Chapter - 7: Cause specific occurrences of episodes

Cause Specific Occurrences of Depressive Episodes

7.1 Introduction:

Depression can be broadly divided as exogenous depression and endogenous depression. Endogenous depression is a kind of depression in which there is no apparent external factor which causes depression. The cause of endogenous depression is not clear even to the psychiatrists. So to study the cause of depression one has to consider the exogenous depression. It is a kind of depression in which there is apparent external factors causing depression. It has been mentioned in Chapter 1, that depression occurs mainly due to stress. Stress can be viewed as an important cause of depression and can be of varying origin as:

1. Social stress – it is a stress induced by various social factors like –

2. Co-morbid medical illness

3. Abuse and dependence on alcohol and other drugs

4. Social isolation

5. Family history

Depression can occur due to a single specific cause or multiple causes, interacting and influencing each other in a complex way leading to appearance of depression. Here two specific causes are considered for the occurrence of depression.

7.2 Materials and methods:

As already mentioned in chapter 6, \{Z (t), t \geq 0\} is stochastic point process. The point of discontinuity of the process Z (t) are epochs at which the system, that is depression is
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either in state 0 or in state 1, i.e., \( \{Z(t), t \geq 0\} \) is a stochastic process with state space \( S = (0, 1) \). The transition is of the type \( 0 \rightarrow 1 \rightarrow 0 \rightarrow 1 \).

Assuming that the system’s occurrence points are uniquely determined, it is easy to see that for this system \( \{Z(t), t \geq 0\} \) is an alternating renewal process.

The probabilistic nature of the process \( \{Z(t), t \geq 0\} \) is determined by the behavior of the patient. As soon as the patient feels that he is suffering from depression he visits a psychiatrist for treatment. Since depression is a psychological disorder, there are various social and environmental factors as mentioned above and these factors vary from person to person and influence the occurrence of depression.

The time epochs \( t_{0i} \) and \( t_{1i} \) represent the occurrence of the events 0 and 1 for \( i^{th} \) time. Two renewal processes such as \( \{Z_0(t), t \geq 0\} \) and \( \{Z_1(t), t \geq 0\} \) can be identified corresponding to \( \{t_{0i}\} \) and \( \{t_{1i}\} \) respectively. The process \( \{Z_1(t), t \geq 0\} \) is considered for study, since the record of visiting a psychiatrist may be available from the ‘patient’s file’. The several factors which are responsible for occurrence of a depressive episode have already been identified.

Let \( A_1, A_2, \ldots A_k \) be the \( k, (k \geq 2) \) independent causes which are responsible for occurrence of a depressive episode. The process \( \{Z_1(t), t \geq 0\} \) is a renewal process corresponding to the distribution function \( F(X_1, X_2, ..., X_k) \) with finite mean. Let \( k = 2 \), i.e., the occurrence of a depressive episode is the result of appearance of causes either \( A_1 \) or \( A_2 \) or both \( A_1 \) and \( A_2 \).

Let \( \{t_{1i}\} \) be i.i.d random variables with joint probability density function \( f_{x,y}(x, y : \lambda, \mu) \), with marginal densities \( f(x, \lambda) \) and \( g(y, \mu) \), \( (x, y) \geq 0, (\lambda, \mu) > 0 \).

Here \( X \) is a random variable representing the time to occurrence of a depressive episode due to cause \( A_1 \) and \( Y \) corresponds to the time to occurrence of a depressive episode due to cause \( A_2 \) (Assuming that the patient visits the doctor as soon as he is under
depression for a certain cause). Therefore the random variable \( t_{ii} = \min (X_i, Y_i) \), indicates time epoch at which a renewal occurs either for cause \( A_1 \) or \( A_2 \).

To observe the process \( \{Z_i(t), t \geq 0\} \) an auxiliary process \( \{X(t), t \geq 0\} \) is defined

where \( X(t) = 0 \), if depressive feelings are not present in a person

\[ =1, \text{ if depression occurs due to cause } A_1 \]

\[ =2, \text{ if depression occurs due to cause } A_2 \]

Hence \( \{X(t), t \geq 0\} \) is a stochastic process with state space \( \mathbb{S} = (0, 1, 2) \).

State 0 may be considered as the state of absence of depressive feelings from both the causes \( A_1 \) and \( A_2 \) and may be denoted by \( 0 = \{0_1, 0_2\} \).

