CHAPTER I

BASIC DEFINITIONS AND ELEMENTARY PROPERTIES

In this chapter we shall assemble all the basic definitions, elementary properties, examples and known results. The best source for all the material presented here is Mahler [13], [14]. Other references include Cohn [5] and Klyek [10].

§ 1. Definition 1.1: Let \( R \) be a commutative ring with identity element \( 1 \). A real valued function \( \omega \) on \( R \) is called a \textit{pseudovaluation} on \( R \) if the following conditions are satisfied.

1. \( \omega(a) \geq 0 \) for all \( a \in R \); \( \omega(0) = 0 \); \( \omega(1) > 0 \).
2. \( \omega(ab) \leq \omega(a) \omega(b) \) for all \( a, b \in R \).
3. \( \omega(a - b) \leq \omega(a) + \omega(b) \) for all \( a, b \in R \).

Thus a pseudovaluation is more general than a valuation. For, any valuation \( \omega \) satisfies the equality \( \omega(ab) = \omega(a) \omega(b) \) for all \( a, b \in R \) instead of (ii) and \( \omega(a) \neq 0 \) for all \( a \neq 0 \) in \( R \).

Definition 1.2: A pseudovaluation \( \omega \) on \( R \) is called an \textit{almost valuation} if it satisfies

\[
\omega(ab) = \omega(a) \omega(b) \text{ for all } a, b \in R.
\]

Example 1.3: Every ring \( R \) possesses the pseudovaluation \( \omega_0 \) defined by

\[
\omega_0(a) = \begin{cases} 
0 & \text{if } a = 0 \\
1 & \text{if } a \neq 0, \ a \in R.
\end{cases}
\]
Remark 1.4: It is not true that every ring $R$ possesses a valuation in general. For, if $R$ is a ring with zero divisors and $a, b \in R$ such that $a \neq 0$, $b \neq 0$ and $a \cdot b = 0$ and if $V$ is a valuation of $R$ then

$$0 = V(0) = V(a \cdot b) = V(a) \cdot V(b) \neq 0$$

since $V(a), V(b)$ are non-zero positive real numbers. So a necessary and sufficient condition for a ring to possess a valuation is that it has no zero divisors and hence it has to be an integral domain. In case $R$ is an integral domain then it possesses at least the so called trivial valuation $V_0$, defined by

$$V_0(a) = \begin{cases} 0 & \text{if } a = 0 \\ 1 & \text{if } a \neq 0, \ a \in R. \end{cases}$$

Example 1.5.: Let $R$ be any arbitrary ring and $\mathfrak{a} \neq (0)$ be any proper ideal of $R$. Choose a real number $\alpha$ with $0 < \alpha < 1$. For each $x \in R$ set

$$\omega_\alpha(x) = \begin{cases} \alpha^n & \text{if } x \in \mathfrak{a}^n, \ x \notin \mathfrak{a}^{n+1} \\ 0 & \text{if } x \in \mathfrak{a}^n \text{ for all } n. \end{cases}$$

Then $\omega_\alpha$ is a pseudovaluation on $R$. We note that $\omega_\alpha$ actually satisfies a requirement which is stronger than that of (iii) in the definition of pseudovaluation namely

$$\omega_\alpha(x - y) \leq \text{Max} \{ \omega_\alpha(x), \omega_\alpha(y) \} \text{ for all } x, y \in R.$$
Example 1.6: Let \( R = \mathbb{Z} \) be the ring of rational integers and \( m = p_1 p_2 \cdots p_r \), \( r \geq 2 \) be a square free natural number with \( p_1, p_2, \ldots, p_r \) distinct prime numbers. Choose a real number \( c \) satisfying \( 0 < c < 1 \). Each non-zero element \( a \) can be written uniquely in the form \( a = m^t b \) where \( t \geq 0 \) and \( b \) are integers and \( b \) is not divisible by \( m \). Now we set

\[
\omega_m(a) = c^t
\]

and

\[
\omega_m(c) = c.
\]

Then \( \omega_m \) is a pseudovaluation on \( R \). \( \omega_m \) actually satisfies a requirement which is stronger than that of (ii) in the definition of pseudovaluation, namely,

\[
\omega_m(a^n) = [\omega_m(a)]^n \quad \text{for all positive integers } n \text{ and } a \in R.
\]

Definition 1.7: (Homogeneous pseudovaluation). A pseudovaluation \( \omega \) on a ring \( R \) is called a homogeneous pseudovaluation if

\[
\omega(a^n) = [\omega(a)]^n \quad \text{for all } a \in R \text{ and positive integers } n.
\]

It may be remarked that every valuation on a ring or a field is a homogeneous pseudovaluation.

