CHAPTER 1
CHAPTER - I
A UNIQUENESS PROBLEM IN VALUED FUNCTION FIELDS OF CONICS

§1. INTRODUCTION

Throughout this chapter $v_0$ is a (Krull) valuation of a field $K_0$ with value group $G_0$ and residue field $k_0$. Let $K/K_0$ be a finitely generated field extension of transcendence degree one and let $v$ be a valuation of $K$ extending $v_0$ with value group $G$ such that $G/G_0$ is not a torsion group. By an element with the uniqueness property for $(K,v)/(K_0,v_0)$ (or more precisely for $v/v_0$), we mean an element $t$ of $K$ which satisfies (i) $v(t)$ is non-torsion modulo $G_0$; (ii) the valuation of $K_0(t)$ obtained by restricting $v$ has a unique extension (upto equivalence) to $K$.

In 1993, Khanduja [12] considered the following uniqueness problem for simple transcendental extensions:

"Suppose that $K$ is a simple transcendental extension of $K_0$ and $v$ is an extension of $v_0$ with value group $G$ such that $G/G_0$ is not a torsion group. Does there exist an element of $K$ which satisfies the uniqueness property for $v/v_0"\text{?}"

It was proved that the answer to the above question is "yes" if (a) either $(K_0,v_0)$ is henselian (b) or $v_0$ is of rank 1 with $G_0$, a cofinal subset of $G$ (cf.[12, Theorem 2.1]). The
example in [12, §5] shows that the hypothesis (a), (b) is not superfluous.

In this chapter, we deal with the uniqueness problem in a more general set-up viz. when $K$ is a function field of a conic over $K_0$, i.e, $K = K_0(x,y)$ where $(x,y)$ satisfies an irreducible polynomial of degree 2 over $K_0$. We shall prove:

THEOREM 1.1. Let $K$ be a function field of a conic over $K_0$. Let $v_0$ be a valuation of $K_0$ with value group $G_0$ and $v$ be an extension of $v_0$ to $K$ with value group $G$ such that $G/G_0$ is not a torsion group. Then there exists an explicitly constructible element of $K$ which satisfies the uniqueness property for $v/v_0$ provided

(i) either $(K_0, v_0)$ is henselian,

(ii) or $v_0$ is of rank 1, the algebraic closure of $K_0$ in $K$ is a purely inseparable extension of $K_0$ and $G_0$ is a cofinal subset of $G$.

§2. DEFINITIONS, NOTATIONS AND SOME PRELIMINARY RESULTS

A field $K$, which is a function field of a conic over $K_0$, is said to be a regular function field of a conic over $K_0$, if (i) $K/K_0$ is a separably generated extension; (ii) $K_0$ is algebraically closed in $K$.

Observe that a simple transcendental extension $K_0(t)$ of a field $K_0$ is a (regular) function field of a conic over $K_0$ which can be visualized on writing $K_0(t)$ as $K_0(t,1/t)$ where
(t, 1/t) satisfies the equation \(XY - 1 = 0\).

We shall use \(X, Y\) for indeterminates and write \(Z\), for the additive group of integers.

As in [30], for any \(c\) in the base field \(K_0\), \(H_c(X, Y)\) will stand for the polynomial \(X^2 - XY - cy^2\) or \(X^2 - cy^2\) according as characteristic of \(K_0\) is 2 or not. It is a routine calculation to check that if \(c\) and \(d\) are non-zero elements of \(K_0\), then the polynomial \(H_c(X, Y) - d\) is irreducible over \(K_0\).

The following Proposition 1.2 and Lemma 1.3 have been proved by an elementary method by Ohm (cf. [30, Theorem 2.3, Corollary 2.10]). We omit their proofs.

**Proposition 1.2.** Let \(K\) be a regular function field of a conic over a field \(K_0\). Then one can determine (by an explicit algorithm) non-zero elements \(c\) and \(d\) in \(K_0\) such that \(K = K_0(x, y)\) where \((x, y)\) satisfies \(H_c(X, Y) - d = 0\).