A patient suffering from depression due to either of the causes \( A_1 \) or \( A_2 \) starts treatment and in the process gets well. But, after a random interval of time he relapses into another episode of depression either due to discontinuation of medicine or some other factors. Diagrammatically the situation can be represented as follows

**Fig: 7.1 Depression due to either of the causes \( A_1 \) or \( A_2 \)**

![Diagram showing the process of depression due to either \( A_1 \) or \( A_2 \).]
Transition from state $1 \rightarrow 0_1$ indicates that the patient is relieved from depression due to cause $A_1$ whereas transition from state $2 \rightarrow 0_2$ indicates that the patient is relieved from depression due to cause $A_2$. States $(1, 2)$ are renewal epochs and are points of discontinuity of $\{X(t)\}$. The renewal states $(1, 2)$ may be denoted by $1$ such that $1 = \{1_1, 1_2\}$.

The process $X(t)$ now can be converted to the renewal process $\{Z_1(t), t \geq 0\}$ generated by the i.i.d sequence $\{t_i\}$ with density $f(x, y)$.

From the above discussion the following cases can be considered

**7.3 When the doctor finds the patient to be under the influence of either of the causes $A_1$ or $A_2$.**

**7.3.1 Hazard rate of $t$**

Hence $t_{ii} = \min(X_i, Y_i)$ indicates time epoch at which a renewal occurs either for cause $A_1$ or $A_2$, where $X$ is the time to occurrence of cause $A_1$ and $Y$ is the time to occurrence of cause $A_2$.

Diagrammatically it can be represented as follows

**Fig: 7.2 X is the time to occurrence of cause $A_1$**
Fig: 7.3 Y is the time to occurrence of cause $A_2$

Then $t_{ii}$ generates a renewal process corresponding to distribution function

$$W(t; \lambda, \mu) = \Pr \{t_{ii} = \min (X_i, Y_i) \leq t \} = 1 - \Pr (X_i > t, Y_i > t)$$

$$= 1 - \{1 - F(t, \lambda)\} \{1 - G(t, \mu)\}$$

$$\therefore W(t; \lambda, \mu) = F(t, \lambda) + G(t, \mu) - F(t, \lambda)G(t, \mu), \quad \text{where} \quad t \geq 0 \quad \text{and} \quad (\lambda, \mu) > 0 \quad \text{... (7.1)}$$

$$1 - W(t; \lambda, \mu) = \{1 - F(t, \lambda)\} \{1 - G(t, \mu)\} \quad \text{... (7.2)}$$

The corresponding density function is given by-

$$w(t; \lambda, \mu) = f(t, \lambda) + g(t, \mu) - f(t, \lambda)G(t, \mu) - F(t, \lambda)g(t, \mu)$$

$$= f(t, \lambda)\{1 - G(t, \mu)\} + g(t, \mu)\{1 - F(t, \lambda)\} \quad \text{... (7.3)}$$

The hazard rate corresponding to $w(t; \lambda, \mu)$ is

$$h(t) \geq \max \{h_\lambda(t), h_\mu(t)\}$$
This is obvious from the fact that both the causes are operating on a patient and are competing with each other. The cause which is operating with higher rate of occurrence is likely to be responsible for making the patient to visit state ‘1’. In fact it may be shown that

\[
h(t) = \frac{w(t; \lambda, \mu)}{1 - W(t; \lambda, \mu)}
\]

\[
= \frac{f(t, \lambda)[1 - G(t, \mu)] + g(t, \lambda)[1 - F(t, \lambda)]}{\{1 - F(t, \lambda)\}\{1 - G(t, \mu)\}}
\]

\[
= \frac{f(t, \lambda)}{\{1 - F(t, \lambda)\}} + \frac{g(t, \lambda)}{\{1 - G(t, \mu)\}}
\]

\[
= h_\lambda(t) + h_\mu(t) \quad \ldots \quad (7.4)
\]

### 7.3.2 Range of t:

Since X and Y are two random variables having different ranges so the range of the observed random variable \( t = \min \{X, Y\} \) may be obtained as follows

**Case (i):** If \( 0 \leq X \leq \infty \) and \( 0 \leq Y \leq \infty \)

Range of \( t \) = Range of \( \min \{0 \leq X \leq \infty , 0 \leq Y \leq \infty\} \)

\[
= (0, \infty)
\]