We observe that in example 1.6, if \( m \) were taken to be equal to \( a_1^2 a_2^2 \) where \( a_1 \text{ or } a_2 > 1 \) then clearly the
homogeneity condition is violated.

**Definition 1.8**: A pseudovaluation \( \omega \) on a ring \( R \) is called non-Archimedean if

\[
\omega(a - b) \leq \text{Max} \{ \omega(a), \omega(b) \} \quad \text{for all } a, b \in R.
\]

A pseudovaluation is said to be Archimedean if it is not non-Archimedean.

For example, the pseudovaluation \( \omega \) on the ring of real numbers \( \mathbb{R} \) given by

\[
\omega(a) = c |a| \quad \text{for all } a \in \mathbb{R}
\]

where \( c > 0 \) is a fixed real number and \(|a|\) is ordinary absolute value, is an Archimedean pseudovaluation.

We shall now list a few elementary properties

1.9 (a). From (ii) we have

\[
\omega(1) \leq [\omega(1)]^2.
\]

Since \( \omega(1) > 0 \) we get

\[
\omega(1) \geq 1.
\]

In case \( \omega \) is valuation, we have

\[
\omega(1) = 1.
\]

1.9 (b). By putting \( a = 0 \) and \( b = -a \) in (iii) we get

\[
\omega(-a) \leq \omega(a) \quad \text{and} \quad \omega(a) \leq \omega(-a) \quad \text{for all } a \in R.
\]

Hence \( \omega(a) = \omega(-a) \) for all \( a \in R \).
1.9 (c) \( \omega(a + b) = \omega(a - (-b)) \leq \omega(a) + \omega(-b) \)
\[ = \omega(a) + \omega(b). \]

Thus \( \omega(a + b) \leq \omega(a) + \omega(b) \) for all \( a, b \in R \).

1.9 (d) Taking the pair \( a - b \), \( a \) in (iii) we get
\( \omega((a-b)-a) \leq \omega(a - b) + \omega(a) \) i.e.,
\( \omega(a - b) \geq \omega(-b) - \omega(a) = \omega(b) - \omega(a). \)

Similarly
\( \omega(b - a) \geq \omega(-a) - \omega(b) = \omega(a) - \omega(b). \)

i.e
\[ - (\omega(b) - \omega(a)) \leq \omega(b - a) = \omega(a - b). \]

From these two inequalities we get
\[ |\omega(a) - \omega(b)| \leq \omega(a - b) \]
for all \( a, b \in R \),
where \( | \cdot | \) denotes the ordinary absolute value of real numbers.

1.9 (e) In case \( R \) is field, we have
\( \omega(a) = 0 \) if and only if \( a = 0. \)

For otherwise
\( 0 = \omega(a) = \omega(a \omega^{-1}) \geq \omega(a \omega^{-1}) \geq \omega(1) \geq 1 \)
a contradiction.

1.9 (f) The set
\[ \eta_\omega = \{ a \in R : \omega(a) = 0 \} \]
forms an ideal of \( R \). Let \( \varphi \) be the canonical homomorphism from \( R \)
onto $\mathbb{R}/\eta_\omega$. Then the function $\tilde{\omega}$ on $\mathbb{R}/\eta_\omega$ given by

$$\tilde{\omega} \left( \phi(x) \right) = \omega(x) \text{ for all } x \in \mathbb{R}$$

is well defined and is a pseudovaluation on $\mathbb{R}/\eta_\omega$. This $\tilde{\omega}$ has the property that

$$\tilde{\omega}(y) = 0 \text{ if and only if } y = 0, y \in \mathbb{R}/\eta_\omega.$$ 

§ 2. The topology induced by a pseudovaluation

Let $\omega$ be a function on a ring $R$ satisfying the axioms (1) and (111) for a pseudovaluation. By means of this function we define a topology in $R$ by stipulating that

$$U_n(a) = \{ b \in R/ \omega(b - a) < 1/n \text{ where } n \text{ is a natural number} \}$$

forms a basis of neighbourhoods of the point $a \in R$, as $n$ varies over the natural numbers.

This topology is not Hausdorff in general. A necessary and sufficient condition for a pseudovaluation $\omega$ on a ring $R$ to induce a Hausdorff topology is that

$$\omega(a) = 0, a \in R \text{ if and only if } a = 0.$$ 

Thus if $K$ is a field with a pseudovaluation $\omega$, then the topology induced by $\omega$ on $K$ in the above manner is a Hausdorff topology.
If $R$ is a ring with a pseudovaluation $\omega$ defined on it, then the topology induced by $\omega$ as above is compatible with the ring structure and thus $R$ is made into a topological ring.