**Lemma 1.3.** Let \(K = K_0(x, y)\) be a regular function field of a conic over \(K_0\) with \(x, y\) related by \(H_c(x, y) - d = 0\) for some non-zero elements \(c, d\) in \(K_0\). Let \(\alpha\) be a root of the quadratic polynomial \(H_c(X, 1)\). Then \(K(\alpha)\) is a simple transcendental extension of \(K_0(\alpha)\) having \(z = x - \alpha y\) as a generator.

**Fundamental Inequality 1.4.** If \(u_0\) is a valuation of field \(F_0\) and \(u_1, u_2, \ldots, u_s\) are all the non-equivalent prolongations of \(u_0\) to a finite extension \(F\) of \(F_0\), then it is well known...
(cf. [7, §8.3, Theorem 1]) that
\[ \sum_{i=1}^{m} e_i f_i \leq [F : F_0] \]

where \( e_i, f_i \) denote respectively the ramification index and the residual degree of the extension \( u_i/u_0 \).

The following lemma proves Theorem 1.1 in a special case.

**Lemma 1.5.** Let \( K, K, \alpha \) and \( z \) be as in Lemma 1.3. Let \( v_0 \) be a valuation of a field \( K_0 \) with value group \( G_0 \) and \( v \) be a prolongation of \( v_0 \) to \( K \) such that \( G/G_0 \) is a non-torsion group. Assume that (i) either \( (K_0, v) \) is henselian; (ii) or \( v_0 \) is of rank 1 with \( G_0, a \) cofinal subset of \( G \). If \( w \) is an extension of \( v \) to \( K(\alpha) \) such that \( w(z) \) is non-torsion modulo \( G_0 \), then one can explicitly construct an element of \( K \), which satisfies the uniqueness property for \( v/v_0 \).

**Proof.** The lemma needs to be proved only when \( \alpha \not \in K_0 \), for if \( \alpha \in K_0 \), then clearly \( z \) satisfies the uniqueness property for \( v/v_0 \). Let \( \bar{\alpha} \) denote the other root of the quadratic polynomial \( H_c(x, 1) \) and \( \bar{z} \), the element \( x - \bar{\alpha}y \) of \( K(\alpha) \) and \( \sigma \), the automorphism of \( K(\alpha)/K \) defined by \( \sigma(\alpha) = \bar{\alpha} \). Observe that if \( \alpha \not \in K_0 \), then \( \sigma \) is the unique non-trivial automorphism of \( K(\alpha)/K \).

Assume first that \( v_0 \) has a unique extension, \( v_1 \) (say) to \( K_0(\alpha) \). We show that the element \( t = z + \bar{z} \) of \( K \) satisfies the uniqueness property for \( v/v_0 \) in this case. Keeping in view
that $z\tilde{z}=d$ and $w(z)$ is non-torsion modulo $G\omega$, it can be easily seen that $w(z) \neq w(\tilde{z})$ and $w(\tilde{z})$ is non-torsion modulo $G\omega$. Hence, by the strong triangle law

$$v(t) = w(t) = w(z+\tilde{z}) = \min\{w(z), w(\tilde{z})\},$$

which implies that $v(t)$ is non-torsion modulo $G\omega$. So the valuation $v^t_\omega$ of the field $K_\omega(t)$, obtained by restricting $v$, is defined on $K_\omega[t]$ by

$$v^t_\omega(\sum a_it^i) = \min\{v_\omega(a_i) + iv(t)\}. \quad (1)$$

By hypothesis, $v_1$ is the only prolongation of $v_\omega$ to $K_\omega(\alpha)$. It is immediate from (1) that $v^t_\omega$ has a unique prolongation to $K_\omega(\alpha,t)$, which we denote by $v^t_1$. The valuations $w$ and $w_\omega$ of $K(\alpha)$, being prolongations of $v^t_\omega$, extend the valuation $v^t_1$. Keeping in view that $[K(\alpha):K_\omega(\alpha,t)] = [K_\omega(\alpha,z):K_\omega(\alpha,t)] = 2$, it now follows from the Fundamental Inequality 1.4 that $w$ and $w_\omega$ are the only two extensions of $v^t_1$ (and hence of $v^t_\omega$) to $K(\alpha)$. Since $w$ and $w_\omega$ coincide with $v$ on $K$, it is now clear that $v$ is the unique extension of $v^t_\omega$ to $K$ and the lemma is proved in the first case.