**Case (ii):** If \( 0 \leq X \leq \infty \) and \( 0 \leq Y \leq b \)

Range of \( t \) = Range of \( \min \{0 \leq X \leq \infty , 0 \leq Y \leq b\} \)

\[
= \text{Range of } \min \{(0, \infty), (0, b)\}
\]

\[
= (0, b)
\]
Case (iii): If $0 \leq X \leq a$ and $0 \leq Y \leq b$

Range of $t = \text{Range of } \min \{0 \leq X \leq a, 0 \leq Y \leq b\}$

$= \text{Range of } \min \{(0, a), (0, b)\}$

$= \{0, \min (a, b)\}$

$= \{0, k\}$, where $k = \min (a, b)$

7.4 Renewal Process corresponding to $W(t)$

If $\{N(t), t \geq 0\}$ be a renewal process corresponding to $W(t)$ with finite mean, then

$$M(s) = \frac{w(s)}{s\{1 - w(s)\}}$$

Depending upon different forms of $f(t, \lambda), G(t, \lambda), w(t)$ and corresponding $w(s)$ may be determined so as to get the Laplace transform of expected number of renewals $M(t)$.

Similarly $E(t)$ may be obtained provided $w(t)$ is available.

7.4.1 Both the causes affecting the patient with constant hazard rate

When $h_\lambda(x) = \lambda$ and $h_\mu(y) = \mu$, then

$$f(x, \lambda) = \lambda e^{-\lambda x}, \quad \lambda > 0, x \geq 0 \quad \text{and} \quad \ldots \quad (7.5)$$

$$g(y, \mu) = \mu e^{-\mu y}, \quad \mu > 0, y \geq 0 \quad \ldots \quad (7.6)$$

$$w(t) = f(t, \lambda) + g(t, \mu) - f(t, \lambda)G(t, \mu) - g(t, \mu)F(t, \lambda)$$
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Hazard rate of \( W_0(t) \) is \( h_0(t) = (\lambda + \mu) \) ... (7.7)

\[ w_0(t) = (\lambda + \mu) \exp \left\{ - \int_0^t h(u) \, du \right\} \]

\[ = (\lambda + \mu) e^{- (\lambda + \mu) t} \] ... (7.8)

Here \( t \) follows exponential distribution with parameter \((\lambda + \mu)\). Hence the renewal process turns out to be a Poisson process with parameter \((\lambda + \mu)\).

Here \( M_0(t) = (\lambda + \mu) t \) ... (7.9)

7.4.2 Both the causes affecting the patient with linearly increasing hazard rate.

When \( h_\lambda(x) = \lambda x \), \( h_\mu(y) = \mu x \)

Then the hazard rate of \( W_1(t) \) is \( h_1(t) = (\lambda + \mu) t \) ... (7.10)

\[ W_1(t) = 1 - e^{- (\lambda + \mu) t^2 / 2} \], where \( t \geq 0, \lambda > 0, \mu > 0 \) ... (7.11)

The corresponding density function is

\[ w_1(t) = (\lambda + \mu) t e^{- (\lambda + \mu) t^2 / 2} \], where \( t \geq 0, \lambda > 0, \mu > 0 \) .....(7.12)

The renewal function corresponding to \( W_1(t) \) may be obtained from chapter 5 by replacing \( \lambda \) by \((\lambda + \mu)\)

Hence \( M_1(t) = (\lambda + \mu) \frac{t^4}{4!} + 3!(\lambda + \mu)^3 \frac{t^6}{6!} + .... \) ..... (7.13)

\[ E_1(t) = \int_0^\infty t(\lambda + \mu) t e^{- (\lambda + \mu) t^2 / 2} \, dt \]
\[ E_1(t) = (\lambda + \mu) \int_{0}^{\infty} t e^{-\frac{(\lambda + \mu) t}{2}} dt \]  

Substituting \( \frac{(\lambda + \mu) t^2}{2} = u \), we get

\[ E_1(t) = \frac{\sqrt{2}}{\sqrt{\lambda + \mu}} \int_{0}^{\infty} u^{\frac{1}{2}} e^{-u} du \]

\[ = \frac{2}{\sqrt{\lambda + \mu}} \Gamma\left( \frac{1}{2} + 1 \right) \]

\[ = \frac{2}{\sqrt{\lambda + \mu}} \Gamma \frac{1}{2} \]

\[ = \frac{1}{2} \sqrt{\frac{2}{\lambda + \mu}} \sqrt{\pi} \]

\[ E_1(t) = \left[ \frac{\pi}{2(\lambda + \mu)} \right]^{\frac{1}{2}} \]  

