Appendix
Appendix

The component wise equations for stresses of the XPP constitutive equation in terms of dimensionless variables for the \(i^{th}\) mode are outlined in this section.

The multimode XPP equation is written in single equation form below as:

\[
\nabla \lambda_i^{-1} \tau_i + \lambda_i^{-1} \frac{\tau_i}{\tau_i} = 2G_iD
\]

\[\text{(A1.1)}\]

\[
\lambda_i^{-1} = \frac{1}{\lambda_{b,i}} \left\{ \frac{\alpha_i}{G_i} \left[ F(\tau_i) \frac{I}{2} + G_i[F(\tau_i) - 1] \right] \right\}
\]

\[\text{(A1.2)}\]

\[
\frac{1}{\lambda_{b,i}} F(\tau_i) = \frac{2}{\lambda_{x,i}} e^{\nu_i(A_i - 1)} \left( 1 - \frac{1}{A_i} \right) + \frac{1}{\lambda_{b,i} A_i^2} \left( 1 - \frac{\alpha_i I x_i}{3G_i^2} \right) ; \Lambda_i < \frac{q_i}{\nu_i} = \frac{2}{q_i}
\]

\[\text{(A1.3)}\]

\[
\Lambda_i = \sqrt{1 + \frac{I x_i}{3G_i}}
\]

\[\text{(A1.4)}\]

The various symbols in equations (A1.1 – A1.4) have the usual meanings: \(\nabla \) represents upper convected derivative, the subscript \(i\) indicates the \(i^{th}\) relaxation mode, and for this mode, \(G_i\) is the relaxation modulus, \(\lambda_{b,i}\) is the orientation relaxation time of the backbone (taken as the mode relaxation time in the relaxation spectrum obtained from linear rheology; Table 3), \(\lambda_{x,i}\) is the backbone stretch relaxation time, \(\alpha_i\) is the backbone tube stretch and \(q_i\) is the number of arms attached to the backbone of the \(i^{th}\) mode pom-pom. Also,
\[ l_{\tau_j} = tr(\tau_j), \ l_{\xi,\tau} = tr(\tau_{ji} \tau_j) \text{ and } \alpha \text{ is a scalar parameter. The net deviatoric stress is given by } \tau = \sum_i \tau_i. \]

The XPP constitutive equation is written below slightly differently as:

\[
F\left(\lambda_i, \tau_i\right) + \lambda_{b,i} \tau_i + G_i \left[f\left(\lambda_i, \tau_i\right) - 1\right] + \frac{\alpha_i}{G_i} \tau_i \tau_i = 2\lambda_{b,i} G_i D_i
\]

where

\[
F\left(\lambda_i, \tau_i\right) = 2 \frac{\lambda_{b,i}}{\lambda_{s,i}} e^{\nu_i(\Lambda_i - 1)} \left(1 - \frac{1}{\Lambda_i}\right) + \frac{1}{\Lambda_i^2} \left(1 - \frac{\alpha_i \tau_i \tau_i}{3G_i^2}\right)
\]

The above equation (A1.6) can be written using the dimensionless forms explained earlier in chapter 5 as:

\[
\Delta = F\left(\lambda_i, \tau_i\right) = 2 \frac{\lambda_{b,i}}{\lambda_{s,i}} e^{\nu_i(\Lambda_i - 1)} \left(1 - \frac{1}{\Lambda_i}\right) + \frac{1}{\Lambda_i^2} \left(1 - \frac{\alpha_i tr(\tau_i \tau_i)}{3G_i^2}\right)
\]

\[
= 2 \frac{D e_{b,i}}{D e_{s,i}} e^{\nu_i(\Lambda_i - 1)} \left(1 - \frac{1}{\Lambda_i}\right)
\]

\[
+ \frac{1}{\Lambda_i^2} \left(1 - \frac{\alpha_i (\tilde{\tau}_{11,i}^2 + \tilde{\tau}_{22,i}^2 + \tilde{\tau}_{33,i}^2) D e_{b,i}}{3G_i^2}\right)
\]

where

\[
\Lambda_i = \sqrt{1 + \frac{D e_{b,i}}{E_i} (\tilde{\tau}_{11,i} + \tilde{\tau}_{22,i} + \tilde{\tau}_{33,i})} = \sqrt{1 + \frac{1}{G_i} (\tilde{\tau}_{11,i} + \tilde{\tau}_{22,i} + \tilde{\tau}_{33,i})}
\]

In equation (A1.8), the dimensionless numbers \(D e_{b,i}\) and \(D e_{s,i}\) correspond to the Deborah numbers for the Pom-Pom backbone orientation relaxation time and the backbone stretch relaxation time, respectively.
wherein \( D e_{b,i} = \frac{\lambda_{b,i} u_0}{X} \); \( D e_{s,i} = \frac{\lambda_{s,i} u_0}{X} \). Also, \( E_i = \tilde{g}_i D e_{b,i} \) is the dimensionless force corresponding to the \( i^{th} \) relaxation mode based on \( \lambda_{b,i} \) for the XPP equation.