Consider now the case, when $v_\omega$ has two extensions to $K_\omega(\alpha)$. As $(K_\omega, v_\omega)$ won’t be henselian in this case, in view of the hypothesis, $v_\omega$ is of rank 1 and $G_\omega$ is cofinal in $G$. We claim that the element

$$t = (z+a+b)(\tilde{z}+\tilde{a}+\tilde{b})/(z+b)(\tilde{z}+\tilde{b}) \quad (2)$$

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of $K$ satisfies the uniqueness property for $v/v_0$ for suitably chosen $a,b$ in $K_0(\alpha)$. 

Let $v_1$ denote the prolongation of $v_0$ to $K_0(\alpha)$, obtained by restricting $w$ to $K_0(\alpha)$, then the other prolongation $v_2$ of $v_0$ to $K_0(\alpha)$ is given by 

$$v_2(\beta) = v_1(\tilde{\beta}), \; \beta \in K_0(\alpha).$$  \hspace{1cm} (3)$$

Since $G_0$ is a cofinal subset of $G$, by the Approximation theorem [10, Page 24(3.13)], we can choose $b \in K_0(\alpha)$, satisfying 

$$v_1(b) > w(z), \hspace{1cm} (4)$$

$$v_2(b) < v_0(d) - w(z). \hspace{1cm} (5)$$

Further choose $a \in K_0(\alpha)$ such that 

$$v_1(a) < w(z), \hspace{1cm} (6)$$

$$v_2(a+b) > v_0(d) - w(z). \hspace{1cm} (7)$$

By (4), (6) and the strong triangle law, we have 

$$w(z+a+b) = v_1(a). \hspace{1cm} (8)$$

Using the relation $z\bar{z} = d$, we have 

$$w(\bar{z}+\bar{a}+\bar{b}) = w(d+z(a+b))-w(z).$$

It now follows from (3), (7) and the strong triangle law that
In view of (8) and (9), the \( w \)-valuation of the element \( t \) defined by (2) is given by

\[
w(t) = v_1(a) + v_o(d) - w(z) - w(z+b) - w(z+b).
\]

By virtue of (4) and the strong triangle law, we see that

\[
w(z+b) = w(z).
\]

Using \( z^2 = d \), (3), (5) and the strong triangle law, we have

\[
w(z+b) = w(d+bz) - w(z) = v_2(b).
\]

Consequently

\[
w(t) = v_1(a) + v_o(d) - v_2(b) - 2w(z). \tag{10}
\]

In particular \( w(t) \) is non-torsion modulo \( G_o \), as \( w(z) \) is given to be so.

Let \( v_o^t, v_1^t \) denote the restrictions of \( w \) to \( K_o(t) \) and \( K_o(\alpha,t) \) respectively. Since \( v_o \) has two prolongations to the quadratic extension \( K_o(\alpha) \) of \( K_o \), it follows from Fundamental Inequality 1.4 that the value group of \( v_1 \) is \( G_o \) and hence by virtue of (10), the value group of \( v_1^t \) is \( G_o + Z2w(z) \). As the value group of \( w \) is \( G_o + Zw(z) \), the index of ramification of \( w/v_1^t \) is 2. Since, by Lüroth Lemma ([41, Page 197]), \([K(\alpha):K_o(\alpha,t)]\)=2, in view of the Fundamental Inequality 1.4,
it is now clear that \( w \) is the only extension of \( \nu^t_1 \) to \( K(\alpha) \). As \( \sigma \) maps \( K_0(\alpha,t) \) onto itself, we conclude that \( w \sigma \) is the only extension of the valuation \( \nu^t_2 \), obtained by restricting \( w \sigma \) to \( K_0(\alpha,t) \). The valuations \( \nu^t_1 \) and \( \nu^t_2 \) being the only distinct extensions of \( \nu^t_0 \) to \( K_0(\alpha,t) \), it now follows that \( w \) and \( w \sigma \) are the only extensions of \( \nu^t_0 \) to \( K(\alpha) \). As these valuations coincide with \( \nu \) on \( K \), it is the only extension of \( \nu^t_0 \) to \( K \).