7.4.3 Both the causes affecting the patient with hazard rates valid in a range depending on parameters

The distribution of \( t \) for range depending on the parameter is assumed to be uniform in \((0, \lambda)\) and \((0, \mu)\) where
\[ f(x, \lambda) = \frac{1}{\lambda}, \quad 0 \leq x \leq \lambda \] \quad \ldots (7.16)

\[ g(y, \mu) = \frac{1}{\mu}, \quad 0 \leq y \leq \mu \] \quad \ldots (7.17)

The corresponding distribution functions are

\[ F(t, \lambda) = \frac{1}{\lambda} \int_0^t dx = \frac{t}{\lambda}, \quad \lambda > 0, \quad 0 < t \leq \lambda \quad \text{and} \quad \ldots (7.18) \]

\[ G(t, \mu) = \frac{t}{\mu}, \quad \mu > 0, \quad 0 < t \leq \mu \] \quad \ldots (7.19)

\[ \therefore h_1(t) = \frac{f(t, \lambda)}{1 - F(t, \lambda)} = \frac{1}{(\lambda - t)}, \quad 0 < t < \lambda \] \quad \ldots (7.20)

Similarly \[ h_\mu(t) = \frac{1}{(\mu - t)}, \quad 0 < t < \mu \] \quad \ldots (7.21)

\[ h_2(t) = h_1(t) + h_\mu(t) = \frac{1}{(\lambda - t)} + \frac{1}{(\mu - t)} \] \quad \ldots (7.22)

Consequently the density function is given by

\[ w_2(t) = \left\{ \frac{1}{(\lambda - t)} + \frac{1}{(\mu - t)} \right\} \exp \left[ -\int_0^t \left\{ \frac{1}{(\lambda - u)} + \frac{1}{(\mu - u)} \right\} du \right] \]

\[ = \left\{ \frac{1}{(\lambda - t)} + \frac{1}{(\mu - t)} \right\} \exp \left\{ -\int_0^t \frac{1}{(\lambda - u)} du - \int_0^t \frac{1}{(\mu - u)} du \right\} \]

\[ = \left\{ \frac{1}{(\lambda - t)} + \frac{1}{(\mu - t)} \right\} \exp \left[ \log(\lambda - t) - \log(\lambda) + \log(\mu - t) - \log(\mu) \right] \]

\[ = \left\{ \frac{1}{(\lambda - t)} + \frac{1}{(\mu - t)} \right\} \exp \left[ \log \frac{(\lambda - t)}{\lambda} + \log \frac{(\mu - t)}{\mu} \right] \]
\[
\begin{align*}
&= \left\{ \frac{1}{\lambda - t} + \frac{1}{\mu - t} \right\} \exp \left[ \log \left\{ \frac{\lambda - t}{\lambda} \right\} \left\{ \frac{\mu - t}{\mu} \right\} \right] \\
&= \left\{ \frac{1}{\lambda - t} + \frac{1}{\mu - t} \right\} \left\{ \frac{(\lambda - t)}{\lambda} \right\} \left\{ \frac{(\mu - t)}{\mu} \right\} \\
&= \left\{ \frac{(\lambda - t) + (\mu - t)}{\lambda \mu} \right\} \\
&= g(t, \mu)[1 - F(t, \lambda)] + f(t, \lambda)[1 - G(t, \mu)], \; 0 < t \leq \min(\lambda, \mu)
\end{align*}
\]

\[
\therefore w_s(t) = \int_0^s e^{-s} w_s(t) dt
\]

\[
= \frac{(\lambda + \mu)}{\lambda \mu} \frac{1}{s} - \frac{2}{\lambda \mu} \frac{1}{s^2}
\]

\[
= \frac{1}{s \lambda \mu} \left[ \lambda + \mu - \frac{2}{s} \right]
\]

\[
E_s(t) = \frac{1}{\lambda \mu} \int_0^t (\lambda - t + \mu - t) dt
\]

\[
= \frac{1}{\lambda \mu} \int_0^t [(\lambda + \mu)t - 2t^2] dt
\]

\[
= \frac{1}{\lambda \mu} \left( \lambda + \mu \right) \frac{t^3}{3} - \frac{2}{\lambda \mu} \frac{t^2}{2}
\]

\[
= \frac{(\lambda + \mu) k^2}{\lambda \mu} - \frac{2 k^3}{\lambda \mu} \frac{3}{3}
\]

