Chapter - 8: Joint effect of simultaneous occurrence of two causes

Cause specific depression: Joint effect of simultaneous Occurrence of two causes

8.1 Introduction:
Depressive episode appears due to the joint effect of both the causes A₁ and A₂, where the time to occurrence of the first cause may be apparent to the psychiatrist and the other cause may not be prominent at that particular time. A patient may be under the influence of both the causes A₁ and A₂. For example - A person who is suffering from chronic physical disease may also suffer from certain stressful conditions like financial crises or emotional crises or crises relating to social readjustment from time to time. Whenever the physical chronic disease occurs simultaneously with any one of these stressful conditions, a depressive episode may take place and he or she visits a psychiatrist. Though financial stress or any other above mentioned stress and chronic disease are independent of each other, their joint effect generates a renewal process corresponding to occurrence of a depressive episode.

8.2 Materials & method
The renewal processes corresponding to \( t_{i} = \min (X_{i}, Y_{i}) \), where \( X_{i} \) and \( Y_{i} \) indicate time epoch at which a renewal occurs either for cause A₁ or A₂ has already been discussed. However in many situations when the doctor finds the patient under the influence of both the causes A₁ and A₂, then it is reasonable to put forward the probabilistic statement concerning two or more random variables as their joint probability distributions. Then it is possible to observe a renewal process \( \{Z_{i}(t), t \geq 0\} \), with \( Z(t) = i \), where \( i \) indicates the \( i^{th} \) depressive episode appearing due to simultaneous effect of A₁ and A₂ at time \( t \). That is \( t_{i} \) generates a renewal process corresponding to distribution function \( W(x, y : \lambda, \mu) = F(x, y : \lambda, \mu) \).
It is sometimes necessary to obtain the joint distribution of random variables \( Z_1 \) and \( Z_2 \), which arise as the functions of \( X \) and \( Y \) viz. \( Z_1 = g_1(X, Y) \), \( Z_2 = g_2(X, Y) \). When \( E(Z_i) \) for \( i=1,2 \) exists, the existence of renewal processes corresponding to \( Z_i \) for \( i=1,2 \) may be justified. However the closed form expression for \( \Pr \{N(t) = k\} \) for the renewal process \( \Pr \{N(t); t \geq 0\} \) may not be always available, in that case we can use \( E(Z_i) \) to compute renewal rate and expected number of renewals per unit of time for large \( t \).

The characteristics of interest in case of studies of depression are given below

1) \( Z_1 = X + Y \) and \( Z_2 = X - Y \)

2) \( Z_1 = XY \) and \( Z_2 = \frac{X}{Y} \)

3) \( Z_1 = X + Y \) and \( Z_2 = \frac{(X + Y)}{X} \)

4) \( \Pr \{ X \leq x, Y \leq y \} \)

5) \( \Pr \{ X > k \mid Y < k \} \)

It is assumed that the functions \( g_i \) satisfy the following conditions (Sheldon Ross 2002).

1) The equations \( Z_1 = g_1(x, y) \) and \( Z_2 = g_2(x, y) \) can be uniquely solved for \( x \) and \( y \) in terms of \( Z_1 \) and \( Z_2 \) with solutions given by \( X_1 = k_1(Z_1, Z_2) \) and \( X_2 = k_2(Z_1, Z_2) \)

2) The functions \( g_1 \) and \( g_2 \) have continuous partial derivatives at all points \( (x, y) \), such that the following \( 2 \times 2 \) determinant \( J(x, y) \) is given by
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\[ J(x, y) = \begin{vmatrix} \frac{dg_1}{dx} & \frac{dg_1}{dy} \\ \frac{dg_2}{dx} & \frac{dg_2}{dy} \end{vmatrix} = \frac{dg_1}{dx} \frac{dg_2}{dy} - \frac{dg_2}{dx} \frac{dg_1}{dy} \neq 0 \] at all points \((x, y)\) may be computed.