If $F$ is a field with a pseudovaluation $\omega$ defined on it, then $F$ becomes a Hausdorff topological field. Here we need to verify that inversion is a continuous operation and this we do as follows.

We first show that $x \to x^{-1}$ is continuous at 1. Let $U_0(1)$ be given. Choose an integer $m$ satisfying $1/m \leq 1/3n$. This $m$ is clearly $\geq 3$. Let $y \in U_m(1)$, then $\omega(y - 1) < 1/m$.

Put

$$y_0 = 1$$
$$y_1 = 1 + (1 - y)$$
$$y_2 = 1 + (1 - y) + (1 - y)^2$$
$$\ldots$$
$$y_j = 1 + (1 - y) + (1 - y)^2 + \ldots + (1 - y)^j$$

Then by induction we see that

$$1 - y_j y = (1 - y)^{j+1} \text{ for all } i = 0, 1, 2, \ldots, j.$$ 

Therefore

$$y^{-1} - y_i = y^{-1} (1 - y y_i)$$

so that

$$\omega(y^{-1} - y_i) \leq \omega(y^{-1}) \omega(1 - y y_i)$$

$$\leq \omega(y^{-1}) \omega(1 - y)^{i+1}.$$
Now choose \( j \) such that

\[
\omega(y^{-1}) \cdot (1/m)^{j+1} \leq 1/m.
\]

Then

\[
\omega(1 - y^{-1}) = \omega(1 - y_j + y_j - y^{-1})
\]
\[
\leq \omega(1 - y_j) + \omega(y_j - y^{-1})
\]
\[
= \omega(1 - y) - (1 - y)^2 \cdots (1 - y)^d + \omega(y_j - y^{-1})
\]
\[
\leq \omega(1 - y) + (\omega(1 - y))^2 + \cdots + (\omega(1 - y))^d + \omega(y_j - y^{-1})
\]
\[
\leq 1/m + 1/m^2 + \cdots + 1/m^d + \omega(y^{-1}) \cdot (1/m)^{d+1}
\]
\[
\leq \frac{1/m (1 - 1/m^d)}{1 - 1/m} + 1/m.
\]
\[
< \frac{1/m - 1}{1/m} \leq 3/m \leq 1/m \quad \text{(because } m \geq 3)\]

i.e., \( \omega(1 - y^{-1}) \leq 1/m \) and this implies \( y^{-1} \in U_n(1) \) which proves our assertion.

Now we show that \( x \to x^{-1} \) is continuous at each point \( a \in F \), \( a \neq 0 \). Let \( U_n(a^{-1}) \) be given, then there exists an integer \( m \) such that whenever \( \omega(x - 1) < 1/m \) Then \( \omega(x^{-1}) < 1/m \cdot \omega(a^{-1}) \).

Now choose an integer \( t \) satisfying \( \omega(a^{-1})/t \leq 1/m \). Let \( y \in U_n(a) \).

Then \( \omega(y - a) < 1/t \). Since

\[
\omega(y a^{-1} - 1) = \omega(a^{-1}(y - a)) \leq \omega(a^{-1}) \omega(y - a) < \omega(a^{-1})/t \leq 1/m
\]
we find that
\[ \omega(y^{-1} a - 1) < \frac{1}{n \omega(a^{-1})}. \]

Now
\[ \omega(y^{-1} a - 1) = \omega(a^{-1}(y^{-1} a - 1)) \leq \omega(a^{-1}) \omega(y^{-1} a - 1) < \frac{1}{n} \]
which complete the proof.

§ 3. Containment, equivalence and independence among pseudovaluations.

Definition 3.1: Let \( \omega_1, \omega_2 \) be two pseudovaluations on a ring \( R \). Then \( \omega_1 \) is said to be contained in \( \omega_2 \) (in symbol \( \omega_1 \subseteq \omega_2 \)) if the topology induced by \( \omega_2 \) is finer than the topology induced by \( \omega_1 \) in \( R \), i.e. every \( \omega_1 \)-open set is also an \( \omega_2 \)-open set. \( \omega_1 \) is said to be equivalent to \( \omega_2 \) (in symbol \( \omega_1 \equiv \omega_2 \)) if \( \omega_1 \) and \( \omega_2 \) induce the same topology in \( R \), i.e. \( \omega_1 \subseteq \omega_2 \) and \( \omega_2 \subseteq \omega_1 \).

If two pseudovaluations \( \omega_1 \) and \( \omega_2 \) on \( R \) are not equivalent, then we say that these are inequivalent.