\[
E_s(t) = \frac{k^2}{2 \lambda \mu} \left( \lambda + \mu - \frac{4k}{3} \right), \text{ where } k = \min(\lambda, \mu)
\]

\[
\therefore \text{ (7.25)}
\]
i.e. the mean of \( W_2(t) \) exists, which shows the existence of a renewal process corresponding to \( W_2(t) \) with renewal rate and expected number of renewals per unit of time for large \('t'\) both equal to \( E_2(t)^{-1} \)

\[
M_2(s) = \frac{w_2(s)}{s(1 - w_2(s))}
\]

\[
= \frac{1}{s\lambda\mu} \left[ \frac{(\lambda + \mu) - \frac{2}{s}}{s\lambda\mu} \right] = \frac{s(\lambda + \mu) - 2}{\lambda\mu s^2 - (\lambda + \mu)s + 2}
\]

\[
= \frac{As + K}{(s - \alpha)(s - \beta)} = \frac{As}{(s - \alpha)} + \frac{K}{(s - \beta)}
\]

\[
= A \left( 1 + \frac{\alpha}{s - \alpha} \right) + \frac{K}{s - \beta}
\]

\[
M_2(t) = A \alpha e^{\alpha t} + Ke^{\beta t} + Ar(t), \quad \ldots \quad (7.26)
\]

where \( r(t) \) is a function of \( t \) defined at \( t = 0 \)

*Here* \( K - A \beta = (\lambda + \mu) \), \( K \alpha = 2 \) \( \Rightarrow \) \( K = \frac{2}{\alpha} \)

\[
:\therefore \quad A = \frac{2 - \alpha (\lambda + \mu)}{\alpha\beta}
\]

\( \alpha \) and \( \beta \) are the roots of \( s \) which is given by

\[
s = \frac{(\lambda + \mu) \pm \sqrt{\lambda^2 + \mu^2 - 6\lambda\mu}}{2\lambda\mu}
\]
The roots are real if \( \sqrt{\lambda^2 + \mu^2 - 6\lambda\mu} \geq 0 \)

Graphically \( h_2(t) \) can be obtained as follows:

**Fig: 7.4 Causes affecting hazard rates valid in a range depending on parameters**

From the above graph it is observed that \( h_2(t) \) is increasing and \( w_2(t) \) follows IFR distribution.

**7.4.4 One of the causes affecting the patient with constant hazard rate other with the range depending on parameter**

Here the distribution of \( y \) is taken as Uniform \((0, \mu)\)

The distribution with constant hazard is given by

\[
F(x, \lambda) = 1 - e^{-\lambda x}, \quad \lambda > 0, \quad x \geq 0 \quad \text{and} \quad \ldots \quad \text{(7.27)}
\]

\[
G(y, \mu) = \frac{y}{\mu}, \quad 0 < y \leq \mu \quad \ldots \quad \text{(7.28)}
\]

Consequently range of \( t = \min(x, y) \) is given by \((0, \mu]\)

The distribution function of \( t \) is given by

\[
W(t; \lambda, \mu) = F(t, \lambda) + G(t, \mu) - F(t, \lambda)G(t, \mu).
\]
where $0 < t \leq \mu$ and $\lambda, \mu > 0$

The corresponding density function is given by

$$w(t; \lambda, \mu) = f(t, \lambda) + g(t, \mu) - f(t, \lambda)G(t, \mu) - F(t, \lambda)g(t, \mu)$$

where $0 < t \leq \mu$, and $\lambda, \mu > 0$

Though the range of $t$ for the first cause is $(0, \infty)$, while coming into operation with the second cause where $0 < t < \mu$, the distribution of minimum is ranged as $(0, \mu)$.

Here $f(t, \lambda) = \frac{\lambda e^{-\lambda t}}{1 - e^{-\lambda \mu}}$  \[\text{..... (7.29)}\]

where $0 < t \leq \mu$ and $\lambda, \mu > 0$

The equation (7.28) is the density function of truncated exponential distribution truncated at $\mu$.