This completes the proof of the lemma.

**DEFINITION 1.6.** For a finite extension \((F,u)/(F_0,u_0)\) of valued fields, the **henselian defect** of the extension is defined to be \([F^h:F_0^h]/e_f\) where 'h' stands for the henselization with respect to the underlying valuation and \( e, f \) for the ramification index and the residual degree of \( u/u_0 \).

**A FUNDAMENTAL EQUALITY 1.7.** If \( u_0 \) is a valuation of a field \( F_0 \) and \( u_1, u_2, \ldots, u_m \) are all the non-equivalent prolongations of \( u_0 \) to a finite extension \( F \) of \( F_0 \), then it is well known [cf. (10, Page 125(17.3))] that

\[
[F:F_0] = \sum_{i=1}^{m} \left[ (F,u_i)^h:(F_0,u_0)^h \right] = \sum_{i=1}^{m} e_i f_i d_i
\]

where 'h' stands for the henselization and \( e_i, f_i, d_i \) denote respectively the ramification index, the residual degree and the henselian defect of the extension \( u_i/u_0 \).

**NOTATIONS.** Let \( w_0 \) be a valuation of a field \( L_0 \) with value group \( H_0 \) and residue field \( \Lambda_0 \), \( w \) be a prolongation of \( w_0 \) to a
simple transcendental extension $L = L_0(x)$ with residue field $\Delta$ and value group $H$ such that $H/H_0$ is a non-torsion group. As in [12, §2], it can be easily seen that $\Delta/\Delta_0$ is a finite extension and $H_1/H_0$ is a finite group where

$$H_1 = \{ h \in H | h \text{ is torsion modulo } H_0 \}.$$

We shall denote by $N, S, I$ (to be more precise by $N(w/w_0)$ etc.), the natural numbers defined by

$$N = \min \{ \deg f | f \in L_0[x], w(f) \text{ is non-torsion modulo } H_0 \},$$

$$S = [\Delta : \Delta_0],$$

$$I = [H_1 : H_0].$$

Corresponding to an element $t \in L$, with $w(t)$ non-torsion modulo $H_0$, let $D_h(t)$ denote the henselian defect of the extension $(L, w)/(L_0(t), w_0^t)$, where $w_0^t$ is the restriction of the valuation $w$ to $L_0(t)$. It has been proved by Kuhlmann (cf. [17, Theorem 5.4]) or (cf. [18]) that $D_h(t)$ is independent of the choice of the element $t$ in $L$, whenever $w(t)$ is non-torsion modulo $H_0$. We shall denote $D_h(t)$ by $D_h$ or more precisely by $D_h(w/w_0)$.

The following results labelled as Theorems A, B are already known (cf. [12, Theorems 2.1, 2.2]) and are quoted for future reference.
**THEOREM A.** Let $L \subseteq L$, $w_0, w, N, S, I$ and $D_h$ be as above. Then there exists an element of $L$ which satisfies the uniqueness property for $w/w_0$, if and only if, $N = ISD_h$ holds.

**THEOREM B.** Let $L \subseteq L$, $w_0$, $w$ and $H_0 \subseteq H$ be as before. Suppose that

(i) either $(L_0, w_0)$ is henselian,

(ii) or $(L_0, w_0)$ has rank 1 and $H_0$ is a cofinal subset of $H$. Then any $t \in L_0[x]$, of minimum degree $N$ such that $w(t)$ is non-torsion modulo $H_0$, satisfies the uniqueness property for $w/w_0$.