Under these two conditions the random variables \(Z_1\) and \(Z_2\) are jointly continuous with joint density function given by

\[ f(z_1, z_2) = f(x_1, x_2)J(x, y)^{-1} \quad \text{.... (8.1)} \]

Consequently

\[ \Pr\{Z_1 \leq z_1, Z_2 \leq z_2\} = \iint_{\{x, y\} \leq z_1, g_1(x, y) \leq z_2, g_2(x, y) \leq z_2} f(x, y)dxdy \]

The random variables \(X\) and \(Y\) are considered to be jointly continuous with p.d.f. \(f(x, y)\)

8.3: Characteristics obtained under various joint dist of \(X \& Y\)

If \(X \sim \text{Exp}(\lambda_1)\) and \(Y \sim \text{Exp}(\lambda_2)\)

\[ f(x, y) = (\lambda_1 \lambda_2) \exp\{- (\lambda_1 x + \lambda_2 y)\} \quad \text{.... (8.2)} \]

where \((\lambda_1, \lambda_2) > 0\) and \(x, y \geq 0\)

Case 1:

Let \(Z_1 = X + Y\), and \(Z_2 = X - Y\)

i.e. \(Z_1\) is the sum of the time to occurrence of depression due to causes \(A_1\) and \(A_2\) and \(Z_2\) is the difference between time to occurrence of depression due to causes \(A_1\) and \(A_2\).
i.e. \( z_1 = g_1(x, y) = x + y \)  

\[ \text{..... (8.3)} \]

and \( z_2 = g_2(x, y) = x - y \)  

\[ \text{..... (8.4)} \]

\[ \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \]

then \( J(x, y) = \frac{1}{1 - 1} = -2 \)

The equations (8.3) and (8.4) also have their solutions as

\[ x = \frac{z_1 + z_2}{2} \text{, and } y = \frac{z_1 - z_2}{2} \]

\[ f(z_1, z_2) = \frac{1}{2} f\left( \frac{z_1 + z_2}{2}, \frac{z_1 - z_2}{2} \right) \]

\[ f(x, y) = \lambda_1 \lambda_2 e^{-\lambda_1(x) - \lambda_2(y)} \]

\[ f(z_1, z_2) = \frac{\lambda_1 \lambda_2}{2} e^{-\lambda_1 \frac{(z_1 + z_2)}{2} - \lambda_2 \frac{(z_1 - z_2)}{2}} \]

\[ \text{..... (8.6)} \]

\[ z_1 + z_2 \geq 0, \ z_1 - z_2 \geq 0 \]

\[ E(z_1) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \]

\[ \text{..... (8.7)} \]

\[ E(z_2) = \frac{1}{\lambda_1} - \frac{1}{\lambda_2} \]

\[ \text{..... (8.8)} \]

**Case 2:**

Let \( Z_1 = XY \), and \( Z_2 = \frac{X}{Y} \)
$Z_2$ is the ratio of the time to occurrence of depression due to causes $A_1$ and $A_2$ respectively, which may be regarded as the focus point of depression studies. Since $(X, Y)$ is a bivariate random variable it is not possible to obtain the distribution of $Z_2$ alone. Hence we have to obtain the joint distribution of $Z_1$ and $Z_2$ so that $X$ & $Y$ are expressible in terms of $Z_1$ and $Z_2$.

i.e. $Z_1 = g_1(x, y) = xy$ \hspace{1cm} .... (8.9)

and $Z_2 = g_2(x, y) = \frac{x}{y}$, where $y > 0$ \hspace{1cm} .... (8.10)

$$J(x, y) = \begin{vmatrix} y & x \\ 1 - \frac{x}{y^2} & y \end{vmatrix} = -2 \frac{x}{y} = -2z_2$$

$$[J(x, y)]^{-1} = -\frac{1}{2z_2} = \frac{1}{2z_2}$$

The equations (8.9) and (8.10) also have their solutions as

$$x = (z_1z_2)^{\frac{1}{2}} \text{ and } y = \left( \frac{z_1}{z_2} \right)^{\frac{1}{2}}$$

From (8.1)

$$f(z_1, z_2) = \lambda_1 \lambda_2 e^{-\lambda_1 \left( z_1z_2 \right)^{\frac{1}{2}} - \lambda_2 \left( \frac{z_1}{z_2} \right)^{\frac{1}{2}}} \left| (J)^{-1} \right|$$
\[= \lambda_1 \lambda_2 e^{-z_1^{1/2} \left[ \lambda_1 (z_2)^{1/2} + \lambda_2 (z_2)^{-1/2} \right]} \frac{1}{2z_2} \]

\[= \frac{\lambda_1 \lambda_2}{2} (z_2)^{-1} e^{-z_1^{1/2} M(z_2)}, \quad \text{.... (8.11)} \]