$$\therefore F(t, \lambda) = \frac{1 - e^{-\lambda t}}{1 - e^{-\lambda \mu}}$$  \[\text{..... (7.30)}\]

$$1 - F(t, \lambda) = 1 - \frac{1 - e^{-\lambda t}}{1 - e^{-\lambda \mu}}$$

$$= \frac{(e^{-\lambda t} - e^{-\lambda \mu})}{(1 - e^{-\lambda \mu})}$$  \[\text{..... (7.31)}\]

$$E_\lambda(t) = \int_0^\mu \frac{\lambda e^{-\lambda t}}{(1 - e^{-\lambda \mu})} dt$$

$$= \frac{\lambda}{(1 - e^{-\lambda \mu})} \left[ t e^{-\lambda t} - \int_0^\mu \left\{ \frac{d}{dt} t e^{-\lambda t} dt \right\} dt \right]$$
\[
E_\lambda(t) = \frac{1}{\lambda} - \frac{\mu e^{-\lambda t}}{(1 - e^{-\lambda t})} \\
E_\lambda(t^2) = \frac{\lambda}{(1 - e^{-\lambda t})} \int_0^\mu t^2 e^{-\lambda t} dt \\
= \frac{\lambda}{(1 - e^{-\lambda t})} \left[ t^2 \int_0^\mu e^{-\lambda t} dt - \int_0^\mu \left\{ \frac{d}{dt} t^2 \int_0^\mu e^{-\lambda t} dt \right\} dt \right] \\
= \frac{\lambda}{(1 - e^{-\lambda t})} \left[ \frac{\mu^2 e^{-\lambda t}}{(-\lambda)} - \frac{2}{(-\lambda)} \left\{ \mu e^{-\lambda t} \right\} dt \right] \\
= \frac{\lambda}{(1 - e^{-\lambda t})} \left[ \frac{\mu^2 e^{-\lambda t}}{(-\lambda)} - \frac{2}{(-\lambda)} \left\{ \mu e^{-\lambda t} - \frac{1}{\lambda^2} (e^{-\lambda t} - 1) \right\} \right] \\
= -\frac{\mu e^{-\lambda t}}{(1 - e^{-\lambda t})} - \frac{2\mu e^{-\lambda t}}{\lambda(1 - e^{-\lambda t})} + \frac{2(1 - e^{-\lambda t})}{\lambda^2(1 - e^{-\lambda t})} \\
= \frac{2}{\lambda^2} - \frac{2\mu e^{-\lambda t}}{\lambda(1 - e^{-\lambda t})} - \frac{\mu^2 e^{-\lambda t}}{(1 - e^{-\lambda t})} \\
E_\lambda(t^2) = \frac{2}{\lambda^2} - \frac{\mu e^{-\lambda t}}{(1 - e^{-\lambda t})} \left[ \frac{2}{\lambda} + \mu \right] 
\]
The hazard rate corresponding to \( f_i(t, \lambda) \) is

\[
h_i(t) = \frac{f_i(t, \lambda)}{1 - F_i(t, \lambda)}
\]

\[
h_i(t) = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t} - e^{-\lambda \mu}} \quad \text{....(7.34)}
\]

The distribution of \( y \) which is taken as Uniform \((0, \mu)\) is given by

\[
g(t, \mu) = \frac{1}{\mu}, \quad 0 < t \leq \mu \quad \text{.... (7.35)}
\]

\[
G(t, \mu) = \frac{t}{\mu}, \quad 0 < t \leq \mu \quad \text{.... (7.36)}
\]

\[
E_{\mu}(t) = \frac{\mu}{2} \quad \text{.... (7.37)}
\]

The hazard rate corresponding to \( g(t, \mu) \) is

\[
h_\mu(t) = \frac{g(t, \mu)}{1 - G(t, \mu)}
\]

\[
= \frac{1}{(\mu - t)} \quad \text{.... (7.38)}
\]

\[
\therefore h_3(t) = h_\lambda(t) + h_\mu(t) = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t} - e^{-\lambda \mu}} + \frac{1}{(\mu - t)} \quad \text{.... (7.39)}
\]

Consequently the density function is given by

\[
w_3(t) = \{h_\lambda(t) + h_\mu(t)\}\exp\left[-\int_0^t \{h_\lambda(z) + h_\mu(z)\}dz\right]
\]

\[
= \{h_\lambda(t) + h_\mu(t)\}\exp\{-\{I_1 + I_2\}\}
\]
Where $I_1 = \int_0^i h_\lambda(z)dz$ and $I_2 = \int_0^i h_\mu(z)dz$

Now $I_1 = \int_0^i h_\lambda(z)dz,$

$$= \int_0^i \frac{\lambda e^{-\lambda z}}{e^{-\lambda z} - B} dz,$$ where $B = e^{-\lambda \mu}$