§3. PROOF OF THEOREM 1.1

In view of the fact that any valuation of a field $F$ has a unique prolongation to a purely inseparable extension of $F$ (see[10,Page 98(13.8)]), it may be assumed that $K_0$ is algebraically closed in $K$. Without loss of generality, we may assume that $K$ is a regular function field of a conic over $K_0$, for if $K = K_0(x', y')$ where $(x', y')$ satisfies an irreducible polynomial relation of degree 2 over $K_0$ and $x'$ is transcendental over $K_0$, then the assumption $K/K_0$, not separable, would lead to $y'$, being algebraic and purely inseparable over $K_0(x')$; consequently, any element of $K_0(x')$ which satisfies the uniqueness property for $K_0(x')/K_0$ would do so for $K/K_0$ also.

Keeping in view Proposition 1.2, one can find (by an
explicit algorithm) non-zero elements c and d in K₀ such that
K = K₀(x,y) where (x,y) satisfies the equation H₉(X,Y)-d = 0.
As before, α₁, α will stand for the roots of the polynomial
H₉(X,1), z, z for the generators x - α₁y, x - αy of the simple
transcendental extension K(α)/K₀(α) and σ for the automorphism
of K(α)/K defined by σ(α) = α₁.

We fix a prolongation w of v to K₀ = K₀(z), where K₀ is
the algebraic closure of K₀. Let w₀, w be the restrictions of
w to K₀(α), K(α) respectively; H₀, H be their value groups and
Δ₀, Δ their respective residue fields. In what follows, N, S, I
and Dₗ will stand for N(w/w₀) etc. relative to the extension
(K(α), w)/(K₀(α), w₀). By Theorems A and B, we have

N = ISDₗ.

(11)

If w(z) is non-torsion modulo G₀, then in view of Lemma
1.5 the theorem is already proved. So from now onwards, we may
assume that w(z) is torsion modulo G₀.

We denote by D, the non-empty subset of K₀, defined by

D = { γ ∈ K₀ | w(z-γ) is non-torsion modulo G₀).

Choose an element β of D such that

[K₀(α, β):K₀(α)] ≤ [K₀(α, γ):K₀(α)]

for all γ in D. Since w(z) is torsion modulo G₀, β can't be
zero. We denote by P(X) = a₀ + a₁X + a₂X² + .............. + Xₙ, the
minimal polynomial of $\beta$ over $K_\circ(a)$ and by $\mu$, the $\tilde{w}$-valuation of $(z-\beta)$. The roots $\beta_1 = \beta, \beta_2, \ldots, \beta_n$ of $P(X)$ are arranged so that $\tilde{w}(z-\beta_i)$ is non-torsion modulo $G_\circ$ for $1 \leq i \leq m$ and $\tilde{w}(z-\beta_i)$ is torsion modulo $G_\circ$ for $i \geq m + 1$.

Since $\tilde{w}(z-\beta)$ is non-torsion modulo $G_\circ$, for any $\delta \in \bar{K}_\circ$, we see that $\tilde{w}(\beta-\delta) \neq \tilde{w}(z-\beta) = \mu$ and hence

$$\tilde{w}(z-\delta) = \min \{\mu, \tilde{w}(\beta-\delta)\}. \quad (12)$$

Consequently

$$w(P(z)) = \sum_{i=1}^{n} \tilde{w}(z-\beta_i) = m\mu + \sum_{i=m+1}^{n} \tilde{w}(\beta-\beta_i) = \vartheta(\text{say}) \quad (13)$$

is non-torsion modulo $G_\circ$, which implies that $n \geq N$. It is clear from the definition of $N$ that $\tilde{w}(z-\gamma)$ is non-torsion modulo $G_\circ$ for some root $\gamma$ of a polynomial of degree $N$ over $K_\circ(a)$. Thus $n = N$.