where \( M(z_2) = \left[ \lambda_1 (z_2)^{1/2} + \lambda_2 (z_2)^{-1/2} \right] \)

\[E(z_1) = E(x)E(y) = \frac{1}{\lambda_1} \{E(y)\} \quad \text{.... (8.12)} \]

\[E(z_2) = \frac{1}{\lambda_1} \left\{E\left(\frac{1}{y}\right)\right\}, \text{ where } y > 0 \quad \text{.... (8.13)} \]

\[E\left(\frac{1}{y}\right) = \int_0^\infty \frac{1}{y} e^{-\lambda_2 y} dy \quad \text{.... (8.14)} \]

\(E(y)\) and \(E\left(\frac{1}{y}\right)\) belong to class of incomplete gamma functions which may be obtained by numerical integration.

Though \(X\) and \(Y\) are independent the random variable \(z_1\) and \(z_2\) are not independent (Sheldon Ross 2002).

**Case 3:**

Let \(Z_1 = X + Y\) and \(Z_2 = \frac{(X + Y)}{X}\)

Here \(Z_1\) is the sum of the time to occurrence of depression due to causes \(A_1\) and \(A_2\) and \(Z_2\) is the sum of the time to occurrence of \(X\) and \(Y\) provided \(X\) is known.

i.e. \(z_i = g_2(x, y) = x + y \quad \text{.... (8.15)} \)
\[ z_2 = g_2(x, y) = \left(\frac{x + y}{x}\right), \quad \text{where} \ x > 0 \quad \ldots \ (8.16) \]

From the above equations

\[ x = \frac{z_1}{z_2}, \quad \text{and} \quad y = z_1 \left(1 - \frac{1}{z_2}\right), \quad \text{where} \ Z_1 > 0 \quad \text{and} \ Z_2 > 0 \]

\[ J(x, y) = \left| \begin{array}{cc} -\frac{1}{x^2} & 1 \\ \frac{1}{x} & 1 \end{array} \right| = \frac{1}{x} \left[1 + \frac{y}{x}\right] = \frac{z_2^2}{z_1} \]

\[ f(z_1, z_2) = \lambda_1 \lambda_2 e^{-\lambda_1 \left(\frac{z_1}{z_2}\right)^{1/2}} - \lambda_2 \left[ z_1 \left(1 - \frac{1}{z_2}\right) \right]^{1/2} |(J)^{-1}| \]

\[ f(z_1, z_2) = \lambda_1 \lambda_2 \frac{z_1}{z_2} e^{-\lambda_1 \left(\frac{z_1}{z_2}\right)^{1/2}} - \lambda_2 \left[ z_1 \left(1 - \frac{1}{z_2}\right) \right]^{1/2} \quad \ldots \ (8.17) \]

\[ E(z_1) = E(x) + E(y) \quad \ldots \ (8.18) \]

\[ E(z_2) = 1 + \lambda_2 E\left(\frac{1}{x}\right), \quad \text{where} \ x > 0 \quad \ldots \ (8.19) \]

\( E(x) \) and \( E\left(\frac{1}{x}\right) \) belong to class of incomplete gamma functions which may be obtained by numerical integration.

**Case 4**

Let the joint density of \( X \) and \( Y \) be given by

\[ f(x, y) = (\lambda_1 \lambda_2) \exp\{- (\lambda_1 x + \lambda_2 y)\}, \quad \ldots \ (8.20) \]
where \((\lambda_1, \lambda_2) > 0\) and \(x, y \geq 0\)

\[
\Pr\{X \leq x, Y \leq y\} = \lambda_1 \lambda_2 \int_0^x \int_0^y e^{-\lambda_1 u - \lambda_2 v} \, du \, dv
\]

\[
= \lambda_1 \int_0^x e^{-\lambda_1 u} \, du \cdot \lambda_2 \int_0^y e^{-\lambda_2 v} \, dv
\]

\[
= \left(1 - e^{-\lambda_1 x}\right) \left(1 - e^{-\lambda_2 y}\right), \quad \text{where } (\lambda_1, \lambda_2) > 0 \text{ and } x, y \geq 0 \quad \ldots (8.21)
\]