For EFC flow kinematics, the rate of deformation tensor is written as:

\[
\begin{bmatrix}
2 \frac{du}{dx} & 0 & 0 \\
0 & 2f(x) & 0 \\
0 & 0 & 2g(x)
\end{bmatrix}
= \begin{bmatrix}
2 \frac{du}{dx} & 0 & 0 \\
0 & \frac{u dL}{L dx} & 0 \\
0 & 0 & \frac{u de}{e dx}
\end{bmatrix}
\]  
(A1.9)

The upper convected derivative of the stress tensor is written as:

\[
\frac{d \tau_i}{dt} = \frac{d \tau_i}{dt} - \left[ (\nabla \nabla)^T \cdot \tau_i \right] - \left[ \tau_i (\nabla \nabla) \right]
\]  
(A1.10)

In terms of Einstein notations, the \( d \tau_i/dt \) term is written under the given flow kinematics in multimode form as:

\[
\frac{d \tau_i}{dt} = \frac{\partial \tau_{11,i}}{\partial t} \hat{e}_1 \hat{e}_1 + \frac{\partial \tau_{22,i}}{\partial t} \hat{e}_2 \hat{e}_2 + \frac{\partial \tau_{33,i}}{\partial t} \hat{e}_3 \hat{e}_3 + v_1 \frac{\partial \tau_{11,i}}{\partial x_1} \hat{e}_1 \hat{e}_1 + v_1 \frac{\partial \tau_{22,i}}{\partial x_1} \hat{e}_2 \hat{e}_2 \\
+ v_1 \frac{\partial \tau_{33,i}}{\partial x_1} \hat{e}_3 \hat{e}_3
\]  
(A1.11)

Following a similar methodology as the one used in deriving the component wise equations for mode stresses for the RP-S constitutive equation (see section 5.1 under chapter 5) and using the directions \((x, y, z)\) for \((1,2,3)\), the mode stresses for the XPP constitutive equation are derived and written below as:
Appendix

xx- terms:
\[
\tilde{u} De_{b,i} \frac{d \tilde{\tau}_{xx,i}}{d \tilde{x}} + \left[ -2E_i \tilde{u} - 2\tilde{\tau}_{xx,i} De_{b,i} \right] \frac{d \tilde{u}}{d \tilde{x}}
= \alpha_i \frac{De_{b,i}}{E_i} \tilde{\tau}_{xx,i}^2 - \left[ \Delta \tilde{\tau}_{xx,i} - \left( \frac{E_i}{De_{b,i}} \right) \right] (\Delta - 1)
\]
(A1.12)

yy- terms:
\[
\tilde{u} De_{b,i} \frac{d \tilde{\tau}_{yy,i}}{d \tilde{x}} + \left[ -2E_i \frac{\tilde{u}}{L} - 2\tilde{\tau}_{yy,i} De_{b,i} \tilde{L} \right] \frac{d \tilde{L}}{d \tilde{x}}
= -\alpha_i \frac{De_{b,i}}{E_i} \tilde{\tau}_{yy,i}^2 - \left[ \Delta \tilde{\tau}_{yy,i} - \left( \frac{E_i}{De_{b,i}} \right) \right] (\Delta - 1)
\]
(A1.13)

zz- terms:
\[
\tilde{u} De_{b,i} \frac{d \tilde{\tau}_{zz,i}}{d \tilde{x}} + \left[ -2E_i \frac{\tilde{u}}{\tilde{e}} - 2\tilde{\tau}_{zz,i} De_{b,i} \tilde{e} \right] \frac{d \tilde{e}}{d \tilde{x}}
= -\alpha_i \frac{De_{b,i}}{E_i} \tilde{\tau}_{zz,i}^2 - \left[ \Delta \tilde{\tau}_{zz,i} - \left( \frac{E_i}{De_{b,i}} \right) \right] (\Delta - 1)
\]
(A1.14)

Equations (A1.12 to A1.14) represent the dimensionless form of the stress components in the three directions under the given flow kinematics where the polymer flow is modeled using the XPP constitutive equation.

Equations (5.16-5.18) as described in Chapter 5 along with the above equations (A1.12 to A1.14) form a stiff system of simultaneous differential equations for the given unknowns that is solved numerically using Matlab®.