Since $z \tilde{w} = d$, the monic polynomial $P_1(z)$ (in the transcendental element $z$ over $K_\circ(a)$), defined by

$$\sigma(P(z)) = \sum_{i=0}^{N-1} \tilde{a}_i \tilde{z}^i + \tilde{z}^N = \tilde{a}_0 P_1(z)/z^N \quad (14)$$

has degree $N$ and is not divisible by $z$. It follows from Lüroth Lemma [41, Page 197] and (11) that if $t = P(z)\sigma(P(z))$, then

$$[K_\circ(a,z):K_\circ(a,t)] = 2N = 2ISD_h. \quad (15)$$

Two cases are distinguished:
Case I. $w$ is the only extension of $v$ to $K(a)$. We claim that the element $t = P(z)\sigma(P(z))$ of $K$ satisfies the uniqueness property for $v/v$ in this case.

We split two sub cases.

Sub case (i). $w(\sigma(P(z))$ is non-torsion modulo $G$. It is clear from (14) that in the present situation, $w(P_1(z))$ is also non-torsion modulo $G$, by virtue of the fact that $w(z)$ is torsion modulo $G$. Observe that $w(P(z)) = w(P_1(z))$, for otherwise, in view of the strong triangle law,

$$w(P(z) - P_1(z)) = \min\{w(P(z)), w(P_1(z))\}$$

which would imply that the $w$-valuation of the polynomial $P(z) - P_1(z)$ of degree less than $N$ is non-torsion modulo $G$, contradicting the choice of $N$. It now follows from (13) and (14) that

$$w(t) = w(P(z)) + w(\sigma(P(z))) = 2\theta + w(\tilde{a}_0) - w(z^N); \quad (16)$$

in particular, $w(t)$ is non-torsion modulo $G$. By [12, Lemma 3.1], the value group $H$ of $w$ is $H_1 + \mathbb{Z}\theta$, where

$$H_1 = \{h \in H \mid h \text{ is torsion modulo } H_0\}.$$ 

Let $w_t$ denote the valuation of $K(a,t)$, obtained by restricting $w$; in view of (16), the ramification index of $w/w_t$ is given by
As in [7, §10.1, Proposition 1], it can be easily seen that the residue field of \( \omega^t_o \) is same as that of \( \omega_o \) and so the residual degree of \( \omega/\omega^t_o \) is \( S \). Since the henselian defect of \( \omega/\omega^t_o \) is \( D_h \), it is immediate by virtue of Fundamental Equality 1.7 and (15), that \( \omega \) is the only extension of \( \omega^t_o \) to \( K(\alpha) \). In view of the hypothesis of case I, \( \omega^t_o \) is the only extension of \( \nu^t_o \) to \( K_o(\alpha, t) \), therefore \( \nu \) is the only extension of \( \nu^t_o \) to \( K \) as desired.

Subcase (ii). \( w(\sigma(P(z))) \) is torsion modulo \( G_o \).

Then clearly

\[
w(t) = w(P(z)) + w(\sigma(P(z))) = \theta + w(\sigma(P(z))) = \theta + w(\sigma(P(z)))
\]

is non-torsion modulo \( G_o \).

Let \( \omega^t_o \) denote the restriction of \( w \) to \( K_o(\alpha, t) \). Keeping in view formula (1) and the fact that \( (\omega_o \sigma)(\delta) = \omega_o(\delta) \) for \( \delta \in K_o(\alpha) \), it is clear that the valuation \( w_0 \sigma \) of \( K(\alpha) \) also extends \( \omega^t_o \). Arguing as in the previous sub case, the ramification index of \( w/\omega^t_o \) is easily seen to be \( I \). Since \( \sigma \) maps the field \( K_o(\alpha, t) \) onto itself, it follows that the ramification index, the residual degree and the henselian defect of \( w_0 \sigma/\omega^t_o \) are the same as the ones of \( w/\omega^t_o \). It is now clear in view of Fundamental Equality 1.7 and (15) that \( w \) and \( w_0 \sigma \) are the only two extensions of \( \omega^t_o \) (and hence of \( \nu^t_o \)) to \( K(\alpha) \). Since \( w, w_0 \sigma \) coincide with \( \nu \) on \( K \), we conclude that \( \nu \) is the unique extension of \( \nu^t_o \) to \( K \).
Case II. When $\nu_o$ has two extensions to $K_o(\alpha)$.