It may be pointed out that if \( \omega_1 \) and \( \omega_2 \) are two non-trivial valuations on a field \( K \), then \( \omega_1 \subseteq \omega_2 \) if and only if \( \omega_1 \equiv \omega_2 \). But in case \( \omega_1 \) and \( \omega_2 \) are pseudovaluations on a field \( K \) which are not equivalent to the trivial pseudovaluation on \( K \) and \( \omega_1 \subseteq \omega_2 \) then \( \omega_1 \equiv \omega_2 \) is not true in general.

For example see 6.6 of this chapter.
Definition 3.2: Let \( \omega_1, \omega_2, \ldots, \omega_n \) be a finite set of pseudovaluations on a ring \( R \). Then \( \omega_1, \omega_2, \ldots, \omega_n \) are called independent if to each set of \( n \) elements \( a_1, a_2, \ldots, a_n \) from \( R \) and to each real number \( \varepsilon > 0 \), there exists an element \( a \in R \) such that

\[
\omega_i(a - a_i) < \varepsilon \quad \text{for} \quad i = 1, 2, \ldots, n.
\]

It may be pointed out that the notions of independence and inequivalence coincide in case of non-trivial valuations on a field \( K \). We use the analogue of Chinese Remainder Theorem (or what is the same the approximation theorem \([15]\)) for valuations to establish this. In case of pseudovaluations on a field (ring) these two notions of independence and inequivalence are completely different. For examples see 6.3 and 6.4 of this chapter.


Let \( R \) be an arbitrary ring with a pseudovaluation \( \omega \) and let \( \{a_n\} = \{a_1, a_2, \ldots, a_n, \ldots\} \) be an arbitrary infinite sequence of elements of \( R \).

Definition 4.1: (1) \( \{a_n\} \) is said to be a bounded sequence with respect to \( \omega \) if there exist two positive integers \( M \) and \( N \) such that

\[
\omega(a_n) \leq M \quad \text{for all} \quad n \geq N,
\]

\( -15- \)
(11) \( \{a_n\} \) is said to be a fundamental sequence with respect to \( \omega \) if given \( \varepsilon > 0 \) there is a positive number \( N(\varepsilon) \) such that

\[
\omega(a_n - a_m) < \varepsilon \quad \text{for all } n, m \geq N(\varepsilon),
\]

and

(iii) \( \{a_n\} \) is said to be a null sequence with respect to \( \omega \) if given \( \varepsilon > 0 \) there is a positive number \( N(\varepsilon) \) such that

\[
\omega(a_n) < \varepsilon \quad \text{for all } n \geq N(\varepsilon).
\]

The following theorem of Mahler [13] characterizes containment between two pseudovaluations in terms of sequences.

**Theorem 4.1:** Let \( \omega_1, \omega_2 \) be two pseudovaluations on a ring \( R \). Then

(i) \( \omega_1 \preceq \omega_2 \) if and only if for any sequence \( \{a_n\} \) in \( R \) \( \{a_n\} \) is a null sequence with respect to \( \omega_1 \) implies \( \{a_n\} \) is a null sequence with respect to \( \omega_2 \).

(ii) \( \omega_1 \preceq \omega_2 \) if and only if for any sequence \( \{a_n\} \) in \( R \) whenever \( \{a_n\} \) is a null sequence with respect to one of \( \omega_1 \) and \( \omega_2 \), then it is also a null sequence with respect to the other.

§ 5. New pseudovaluations out of old.

From a given pseudovaluation on a ring \( R \) we can obtain
new pseudovaluations in different ways.

Let \( \omega \) be any pseudovaluations on a ring \( R \) and \( \lambda \) be any positive real number. Then \( \omega^\lambda \) defined by \( \omega^\lambda(x) = [\omega(x)]^\lambda \) for all \( x \in R \) is also a pseudovaluation in case \( \omega \) is non-Archimedean. However if \( \omega \) is Archimedean then we need to impose the condition that \( \lambda \leq 1 \) in order that \( \omega^\lambda \) is also a pseudovaluation. If \( c \) is any real number such that \( c \geq 1 \) then \( \omega_c \) defined by

\[
\omega_c(x) = c \cdot (\omega(x)) \quad \text{for all } x \in R
\]

is also a pseudovaluation on \( R \). One can prove without any difficulty that both \( \omega^\lambda \) and \( \omega_c \) are equivalent to \( \omega \).