**Case 5**

Let the joint density of \(X\) and \(Y\) be given by

\[
f(x, y) = \frac{1}{\lambda_1 \lambda_2} e^{-\left(\frac{y}{\lambda_1} + \frac{y}{\lambda_2}\right)}, \quad \ldots (8.22)
\]

where \((\lambda_1, \lambda_2) > 0\) and \(x, y \geq 0\)

\[
\Pr\{X > k \mid Y < k\} = \frac{\Pr\{X > k, Y < k\}}{\Pr\{Y < k\}}
\]

\[
= \frac{\frac{1}{\lambda_1} \int_k^\infty e^{-\frac{y}{\lambda_1}} \, dy \cdot \frac{1}{\lambda_2} \int_k^\infty e^{-\frac{y}{\lambda_2}} \, dy}{\frac{1}{\lambda_2} \int_0^k e^{-\frac{y}{\lambda_2}} \, dy}
\]

\[
= \frac{\frac{1}{\lambda_1} \int_k^\infty e^{-\frac{y}{\lambda_1}} \, dy \left(\frac{1}{\lambda_2} \int_k^\infty e^{-\frac{y}{\lambda_2}} \, dy\right)}{\frac{1}{\lambda_2} \int_0^k e^{-\frac{y}{\lambda_2}} \, dy}
\]

\[
= \frac{1}{\lambda_1} \int_k^\infty e^{-\frac{y}{\lambda_1}} \, dy \left(\frac{1}{\lambda_2} \int_k^\infty e^{-\frac{y}{\lambda_2}} \, dy\right)
\]

\[
= \frac{\left(e^{-\frac{k}{\lambda_1}} \right) \left(e^{-\frac{k}{\lambda_2}}\right)}{e^{-\frac{k}{\lambda_2}}} = e^{-\frac{k}{\lambda_1}}, \quad \lambda_1 > 0 \text{ and } k \geq 0 \quad \ldots (8.23)
\]
8.4: Special case:

Case 1

If X ~ Uniform distribution (0, 1) and Y ~ Uniform distribution (0, 1)

Let $Z_1 = X + Y$, and $Z_2 = X - Y$

i.e. $z_1 = g_1(x, y) = x + y$ \hspace{1cm} (8.24)

and $z_2 = g_2(x, y) = x - y$ \hspace{1cm} (8.25)

then $J(x, y) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$

The equations (7.65) and (7.66) also have their solutions as

$x = \frac{z_1 + z_2}{2}$, and $y = \frac{z_1 - z_2}{2}$

$f(z_1, z_2) = \frac{1}{2} f\left(\frac{z_1 + z_2}{2}, \frac{z_1 - z_2}{2}\right)$

$= \frac{1}{2}$ for $0 < z_1 \leq 1$, $0 < z_1 - z_2 \leq 1$ \hspace{1cm} (8.26)

Case 2

When X and Y are dependent $F(x, y)$ may take various forms, for demonstration purpose the following cases are considered

(i) Let the joint distribution of X and Y be given by

$f(x, y) = \lambda_1 \lambda_2 y^{-1} \exp\left[-\left(\lambda_1 y^{-1} x + \lambda_2 y\right)\right]$, \hspace{1cm} (8.27)

where $\lambda_1, \lambda_2 > 0$, $x, y > 0$
And the marginal distribution of $y$ is given by

$$f(y) = \int_{0}^{\infty} f(x, y)dx = \lambda_2 e^{-\lambda_2 y}, \quad \ldots \ (8.28)$$

where $\lambda_1, \lambda_2 > 0$ and $x > 0, y > 0$

Hence the conditional distribution of $X$ given $Y$ is presented by

$$f(x | y) = \frac{f(x, y)}{\int_{0}^{\infty} f(x, y)dx} = \frac{\lambda_1 \lambda_2 y^{-1} \exp[-(\lambda_1 y^{-1}x + \lambda_2 y)]}{\lambda_2 e^{-\lambda_2 y}}$$

$$= \frac{\lambda_1 y^{-1} \exp[-(\lambda_1 y^{-1}x)]}{\lambda_2 e^{-\lambda_2 y}} \quad \ldots \ (8.29)$$

$$E(X | Y) = \frac{y}{\lambda_1} \quad \ldots \ (8.30)$$

$$E(Y) = \frac{1}{\lambda_2} \quad \ldots \ (8.31)$$

$$E(X) = E\{E(X | Y)\} \quad \text{by definition}$$

$$= \frac{E(Y)}{\lambda_1} = \frac{1}{\lambda_1 \lambda_2} \quad \ldots \ (8.32)$$