As $(K_o, \nu_o)$ won't be henselian in this case, $\nu_o$ is of rank 1 and $G_o$ is cofinal in $G$.

Recall that $w_o$ is the extension of $\nu_o$ to $K_o(\alpha)$, obtained by restricting $w$. The other prolongation $\overline{w}_o$(say) of $\nu_o$ to $K_o(\alpha)$ is given by

$$\overline{w}_o(\beta) = w_o(\sigma(\beta)), \beta \in K_o(\alpha).$$

We discuss two sub cases.

Subcase (i). $w(\sigma(P(z)))$ is non-torsion modulo $G_o$.

We show that $t = P(z)\sigma(P(z))$ satisfies the uniqueness property in this sub case. Proceeding exactly as in the sub case(i) of case I, it can be shown that $w(t)$ is non-torsion modulo $G_o$ and $w$ is the only extension of $w^t_o$ to $K(\alpha)$, where $w^t_o$ denote the restriction of $w$ to $K_o(\alpha,t)$. Consequently, $w^t_o\sigma$ is the only extension of the valuation $\overline{w}_o^t$ obtained by restricting $w^t_o\sigma$ to $K_o(\alpha,t)$. Since $w^t_o, \overline{w}_o^t$ constitute all the extensions of $v^t_o$ to $K_o(\alpha,t)$, it follows that $w$ and $w^t_o\sigma$ are the only extensions of $v^t_o$ to $K(\alpha)$. As $w, w^t_o\sigma$ agree with $\nu$ on $K$, we conclude that $\nu$ is the unique extension of $v^t_o$ to $K$.

Subcase (ii). $w(\sigma(P(z)))$ is torsion modulo $G_o$.

In view of sub case (i) of case II, it is enough to construct an element $\gamma \in K_o(\alpha, \beta) \cap D$, whose minimal polynomial $Q(X)$ (say) over $K_o(\alpha)$ satisfies $w(\sigma(Q(z)))$ is non-torsion
To choose $\gamma$, we consider two possibilities.

Suppose first that $\alpha \notin K_0(\beta)$. Let $\tau$ denote the automorphism of $K_0(\alpha, \beta)/K_0(\beta)$ defined by $\tau(\alpha) = \overline{\alpha}$ and $\omega_1$, the valuation obtained by restricting $\omega$ to $K_0(\alpha, \beta)$. By the Approximation theorem [10, Page 24 (3.13)], applied to the valuations $\omega_1$ and $\omega_1 \tau$, we can find an element $\xi \in K_0(\alpha, \beta)$ such that

\begin{align*}
\omega_1(\xi - \beta^{-1}) &> \max \{\omega_1(\beta^{-1}), \mu - 2\omega_1(\beta)\}, & (19) \\
(\omega_1 \tau)(\xi - \beta \gamma^{-1}) &> \mu - v_0(d). & (20)
\end{align*}

Define $\gamma = 1/\xi$. It is immediate from (19) that $\omega_1(\beta) = \omega_1(\gamma)$ and $\omega_1(\beta - \gamma) > \mu$, consequently $\gamma \in D$.