If $e^{-\lambda z} - B = m,$ then

$$-\lambda e^{-\lambda z} dz = dm$$

$$\therefore I_1 = \int \frac{-dm}{m} = -\int \frac{dm}{m}$$

The range of $m$ -

If $u = 0,$ then $1 - B = m$

If $u = t,$ then $(e^{-\lambda t} - B) = m$

$$\therefore -\int_0^i \frac{dm}{m} = -\left[ \log(e^{-\lambda t} - B) - \log(1 - B) \right]$$

$$= -\left[ \log \frac{e^{-\lambda t} - B}{1 - B} \right] = -\left[ \log \frac{e^{-\lambda t} - e^{-\lambda \mu}}{1 - e^{-\lambda \mu}} \right]$$

Similarly $I_2 = \int_0^i h_\mu(u)du$

$$= \frac{1}{(\mu - u)} du$$

If $(\mu - u) = m,$ then $-du = dm$
\[ I_2 = -\int_0^t \frac{dm}{m} \]

Now if \( u = 0 \), then \( m = \mu \) and

If \( u = t \), then \( m = \mu - t \)

\[ \therefore I_2 = -\int_0^t \frac{dm}{m} = -\left[ \log(\mu - t) - \log \mu \right] \]

\[ = -\left[ \log \left(\frac{\mu - t}{\mu}\right) \right] \]

\[ \therefore w_3(t) = \left[ \frac{\lambda e^{-\lambda t}}{e^{-\lambda t} - e^{-\lambda \mu}} + \frac{1}{(\mu-t)} \right] \left[ \frac{e^{-\lambda t} - e^{-\lambda \mu}}{1-e^{-\lambda \mu}} \times \frac{(\mu-t)}{\mu} \right] \]

\[ = \frac{\lambda e^{-\lambda t}}{(1-e^{-\lambda \mu})} + \frac{e^{-\lambda t}}{\mu(1-e^{-\lambda \mu})} - \frac{\lambda te^{-\lambda t}}{\mu(1-e^{-\lambda \mu})} - \frac{e^{-\lambda \mu}}{\mu(1-e^{-\lambda \mu})} \]

\[ = \frac{e^{-\lambda t}(\lambda \mu + 1)}{\mu(1-e^{-\lambda \mu})} - \frac{\lambda te^{-\lambda t}}{\mu(1-e^{-\lambda \mu})} - \frac{e^{-\lambda \mu}}{\mu(1-e^{-\lambda \mu})} \]

\[ = f(t, \lambda)[1-G(t, \mu)] + g(t, \mu)[1-F(t, \lambda)] \]

\( w_3(t) \) satisfies the result obtained in equation (7.3)

It may be shown that

\[ \int w_3(t; \lambda, \mu) dt = 1 \] \hspace{1cm} \ldots (7.41)

If \( \{N(t) \geq 0\} \) be a renewal process corresponding to \( w_3(t) \) with finite mean given by
where \( \left[ 1 - \frac{1}{\lambda \mu} \right] \geq 0 \), i.e. \( \lambda \mu \geq 1 \).

This indicates that the rate of occurrence of cause \( A_1 \) is greater than the inverse of the range of occurrence of cause \( A_2 \).

It is too complicated to obtain the \( \Pr \{ N(t) = n \} \) even in terms of Laplace transform.

The survival function is given below

\[
1 - W(t) = \left[ 1 - F(t) \right] \left[ 1 - G(t) \right]
\]

\[
= \left( e^{-\lambda t} - e^{-\lambda \mu} \right) \left[ 1 - \frac{t}{\mu} \right]
\]

However, \( h_3(t) \) can be plotted graphically as follows:

**Fig: 7.5 Causes affecting with constant hazard rate and with the range depending on parameters**
From the above graph it is observed that $h_3(t)$ is increasing and $W_3(t)$ follows IFR distribution. Since $h_3(t)$ is almost linearly increasing function of $t$ so the renewal process corresponding to $W_3(t)$ may be regarded as close to Poisson process as $t$ increases.

**7.5 Discussion:**

Here the study is confined to two specific causes which can also be done for multiple causes. Different situations have been studied when the specific causes affect the patients with different hazard rates. For each situation different distributions are formulated to study the pattern of the occurrence of depressive episodes. The pattern of the depressive episodes may be of help to the psychiatrists to predict the time to occurrence of depressive episodes and act accordingly.