$$Var(Y) = \frac{1}{\lambda_2^2} \quad \ldots \ (8.33)$$

$$Var(X) = E\{Var(X | Y)\} + Var\{E(X | Y)\} \quad \text{by definition}$$
\[ E \left( Y^2 \right) = E \left( \frac{Y^2}{\lambda_1^2} \right) + \text{Var} \left( \frac{Y}{\lambda_1} \right) \]

\[ = \frac{2}{\lambda_1^2 \lambda_2^2} + \frac{1}{\lambda_1^2 \lambda_2^2} \]

\[ = \frac{3}{\lambda_1^2 \lambda_2^2} \]  

\[ E (X Y) = \int\int x y f(x, y) dx dy \; , \; \text{where} \; x, y > 0 \]

\[ = \int \int \lambda_2 e^{-\lambda_2 y} \left( \lambda_1 e^{-\lambda_1 x} x \right) dx dy \; , \; \text{where} \; \lambda_1 \; \text{and} \; \lambda_2 > 0 \]

\[ = \int \frac{\lambda_2 y^2}{\lambda_1} e^{-\lambda_2 y} dy \]

\[ = \frac{1}{\lambda_1} \int \lambda_2 y^2 e^{-\lambda_2 y} dy \]

\[ = \frac{2}{\lambda_1^2 \lambda_2^2} \]  

\[ \rho_{xy} = \frac{E(XY) - E(X)E(Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} \]

\[ = \frac{2(\lambda_1 \lambda_2^2)^{-1} - (\lambda_1 \lambda_2)^{-1}}{\sqrt{3(\lambda_1^2 \lambda_2^4)^{-1}}} \]
\[ \frac{1}{\sqrt{3}} = 0.5773 \quad \ldots \quad (8.36) \]

i.e. the correlation between the occurrence time of depressive episode due to \( A_1 \) and \( A_2 \) is 0.5773 or 58%

### 8.5 Joint distribution of X and Y

Let the joint distribution of \( X \) and \( Y \) be given by

\[ f(x, y) = K \left( \alpha - \lambda_1 x - \lambda_2 y \right) \]  \text{, where } 0 < x < a, 0 < y < b  \quad \ldots \quad (8.37)

It has already been defined that \( X \) is the time to occurrence of a depressive episode due to cause \( A_1 \) and \( Y \) is the time to occurrence of a depressive episode due to cause \( A_2 \). The time to occurrence of a depressive episode is a linear combination of time to occurrence of an episode due to cause \( A_1 \) and the time to occurrence of an episode due to cause \( A_2 \). That is, the joint density \( f(x, y) \) can occur as a linear combination of \( X \) and \( Y \) due to causes \( A_1 \) and \( A_2 \) respectively, which is represented by \( Z = \lambda_1 x + \lambda_2 y \). Here, \( \lambda_1 \) and \( \lambda_2 \) are the influencing parameters or the parameters responsible for the occurrence of depressive episodes due to causes \( A_1 \) and \( A_2 \) respectively. These influencing parameters may be the socio-demographic factors viz. gender, age, family history, marital status, education, occupation and socio-economic status as discussed in chapter 2.

The parameters \( \lambda_1 \) and \( \lambda_2 \) serve as weights of the causes, whereas \( \alpha \) is the key parameter of the depressive episode influenced by the time due to causes \( A_1 \) and \( A_2 \) respectively. \( K \) is a parameter which is a function of \( (\alpha, \lambda_1, \lambda_2, a \text{ and } b) \), \( a \) and \( b \) are upper limits of \( x \) and \( y \) and are finite.

The density of \( f(x, y) \) can be written as

\[ f(x, y) = K(\alpha - Z) = K(\alpha - \lambda_1 x - \lambda_2 y) \]

Where, \( K \) may be obtained by integrating the density function
\[ \iint f(x, y) \, dx \, dy = 1, \text{ subject to the condition } \alpha > \lambda_1 x + \lambda_2 y. \]

The parameter space of \( \alpha \) is defined when \( f(x, y) \) is positive.