Next let \( \omega \) be any pseudovaluation on a ring \( R \). Then the function

\[
\overline{\omega}(x) = \lim_{n \to \infty} \left[\omega(x^n)\right]^{1/n}
\]

is defined for all \( x \in R \) and is a homogeneous pseudovaluation. This was proved by Rees [21] for non-Archimedean pseudovaluation and by Bergman [2] for arbitrary pseudovaluation. \( \overline{\omega} \) is called the homogenization of \( \omega \). Clearly if \( \omega \) is homogeneous then \( \overline{\omega} = \omega \). An example is given in \S 5 chapter III to show that \( \omega \) is not equivalent to \( \overline{\omega} \) in general.
Suppose $\{ \omega_\alpha \}_{\alpha \in \Lambda}$ is a class of pseudovaluations on a ring $R$ and that for each $x \in R$, $\sup_{\alpha \in \Lambda} \omega_\alpha(x)$ is finite. Then $\omega_\Lambda$ defined by

$$\omega_\Lambda(x) = \sup_{\alpha \in \Lambda} \omega_\alpha(x) \quad \text{for all } x \in R$$

is also a pseudovaluation on $R$. If $\Lambda$ is a finite set then $\omega_\Lambda$ defined by

$$\omega_\Lambda(x) = \sum_{\alpha \in \Lambda} \omega_\alpha(x) \quad \text{for all } x \in R$$

is also a pseudovaluation. One can also show that in this case $\omega_\Lambda$ is equivalent to $\omega_\Lambda^*$. 

§6. Subvaluation:

In this section we introduce the concept of subvaluation and determine completely the classes of subvaluations in the case of a field.

Definition 6.1: A pseudovaluation $\omega$ on a ring $R$ is called a subvaluation if either it is equal to the trivial valuation or there exists a finite set $U_1, \ldots, U_\eta$ of non-trivial, pairwise inequivalent valuations such that

$$\omega(a) = \max \{ U_1(a), \ldots, U_\eta(a) \} \quad \text{for all } a \in R.$$
Theorem 6.2 If 
\\ [\omega = \operatorname{Max} \{ \nu_1, \nu_2, \ldots, \nu_n \}] 
and \\
\\ [\mu = \operatorname{Max} \{ \mu_1, \mu_2, \ldots, \mu_m \}] 
are two subvaluations on a field \( K \), then \( \omega \) is equivalent to \( \mu \) if and only if \( n = m \) and for each \( i, 1 \leq i \leq n \), \( \mu_i = \nu_{\xi} \) for some real number \( \xi > 0 \) after a suitable rearrangement.

Proof: If \( n = m \) and for each \( i, 1 < i < n \), \( \mu_i = \nu_{\xi} \) for some real number \( \xi > 0 \), then \( \nu_{\xi} \sim \mu_i \), therefore \( \omega \sim \mu \).

Conversely, suppose \( \omega \sim \mu \).

Case (1) If \( n = 1 \) and \( \nu_{\xi} \) is equal to the trivial valuation then \( \omega \) is the trivial valuation and since \( \omega \sim \mu \) it follows that \( \mu \) induces the discrete topology on \( K \). Therefore there exists a real number \( \varepsilon > 0 \) such that \( \mu(a) \geq \varepsilon \) for all non-zero \( a \in K \). Now if \( m \neq 1 \) then by definition of subvaluation \( \mu_1, \mu_2, \ldots, \mu_m \) are non-trivial, pairwise inequivalent valuations. Therefore using the approximation theorem [15], we can find a non-zero element \( a \in K \) such that \( \mu_i(a) < \varepsilon \) for all \( i = 1, 2, \ldots, m \) which is not true. Therefore \( m = 1 \) and hence \( \mu = \nu_{\xi} \). Therefore \( \mu_{\xi} = \nu_{\xi} \) and both are the trivial valuations and \( n = m = 1 \).
Case (ii) No $V_i$, $1 \leq i \leq n$ and no $\mu_j$, $1 \leq j \leq m$ is equal to the trivial valuation. $\omega \preceq \mu$ implies $V_i \subset \mu$.

Now our claim is that $V_i \subset \mu_j$ for some $1 \leq j \leq m$. If $V_i$ is not contained in any $\mu_j$, $1 \leq j \leq m$, then $V_i$, $\mu_1$, $\ldots$, $\mu_m$ form a set of pairwise inequivalent valuations none of which is equal to the trivial valuation. Therefore by the approximation theorem for valuations, there exists a sequence $\{a_t\}$ from $K$ such that

$$V_i(a_t - 1) < 1/t$$

$$\mu_j(a_t) < 1/t \quad \text{for all } j = 1, 2, \ldots, m.$$ 

Then $\mu(a_t) < 1/t$ for all $t$ and hence $\{a_t\}$ is a null sequence with respect to $\mu$. But $\{a_t\}$ is not a null sequence with respect to $V_i$, and this is a contradiction to the fact that $V_i \subset \mu$. Therefore there exists $j$, $1 \leq j \leq m$ such that $V_i \subset \mu_j$. Since $V_i$ and $\mu_j$ are both valuations on $K$ none of which is the trivial valuation and whenever $\mu_j(a) < 1$ for some $a \in K$ implies $V_i(a) < 1$ therefore $V_i$ is equivalent to $\mu_j$.

