elements of $K$ gives rise to a neighbourhood base at $0$. For if $U$ is any neighbourhood of $0$, from the boundedness of $B$, we can find a neighbourhood $N$ of $0$ in $K$ such that $N B \subseteq U$. As $J_1$ is not discrete, we can find a $b$ in $N$, $b \neq 0$ such that $b \cdot B \subseteq U$.

Using the continuity of scalar multiplication and the fact that $B$ is a neighbourhood of $0$ we find that $(d^s G) (b B) \subseteq B$ for some non-negative integers $s$ and a non-zero element $b$ in $K$. 
As $d^* G$ is a neighbourhood of $0$ in $K$, so is $b d^* G$. Hence we can find an integer $t$ such that $d^t G \subseteq b d^* G$. Therefore $d^t G B \subseteq B$.

Now set $B_1 = d^t B$. Then clearly $B_1$ is a symmetric, bounded, open set containing $0$. Consequently $\{b B_1\}$ as $b$ varies over the non-zero elements of $K$ forms a neighbourhood basis of $0$.

Finally we define

$$H = \{v \in V : (d^t G)v \subseteq B\}.$$  

Since $B_1 \subseteq H$, $H$ is a neighbourhood of $0$ and by definition $d^t G H \subseteq H$. We assert that $\{d^n H\}_n = 0, 1, \ldots$ forms a basis of neighbourhoods at $0$. For if $U$ is any neighbourhood of $0$, then we can find a non-zero $b$ in $K$ such that $b B_1 \subseteq U$. As $d^n \to 0$, $d^n b^{-1}$ also tends to $0$. Therefore $d^n b^{-1}$ belongs to $d^t G$ for some integer $n$. Thus

$$d^n \in b d^t G.$$ Hence $d^n H \subseteq (b d^t G) H \subseteq B_1 \subseteq U$.

We now show that $(H, G)$ is a pair gauge set with $d$ as a gauge element.

(i) Since $A_2$ and $B_1$ are symmetric, therefore

$$-1 \in G \quad \text{and} \quad -H = H.$$

(ii) From the definitions of $G$ and $H$, we have

$$G G \subseteq G \quad \text{and} \quad GH \subseteq H.$$
(iii) From the continuity of addition in \( K \) and \( V \), and the fact that \( \{ d^n G \}_n = 0,1, \ldots \) and \( \{ d^n H \}_n = 0,1, \ldots \) are basis of neighbourhoods of 0 and \( \theta \) respectively, we get

\[ G + G \subseteq d^{-p} G \text{ for some integer } p > 0 \]

and

\[ H + H \subseteq d^{-q} H \text{ for some integer } q > 0. \]

Taking \( c = d^{-\max(p,q)} \), we get

\[ G + G \subseteq c G \quad \text{and} \quad H + H \subseteq c H. \]

(iv) Now \( G \neq K \) and \( H \neq V \), since \( G \) and \( H \) are bounded in \( K \) and \( V \) respectively while \( K \) and \( V \) are not.

(v) For any \( a \in K \), \( d^n a \to 0 \) as \( n \to \infty \) since \( d \) is topologically nilpotent. Therefore \( d^n a \in G \) for some integer \( n > 1 \).

Thus \( a \in d^{-n} K \). Hence \( K = \bigcup_{n=1}^{\infty} d^{-n} G \). Similarly for each \( v \in V \), \( d^n v \to 0 \) as \( n \to \infty \). Therefore \( d^n v \in H \) for some integer \( n > 1 \). Thus \( v \in d^{-n} H \). Hence \( V = \bigcup_{n=1}^{\infty} d^{-n} H \).

Thus \( (G, H) \) is a pair gauge set with \( d \) as a gauge element.

The rest of the proof follows from theorem 5.1.

We conclude this section with an example of a topological vector space \( (V, \mathcal{J}) \) over a pseudovaluated field \( (K, \omega) \) such that \( \mathcal{J} \) is not given by any \( \omega \)-pseudonorm.
Example 6.3: Let $K = \mathbb{R}$ be the field of real numbers and let $\omega(a) = c|a|$ where $|a|$ is the ordinary absolute value of $a$ and $c$ is a real number with $c > 1$. Clearly $\omega$ induces a pseudovaluation on $K$. For the vector space $V$ we take the ring of polynomials in one variable $x$ over $K$ and give a topology for $V$ as follows.

Set $\mathcal{N} = \{ \alpha : \alpha = (\alpha_1, \alpha_2, \ldots) \}$ sequence of positive numbers in $\mathbb{R}$. For each $\alpha$ in $\mathcal{N}$, define

$$N_\alpha = \{ f(x) = \sum_{i=1}^{n} a_i x^i : \omega(a_i) < \alpha_i \text{ for all } i \}. $$

It can be verified without difficulty that the set $\mathcal{N} = \{ N_\alpha \}$, gives rise to a neighborhood basis for $0$ in $V$. If $\mathcal{J}$ is the topology given by $\mathcal{N}$ then $(V, \mathcal{J})$ is a topological vector space over $(K, \omega)$. We now show that given any $N_\alpha$ in $\mathcal{N}$, we can find a $N_\beta$ such that for all non-zero $a$ in $K$, $a N_\alpha \not\subset N_\beta$. For this, if $\alpha = (\alpha_1, \alpha_2, \ldots)$ then take $\beta = (\beta_1, \beta_2, \ldots)$ where $\beta_m = 2^n \alpha_n$ for each $n$. Suppose $b$ is a non-zero element of $K$. Then there exists an integer $m$ such that $\omega(b) > \left( \frac{1}{2} \right)^m$. The polynomial $\frac{\beta_m}{2} x^m$ belongs to $N_\beta$, while $b \cdot \frac{\beta_m}{2} x^m$ does not belong to $N_\alpha$. For

$$\omega\left( \frac{b \beta_m}{2} \right) = 0 \quad \frac{b \beta_m}{2} > \frac{1}{2} \left( \frac{1}{2} \right)^{m-1} 2^{m} \alpha_m = \alpha_m.$$ 

Thus $b N_\beta \not\subset N_\alpha$. From this one can easily infer that there does not exist a non-empty open bounded subset of $V$. Therefore, in view of Theorem 6.2, the topology $\mathcal{J}$ for $V$ cannot be induced by any $\omega = \text{pseudonorm}$. 