\[ \iint f(x, y) \, dx \, dy = 1 \]

\[ \Rightarrow \iint K(\alpha - \lambda_1 x - \lambda_2 y) \, dx \, dy = 1 \]

\[ \Rightarrow K \int \left[ \alpha a - \frac{\lambda_1 a^2}{2} - \lambda_2 ay \right] \, dy = 1 \]

\[ \Rightarrow K \left[ \alpha ab - \frac{\lambda_1 a^2 b}{2} - \frac{\lambda_2 ab^2}{2} \right] = 1 \]

\[ \Rightarrow \frac{Kab}{2} (2 \alpha - \lambda_1 a - \lambda_2 b) = 1 \]

\[ \therefore K = \frac{2}{ab (2 \alpha - \lambda_1 a - \lambda_2 b)} \quad \cdots (8.38) \]

\( K > 0, \quad \alpha > \frac{\lambda_1 a + \lambda_2 b}{2} \quad \cdots (A) \)

The distribution function \( F(x, y) \) is given by

\[ F(x, y) = \Pr\{X \leq x, Y \leq y\} \]

The survival function is given by \( 1 - F(x, y) \)

Now, \( F(x, y) = \iint K(\alpha - \lambda_1 u - \lambda_2 v) \, du \, dv \)

\[ = Kxy \left( \alpha - \frac{\lambda_1}{2} x - \frac{\lambda_2}{2} y \right) \quad \cdots (8.39) \]

The hazard rate corresponding to \( f(x, y) \) is given by
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\[ h(x, y) = \frac{f(x, y)}{1 - F(x, y)} \]

\[ = \frac{K(\alpha - \lambda_1 x - \lambda_2 y)}{1 - K_{xy}(\alpha - \frac{\lambda_1}{2} x - \frac{\lambda_2}{2} y)} \]  

\[ h(x, y) \] is the hazard rate of \( f(x, y) \); a three dimensional graph of \( h(x, y) \) is plotted below for \( (\alpha, \lambda_1, \lambda_2, a \text{ and } b) \), computation is provided in Annexure 4.

**Fig: 8.1 Hazard rate \( h(x, y) \) of \( f(x, y) \) for \( (\alpha, \lambda_1, \lambda_2, a \text{ and } b) \)**
The marginal density of $x$ is given by

$$g(x) = \int_{0}^{b} f(x, y)dy = \int_{0}^{b} K(\alpha - \lambda_1 x - \lambda_2 y)dy$$

$$g(x) = Kb \left[ \alpha - \lambda_1 x - \lambda_2 \frac{b}{2} \right] \quad \ldots (8.41)$$

The expected time to occurrence of a depressive episode due to cause $A_1$ i.e. $E(x)$ can be obtained as

$$E(X) = K \int_{0}^{a} x \left[ \alpha b - \lambda_1 bx - \lambda_2 \frac{b^2}{2} \right]dx$$

$$= Ka^2 b \left( \frac{\alpha}{2} - \lambda_1 \frac{a}{3} - \lambda_2 \frac{b}{4} \right)$$

Here it is observed that

$$\frac{\alpha}{2} > \frac{\lambda_1 a}{3} + \frac{\lambda_2 b}{4} \Rightarrow \alpha > .6\lambda_1 a + .5\lambda_2 b \quad \ldots \ldots (B)$$

Substituting the value of $K$ in $E(x)$ the marginal expectation of $x$ is given by

$$E(X) = \frac{a(6\alpha - 4\lambda_1 a - 3\lambda_2 b)}{6(2\alpha - \lambda_1 a - \lambda_2 b)} \quad \ldots (8.42)$$

The conditional distribution of $y$ given $x$ is expressed as

$$\frac{f(x, y)}{g(x)} = \frac{K(\alpha - \lambda_1 x - \lambda_2 y)}{Kb \left( \alpha - \lambda_1 x - \lambda_2 \frac{b}{2} \right)}$$

$$= \frac{(\alpha - \lambda_1 x - \lambda_2 y)}{b \left( \alpha - \lambda_1 x - \lambda_2 \frac{b}{2} \right)} \quad \ldots (8.43)$$
The conditional expectation of $Y$ given $X$ is

$$E(Y \mid X) = \frac{\int_0^b y(\alpha - \lambda_1 x - \lambda_2 y)dy}{b\left(\alpha - \lambda_1 x - \lambda_2 \frac{b}{2}\right)}$$

$$= \frac{\left(\alpha \frac{b^2}{2} - \lambda_1 \frac{b^2}{2} x - \lambda_2 \frac{b^3}{3}\right)}{b\left(\alpha - \lambda_1 x - \lambda_2 \frac{b}{2}\right)}$$

$$E(Y \mid X) = \frac{b(3\alpha - 3\lambda x - 2\lambda y b)}{3(2\alpha - 2\lambda x - \lambda y b)}$$