Now we observe that if $V_i \subset \mu_j$ and $V_i \subset \mu_k$ then $j = k$. Otherwise $\mu_j \preceq \mu_k$, which contradicts the fact that $\mu_1, \ldots, \mu_m$ are pairwise inequivalent. Likewise we can show that every $V_i$ is equivalent to only one $\mu_j$ and vice versa so that $n = m$ and after suitable arrangement each $V_i$ is equivalent to $\mu_i$. 
Since \( V_i \) and \( \mu_i \) are both valuations on a field \( K \), therefore there exists real number \( \alpha_i > 0 \) with \( V_i^{\alpha_i} = \mu_i \).

**Corollary 6.3**: If \( \omega \) is a subvaluation on a field \( K \), then either \( \omega \) is equal to the trivial valuation or there exists a unique finite set \( V_1, V_2, \ldots, V_n \) of non-trivial pairwise inequivalent valuations such that

\[
\omega(a) = \max \{ V_1(a), \ldots, V_n(a) \} \quad \text{for all } a \in K.
\]

**Proof**: Suppose

\[
\omega(a) = \max \{ V_1(a), \ldots, V_n(a) \} = \max \{ \mu_1(a), \ldots, \mu_m(a) \} \quad \text{for all } a \in K,
\]

where \( V_i, 1 \leq i \leq n \) and \( \mu_j, 1 \leq j \leq m \) are non-trivial pairwise inequivalent valuations. From the above theorem we get \( n = m \) and after suitable rearrangement \( \mu_i = V_i^{\alpha_i} \) for each \( i \), \( 1 \leq i \leq n \) and for some real numbers \( \alpha_i > 0 \). It is now enough to prove that \( \alpha_i = 1 \) for \( i = 1, 2, \ldots, n \). If \( m = n = 1 \) then clearly \( \alpha_1 = 1 \). Suppose \( m = n \geq 2 \). Let \( 1 \leq i \leq n \) be any integer. Since \( V_1, \ldots, V_n \) are non-trivial pairwise inequivalent valuations on a field \( K \), therefore there exists an element \( a \in K \) such that

\[
V_i(a) > 1 \text{ and } V_j(a) < 1 \quad \text{for } j = 1, 2, \ldots, n, \; j \neq i.
\]

Therefore

\[
\mu_i(a) > 1 \text{ and } \mu_j(a) < 1 \quad \text{for } j = 1, 2, \ldots, n, \; j \neq i.
\]
Therefore
\[ V_i(a) = \omega(a) = \mu_i(a) \] and since \( V_i(a) \neq 1 \) it follows that \( a_1 = 1 \).

Remark 6.4: Given a subvaluation \( \omega \) on a field \( K \), we have completely determined the equivalence class of \( \omega \) in the set of all subvaluations on \( K \).

The following example will illustrate two important facts. First there may exist a finite set of non-trivial pairwise inequivalent valuations on a ring which are not independent (note that this cannot happen in case of a field) and second, Theorem 6.2 is not true for a ring in general.

Example 6.5: Let \( K \) be a totally real algebraic number field of degree \( n \geq 3 \) (i.e. \( K \) is algebraic extension of degree \( n \) over the field \( \mathbb{Q} \) of rational numbers, and every embedding of \( K \) in the field of complex numbers is contained in the field of real numbers). Let \( \sigma_1, \sigma_2, \ldots, \sigma_n \) be distinct \( \mathbb{Q} \)-isomorphisms of \( K \) in the field of real numbers. Let \( \mathcal{R} \) be the ring of algebraic integers in \( K \). For each \( \sigma \in \mathcal{R} \), we define
\[ U_\sigma(\alpha) = |\sigma_\alpha(\alpha)| \quad h = 1, 2, \ldots, n \]
where \( | \cdot | \) is the ordinary absolute value of real numbers.
Taking $\omega = \max \{ V_1, V_2, \ldots, V_n \}$ and $\mu$ to be the trivial valuation on $K$, one can show that $\omega \mu$ on $R$ (because the system of inequalities $V_h(\alpha) < 1$, $h = 1, 2, \ldots, n$ has no non-trivial solution in $R$).