In view of (20) and the fact that $\tau(\beta) = \beta$, we have

$$\omega_1(\tau(\xi) - \beta) > \mu$$

which combined with $\omega_1(z - \beta) = \mu$ gives

$$\omega_1(\tau(\xi) - z) = \mu,$$

i.e.,

$$\omega_1(z - \tau(\gamma)) = \mu - \omega_1(\tau(\xi)). & (21)$$

We are now in a position to show that $w(\sigma(Q(z)))$ is non-torsion modulo $G_0$, where $Q(X)$ is the minimal polynomial of $\gamma$ over $K_0(\alpha)$. Let $\overline{\sigma}$ be an automorphism of $K\overline{K}_0/K$ which coincides with $\tau$ on $K_0(\alpha, \beta)$; such an automorphism exists.
because K being a regular extension of \( K_0 \) is linearly disjoint from \( \bar{K}_0 \) over \( K_0 \). (cf [20, Chapter 3, §1, Theorem 2]).

Let \( \gamma_1 = \gamma, \gamma_2, \ldots, \gamma_N \) be all the roots of the polynomial \( Q(X) \). Then

\[
\hat{w}(\sigma(Q(z))) = \sum_{i=1}^{N} \hat{w}(z-\sigma(\gamma_i)) = \sum_{i=1}^{N} \hat{w}(d-z\sigma(\gamma_i)) - N\hat{w}(z).
\]

Keeping in view (21), (12) and the assumption \( \hat{w}(z) \) is torsion modulo \( G_0 \), we conclude from the last equation that \( \hat{w}(\sigma(Q(z))) \) is non-torsion modulo \( G_0 \) as desired.

The remaining possibility when " \( \alpha \in K_0(\beta) \) " will be disposed of by showing that we can find an element \( \beta' \) of \( D \) which is algebraic of degree \( N \) over \( K_0(\alpha) \) such that \( \alpha \notin K_0(\beta') \).

Since the valuation \( v_0 \) of \( K_0 \) has two prolongations to the separable quadratic extension \( K_0(\alpha) \), therefore by [10, Page 16(2.12)], \( \alpha \) belongs to the completion \( \hat{K}_0 \) of \( K_0 \) with respect to \( v_0 \). Since \( K_0 \) is dense in \( \hat{K}_0 \), corresponding to the coefficients \( a_0, a_1, \ldots, a_{N-1} \) of \( P(X) \in K_0(\alpha)[X] \), we can choose \( b_0, b_1, \ldots, b_{N-1} \in K_0 \) such that

\[
w_0(b_i-a_i) > Nu + i\lambda, \quad 0 \leq i \leq N-1, \quad (22)
\]

where \( \lambda = \max\{-\hat{w}(\beta), \hat{w}(\beta)\} \). Set
\[ R(X) = X^N + b_{N-1}X^{N-1} + \ldots + b_0. \]

We claim that there exists a root \( \beta' \) of \( R(X) \) such that \( \tilde{w}(\beta - \beta') > \mu \). Suppose not, then for each root \( \alpha_i \) of \( R(X) \), \( \tilde{w}(\beta - \alpha_i) \leq \mu \). Consequently,

\[
\tilde{w}(P(\beta) - R(\beta)) = \tilde{w}(R(\beta)) = \sum_{i=1}^{N} \tilde{w}(\beta - \alpha_i) \leq N\mu;
\]

this is impossible because by (22)

\[
\tilde{w}(P(\beta) - R(\beta)) = \tilde{w}(\sum_{i=0}^{N-1} (a_i - b_i)\beta^i) \geq \min \{ \tilde{w}(a_i - b_i) - i\lambda \} > N\mu.
\]

Hence the claim.

It only remains to be shown that \( \alpha \notin K_0(\beta') \). If \( \alpha \in K_0(\beta') \), then keeping in view that \( \beta' \) is algebraic over \( K_0 \) of degree not exceeding \( N \) and that every element of \( D \) is algebraic over \( K_0(\alpha) \) of degree not less than \( N \), we are led to

\[
N \geq [K_0(\beta') : K_0] = [K_0(\alpha, \beta') : K_0] = [K_0(\alpha, \beta') : K_0(\alpha)][K_0(\alpha) : K_0] \geq 2N.
\]

This contradiction proves the desired result.