The marginal density of $y$ is given by

$$w(y) = \int_0^a f(x, y)dx = \int_0^a K(\alpha - \lambda_1 x - \lambda_2 y)dx$$

$$= Ka\left[\alpha - \lambda_1 \frac{a}{2} - \lambda_2 y\right] , 0 < y < b$$

Similarly the expected time to occurrence of a depressive episode due to cause $A_2$ i.e. $E(Y)$ can be obtained as

$$E(Y) = K \int_0^b y\left[a\alpha - \lambda_1 \frac{a^2}{2} - \lambda_2 ya\right]dy$$

$$= Kab\left[\frac{\alpha}{2} - \lambda_1 \frac{a}{4} - \lambda_2 \frac{b}{3}\right]$$

Here it is observed that $\frac{\alpha}{2} > \frac{\lambda_1 a}{4} + \frac{\lambda_2 b}{3} \Rightarrow \alpha > .5\lambda_1 a + .6\lambda_2 b$ .... (C)
Substituting the value of $K$ in $E(Y)$ the marginal expectation of $Y$ is given by

$$E(Y) = \frac{b(6\alpha - 3\lambda_1 a - 4\lambda_2 b)}{6(2\alpha - \lambda_1 a - \lambda_2 b)} \quad \ldots \ (8.46)$$

Similarly the conditional distribution of $x$ given $y$ is expressed as

$$f(x, y) = \frac{K(\alpha - \lambda_1 x - \lambda_2 y)}{K a \left( \alpha - \lambda_1 \frac{a}{2} - \lambda_2 y \right)} = \frac{(\alpha - \lambda_1 x - \lambda_2 y)}{a \left( \alpha - \lambda_1 \frac{a}{2} - \lambda_2 y \right)} \quad \ldots \ (8.47)$$

The conditional expectation of $x$ given $y$ is

$$E(X \mid Y) = \frac{\int_{0}^{a} x(\alpha - \lambda_1 x - \lambda_2 y) \, dx}{a \left( \alpha - \lambda_1 \frac{a}{2} - \lambda_2 y \right)} = \frac{\left( \alpha \frac{a^2}{2} - \lambda_1 \frac{a^3}{3} - \lambda_2 \frac{a^2}{2} y \right)}{a \left( \alpha - \lambda_1 \frac{a}{2} - \lambda_2 y \right)}$$

$$E(X \mid Y) = \frac{a(3\alpha - 2\lambda_1 a - 3\lambda_2 b)}{3(2\alpha - \lambda_1 a - 2\lambda_2 b)} \quad \ldots \ (8.48)$$

From the above conditions (A), (B), and (C) it may be assumed that $\alpha > \lambda_1 a + \lambda_2 b$ where $\lambda_1, \lambda_2, a, b > 0$

So $E(X)$ and $E(Y)$ are positive, which implies the existence of a renewal process.
8.5.1 Conditions for deriving the influencing parameters $\lambda_1$ and $\lambda_2$

Without loss of generality it can be assumed that $\lambda_1 a + \lambda_2 b = \alpha$

Depressive episodes occur as a linear combination of $x$ and $y$ viz. $Z = \lambda_1 x + \lambda_2 y$. This linear combination may be regarded as convex combination of $x$ and $y$ such that $\lambda_1 + \lambda_2 = 1$.

The linear combinations on parameters are obtained as follows:

1. $2\alpha > \lambda_1 a + \lambda_2 b$
2. $\lambda_1 a + \lambda_2 b = \alpha$
3. $\lambda_1 + \lambda_2 = 1$

Now, combining the combinations (2) and (3) the values of $\lambda_1$ and $\lambda_2$ can be obtained as

$$\lambda_1 = \frac{b - \alpha}{b - a} \quad \text{and} \quad \lambda_2 = \frac{\alpha - a}{b - a},$$

provided $\lambda_1 > 0$, $\lambda_2 > 0$ and $b > a \Rightarrow a < \alpha < b$

Instead of convex combination between $x$ and $y$ some other relations may be taken between $\lambda_1$ and $\lambda_2$ which will determine the limits of the key parameter $\alpha$.