We now show that $V_1, V_2, \ldots, V_n$ are non-trivial, pairwise inequivalent valuations on $R$. Let $u_1, u_2, \ldots, u_n$ be an integral base of $K$. Consider the following linear forms:

\[
L_1(x_1, \ldots, x_n) = x_1 \varphi_1(u_1) + x_2 \varphi_2(u_2) + \cdots + x_n \varphi_n(u_n)
\]

\[
L_2(x_1, \ldots, x_n) = x_1 \varphi_1(u_1) + x_2 \varphi_2(u_2) + \cdots + x_n \varphi_n(u_n)
\]

\[
\vdots
\]

\[
L_n(x_1, \ldots, x_n) = x_1 \varphi_1(u_1) + x_2 \varphi_2(u_2) + \cdots + x_n \varphi_n(u_n)
\]

Let $i, j, (1 \leq i \leq n, 1 \leq j \leq n, i \neq j)$ be two given integers.

We can find by using "Minkowski Theorem" on linear forms rational integers $a_1, a_2, \ldots, a_n$ not all zero satisfying

\[
|L_t(a_1, \ldots, a_n)| < 1 \quad \text{for all} \quad t = 1, 2, \ldots, n;
\]

\[
t \neq j
\]

Put $\alpha = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n$. Clearly $\alpha \neq 0$, $\alpha \in R$ and $V_t(\alpha) < 1$ for all $t = 1, 2, \ldots, n, t \neq j$. Therefore $V_j(\alpha) \geq 1$.

For, otherwise there exists an element $\alpha \neq 0$ in $R$ satisfying

\[
V_h(\alpha) < 1 \quad \text{for all} \quad h = 1, 2, \ldots, n
\]
and this is a contradiction. Therefore $\mathcal{V}_i$, $\mathcal{V}_j$ are inequivalent. Thus $\omega$ and $\mu$ are two equivalent subvaluations on ring $R$ for which theorem 6.2 does not hold. Finally since $\omega$ is equivalent to the trivial valuation on $R$; $\mathcal{V}_i$, $\ldots$, $\mathcal{V}_n$ cannot be independent valuations on $R$.

Now we shall give an example promised in section 3.

Example 6.6. Let $K$ and $\mathcal{V}_1$, $\mathcal{V}_2$, $\ldots$, $\mathcal{V}_n$ be as in above example 6.5.

Let

$$\omega_1 = \mathcal{V}_1$$

and

$$\omega_2 = \text{Max} \{ \mathcal{V}_2, \mathcal{V}_n \}.$$  

One can show directly or by using Theorem 6.2 that $\omega_1$ and $\omega_2$ are two inequivalent pseudovaluations on $K$. Further, $\omega_1$ and $\omega_2$ are both $K$-equivalent to the trivial pseudovaluation on $K$. Since $\omega_1 \subseteq \omega_2$, $\omega_1$, $\omega_2$ cannot be independent.

7. Gauge set and pseudovaluation defined by a gauge set.

P. M. Cohn introduced in [5] the concepts of gauge set. With the help of gauge set, he gave an invariant characterization of pseudovaluations on a field. We shall state the definition of gauge set below and collect some results connecting gauge set and the pseudovaluation induced by it.
Definition 7.1. **Gauge set.** Let $K$ be any field. A subset $G$ of $K$ will be called a **gauge set** of $K$ if

1. $-1 \in G$
2. $G \subseteq G$
3. There exists an element $c$ in $K$ such that $G + G \subseteq cG$
4. $G$ is a proper subset of $K$.
5. There exists a non-zero element $d$ in $G$ such that $K = \bigcup_{n=1}^{\infty} d^n G$.

$[G,G]$ denotes the set of all $g_1g_2$, $g_1 \in G$, $g_2 \in G$ and $G + G$ denotes the set of all $g_1 + g_2$, $g_1 \in G$, $g_2 \in G$]

Any element $d \in K$, $d \neq 0$ satisfying (V) is called a **gauge element** of $G$. Let $c$ be any element of $K$ satisfying (iii), then by (V) $c \in d^{-t}G$ for some integer $t > 0$. Further it is clear from (ii) and (v) that if $d$ is a gauge element then so is $d^t$ ($t > 0$). Therefore, when the gauge element is not fixed in advance, we may choose it so that (iii) takes the form $G + G \subseteq d^{-1}G$ for some gauge element $d$.

To each pseudovaluation $\omega$ on a field $K$ which is not equivalent to the trivial pseudovaluation we can associate a gauge set $G$ in the following manner.

Put $A = \{a \in K : \omega(a) < 1\}$ and $G = \{a \in K : aA \subseteq A\}$. 
Then one can verify that $G$ is a gauge set and each $d \neq 0$, $d \in A$ is a gauge element of $G$. We now show that if $d$ is any fixed gauge element of $G$, then $d^n G$ ($n = 0, 1, 2, \ldots$) is a neighborhood base of zero in the topology defined by $\omega$.