It is also important to find if there exists any correlation between the time to occurrence of a depressive episode due to cause $A_1$ and $A_2$. Here to find the correlation between $X$ and $Y$, it is required to find the $\text{Cov}(X, Y)$, $\sigma_x$ and $\sigma_y$, which is shown below.

$$E(XY) = \iint Kxy(\alpha - \lambda_1 x - \lambda_2 y)dx dy,$$

where $(0 < x < a$ and $0 < y < b)$

$$= K \int_0^b \left( \frac{\alpha a^2}{2} y - \frac{\lambda_1 a^3}{3} y - \frac{\lambda_2 a^2}{2} y \right) dy.$$
\[ E(Y^2) = \frac{b^2(4\alpha - 2\lambda_1 a - 3\lambda_2 b)}{6(2\alpha - \lambda_1 a - \lambda_2 b)} \quad \ldots \quad (8.52) \]
\[ \text{Var}(Y) = E(Y^2) - \{E(Y)\}^2 \]
\[ = \frac{b^2(4\alpha - 2\lambda, a - 3\lambda,b)}{6(2\alpha - \lambda, a - \lambda,b)} - \frac{a(6\alpha - 4\lambda,a - 3\lambda,b)}{6(2\alpha - \lambda,a - \lambda,b)} \] \hspace{1cm} \ldots (8.53)

\[ \text{Cov}(X,Y) = E(XY) - E(X)E(Y) \]

Correlation between X and Y is given by
\[ \rho_{xy} = \frac{E(XY) - E(X)E(Y)}{\sigma_x \sigma_y} \]

8.6 Numerical example:

Here \( \alpha \) is considered to be the key parameter and \( \lambda_1, \lambda_2 \) are the influencing parameters of x and y. It is already shown that if \( Z = \lambda_1 x + \lambda_2 y \) such that \( \lambda_1 + \lambda_2 = 1 \), \( (\lambda_1, \lambda_2) > 0 \) then \( a < \alpha < b \). Now as the \( \alpha \) changes the characteristics \( E(X), E(Y), \) \( \text{Var}(X), \text{Var}(Y) \) and \( \rho_{xy} \) also changes accordingly. The computations of the characteristics are provided in Annexure 5

Diagrammatically they can be presented as follows:

**Fig: 8.2 E(X) as a function of key parameter (\( \alpha \)) lying between [a, b) where a=14, b=21**
It can be observed from the above graph that $E(X)$ is increasing with the increase in the key parameter value and it lies between 4.9 and 5.9.

**Fig.8.3: $E(Y)$ as a function of key parameter ($\alpha$) lying between $[a, b)$**

where $a=14$, $b=21$

From the above graph it can be seen that $E(Y)$ is increasing with the increase in the key parameter value and it lies between 9.75 and 10.1.

**Fig.8.4: $\text{Var} (X)$ as a function of key parameter ($\alpha$) lying between $[a, b)$**

where $a=14$, $b=21$
The above graph shows that $\text{Var}(X)$ decreases as the key parameter value $\alpha$ increases.

**Fig: 8.5** $\text{Var} (Y)$ as a function of key parameter ($\alpha$) lying between $[a, b)$

where $a=14$, $b=21$

Here it is observed from the above graph $\text{Var}(Y)$ increases as the key parameter value $\alpha$ increases.

**Fig: 8.6:** $\rho_{xy}$ as a function of key parameter ($\alpha$) lying between $[a, b)$ where $a=14$, $b=21$
It can be observed from the above graph that correlation of X and Y ($\rho_{xy}$) lies between -0.015 and 0.01. It is also seen that as the key parameter value $\alpha$ increases the value of $\rho_{xy}$ also increases.

Thus it is observed that all the characteristics taken into consideration depend upon the key parameter $\alpha$, which is illustrated by the above graphs.

**8.7: Discussion:**

Here the study is confined to two specific causes which can also be done for multiple causes. Different situations have been studied when the specific causes affect the patients with different hazard rates. For each situation different distributions are formulated to study the pattern of the occurrence of depressive episodes. The pattern of the depressive episodes may be of help to the psychiatrists to predict the time to occurrence of depressive episodes and act accordingly. The characteristics namely mean occurrence time to depressive episodes and the expected number of renewals are identified and presented graphically.