Let $\mathcal{J}$ be the topology defined by $\omega$ in $K$. Then $(K, \mathcal{J})$ is a topological field, and $U_1(0) = A$ is a $\mathcal{J}$-neighborhood of zero. Since $A \subseteq G$, $G$ is also a $\mathcal{J}$-neighborhood of zero and so is $\omega G$ for each $a \neq 0$, $a \in K$. In particular $d^n G$ is a neighborhood of zero for every non-negative integer $n$. Now let $U_\epsilon(0)$ be any given neighborhood of zero, then there is a $b \in K$, $b \neq 0$ such that $\omega(b) \leq 1/\epsilon$. Therefore $b A \subseteq U_\epsilon(0)$. Choose an integer $n$ such that $d^n b^{-1} \in A$, i.e., $d^n \in b A$. Now

$$
\{d^n G\}, \ n = 0, 1, 2, 3, \ldots \text{ is a neighborhood base of zero.}
$$

Conversely let $G$ be any gauge set of $K$. Choose a gauge element $d$ in $K$ such that $G + G \subseteq d^{-1} G$ and a real number $\epsilon$ such that $1 < \epsilon \leq 2$. Note that to each $a \in K$, $a \neq 0$, there exists a greatest integer $n$ such that $a \in d^n G$ (because $\cap_{n=0}^\infty d^n G = \{0\}$). Define a real valued function $\psi$ on $K$ as
\( \psi(a) = e^{-a} \)

and \( \psi(o) = 0 \).

Then \( \psi \) satisfies the following properties

(i) \( \psi(\ast 1) = 1 \)

(ii) \( \psi(xy) \leq \psi(x) \psi(y) \) for all \( x, y \in K \).

(iii) \( \psi(x + y) \leq \text{Max} \{ \psi(x), \psi(y) \} \) for all \( x, y \in K \)

(iv) \( \psi(a^{-1}) = [\psi(a)]^{-1} = e \)

(v) \( \psi(o) = 0 \).

Thus \( \psi \) would be a pseudovaluation except that the triangle inequality is replaced by

\[ \psi(x + y) \leq \text{Max} \{ \psi(x), \psi(y) \}. \]

So \( \psi \) itself is not a pseudovaluation. However we can derive a pseudovaluation from \( \psi \) as follows. Set

\[ \omega(a) = \inf_{\sum_{i=1}^{n} x_i = a} \left\{ \sum_{i=1}^{n} \psi(x_i) \right\} \]

where the lower bound is extended over all decompositions \( a = x_1 + x_2 + \ldots + x_n \) of \( a \) into a sum of \( a \) elements of \( K \). Then \( \omega \) is a pseudovaluation on \( K \). We associate this pseudovaluation \( \omega \) to the given gauge set \( G \).
We note that $\omega$ still depends on the choice of the gauge element $d$, but it follows from the following theorem that different gauge elements give rise to equivalent pseudovaluation.

**Theorem 7.2** (Cohn [5] Theorem 5.1) Let $G$ be a gauge set of $K$ and $\omega$ be any pseudovaluation associated with $G$. If $d$ is any gauge element of $G$, then $\{d^n G, n = 0, 1, 2, \ldots\}$ is a neighbourhood base of $o$ in the topology defined by $\omega$.

From the above discussion it follows that we can associate to each pseudovaluation $\omega$ a gauge set $G$ and to each gauge set $G$ a pseudovaluation $\omega$ in such a way that for each gauge element $d$ of $G$, $\{d^n G, n = 0, 1, 2, \ldots\}$ forms a neighbourhood base of zero in the topology defined by $\omega$. Clearly this association is not one-one.

**Definition 7.3.** Two gauge sets $G_1, G_2$ of $K$ are said to be equivalent (in symbol $G_1 \sim G_2$) if there exist $a_1 \neq o, a_2 \neq o$ in $K$ such that

$$a_1 G_1 \subseteq G_2 \quad \text{and} \quad a_2 G_2 \subseteq G_1.$$

There exists a one-one correspondence between the set of equivalence classes of gauge sets of $K$ and the set of equivalence classes of pseudovaluations on $K$ which are not equivalent to the trivial pseudovaluation on $K$. This follows from the following theorem.
Theorem 7.4 (Cohn [5] Theorem 8.1) Let $G_1$, $G_2$ be two gauge sets of a field $X$. Then the topologies associated with them coincide if and only if $G_1 \subseteq G_2$.

In the above "by the topology associated with a given gauge set $G$" we mean the unique topology defined by taking $\{a \in X, a \neq 0\}$ to be the neighbourhood base of zero.