INTRODUCTION
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The celebrated "Zariski's Cancellation Problem" was proposed by O.Zariski at Paris colloquium on Algebra and Theory of Numbers in 1949.

Zariski's Cancellation Problem: Let $K_1$ and $K_2$ be finitely generated extensions of a field $K$. Assume that the simple transcendental extensions of $K_1$ and $K_2$ are $K$-isomorphic to each other. Does it follow that $K_1$ and $K_2$ are $K$-isomorphic?

Several mathematicians like Segre, Igusa and Samuel [24] dealt with this problem and found the answer in affirmative in certain cases. In 1986, Beauville, Colliot-Thélène, Sansuc and Sir Swinnerton Dyer [6] proved that the answer to the above question is "no" in general. M.Nagata [24], while working on this problem made a conjecture in 1967, which came to be known as the Ruled Residue Conjecture.

Ruled Residue Conjecture: Let $v_0$ be a valuation of a field $K_0$ with residue field $k_0$ and $v$ be a prolongation of $v_0$ to a simple transcendental extension $K_0(x)$ of $K_0$. Then the residue field of $v$ is either an algebraic extension of $k_0$ or it is a simple transcendental extension of a finite extension of $k_0$.

In 1983, J.Ohm [27] proved the Ruled Residue Conjecture.
For a residually transcendental extension \((K_0(x), v)/(K_0, v_0)\), i.e., when the residue field \(k\) of \(v\) is transcendental over the residue field \(k_0\) of \(v_0\), it is easily proved that \([G : G_0]\) is finite (see [26, Page 203]), where \(G_0 \leq G\) are the value groups of \(v_0, v\) respectively. Corresponding to such an extension, Ohm introduced three numbers \(I, R\) and \(E\) (more precisely written as \(I(v/v_0)\) etc.) by

\[I = [G : G_0],\]
\[R = [k'_0 : k_0],\]
\[E = \min\{[K_0(x) : K_0(t)] | t \in K_0(x), v(t) \neq 0, t \text{ transcendental over } k_0\},\]

where for any \(\xi\) in the valuation ring of \(v\), \(\xi^*\) denotes its \(v\)-residue, i.e., the image of \(\xi\) under the canonical homomorphism from the valuation ring of \(v\) onto the residue field of \(v\).

Ohm's proof of the Ruled Residue Conjecture yields the inequality \(E \geq IR\) (see [28, Page 586(1.2)]). In 1985, he [28] conjectured that the above inequality becomes equality if any one of the following conditions is satisfied:

(i) \(v_0\) is henselian and characteristic \(k_0 = 0\)
(ii) \(v_0\) is of rank one and characteristic \(k_0 = 0\)
(iii) \(v_0\) is discrete of rank one.

In 1990, Matignon and Ohm proved the conjecture in all the three cases (cf. [23, Remark 3.4]). The equality "\(E = IR\)" is significant because it immediately implies the existence of an element \(t \in K(x)\) such that \(v\) is the unique prolongation to
\[ K_0(x) \] of the Gaussian valuation \( v^t_0 \) of \( K_0(t) \) defined on \( K_0[t] \) by

\[
v^t_0(\sum_i c_i t^i) = \min\{v_0(c_i)\}; \quad c_i \in K_0.
\]

This led to the following Uniqueness Problem.

**Uniqueness Problem:** Let \((K,v)/(K_0,v_0)\) be a residually transcendental extension where \( K \) is a function field of transcendence degree one over \( K_0 \). Does there exist an element \( t \in K \) such that \( v \) coincides with the Gaussian valuation \( v^t_0 \) on \( K_0(t) \) and \( v \) is the unique extension of \( v^t_0 \) to \( K_0(x) \)? (Such an element \( t \) will be referred to as an element with uniqueness property for \( v/v_0 \)).

In 1988, Matignon and Ohm [22, Theorem 0.1] jointly proved that the answer to the above question is "yes" when \( K = K_0(x) \) is a simple transcendental extension of \( K_0 \) and \( v_0 \) is of rank 1. In 1990, they showed that the answer is also "yes" if \( v_0 \) is henselian and the answer is "no" in general, when \( K = K_0(x) \) (cf. [23, (3.3.1), (2.3.4)]). In 1988, Polzin [35, Proposition 4] generalized Matignon's and Ohm's result by proving the existence of elements with uniqueness property when \( K \) is a function field of a conic over \( K_0 \), \( K_0 \) is algebraically closed in \( K \) and \( v_0 \) is of rank 1. In 1990, Green, Matignon and Pop jointly proved the existence of such elements when \( K \) is a function field of transcendence degree one over \( K_0 \) and \( v_0 \) is henselian of any rank (cf. [11, Theorem 3.1]).
Continuing in this vein, one can pose similar questions for non-torsion extensions \((K,v)/(K_o,v_o)\), i.e., those extensions for which \(G/G_o\) is not a torsion group; \(G_o \subseteq G\) being the value groups of \(v_o, v\) respectively. Keeping in view that a valuation \(v\) coincides with the Gaussian valuation \(v^t_o\) on \(K_o(t)\) is equivalent to saying that the \(v\)-residue \(t^*\) of \(t\) is transcendental over the residue field \(k_o\) of \(v_o\), the following question is analogous to the uniqueness problem.

"Let \((K,v)/(K_o,v_o)\) be a non-torsion extension where \(K\) is a function field of transcendence degree one over \(K_o\). Does there exist an element \(t \in K\) such that \(v(t)\) is non-torsion modulo the value group \(G_o\) of \(v_o\) and \(v\) is the unique extension to \(K\) of the valuation obtained by restricting it to \(K_o(t)\)?"

In 1993, Khanduja [12] considered the above problem for a non-torsion simple transcendental extension. It was proved that the answer is "yes" if \(v_o\) is henselian or if \(v_o\) is of rank 1 with \(G_o\) a cofinal subset of the value group \(G\) of \(v\) in the latter case and the answer is "no" in general. In Chapter I, we have dealt with the above problem when \(K\) is a function field of a conic over \(K_o\) and showed the existence of elements with the uniqueness property if either \((K_o,v_o)\) is henselian or \(v_o\) is of rank 1, the algebraic closure of \(K_o\) in \(K\) is a purely inseparable extension of \(K_o\) and \(G_o\) is a cofinal subset of the value group \(G\) of \(v\). This result has appeared in the Bulletin of London Mathematical Society [14].
However, the problem analogous to one considered in [11] for the non-torsion case is still open. The proofs of all these results depend heavily on the Independence Theorem which was proved by Ohm[29] in 1989 and later generalised by Kuhlmann (see[17] or [18]).

The Ruled Residue Theorem gave rise to another chain of problems. The following natural question arises:

"Given a valuation \( v_0 \) of a field \( K_0 \) with residue field \( k_0 \) and a residually transcendental prolongation of \( v_0 \) to \( K_0(x) \) with residue field \( k \), how to determine \( k \) explicitly, i.e., how to find the algebraic closure \( k'_0 \) of \( k_0 \) in \( k \) and an element \( t \in K_0(x) \) with \( v(t) = 0 \) such that the \( v \)-residue \( t' \) of \( t \) generates the simple transcendental extension \( k/k'_0 \)."

The answer was given by V.Alexandru, N.Popescu and A.Zaharescu in 1988 (see[1]). In 1990, They gave a description of all residually transcendental prolongations and non-torsion prolongations of \( v_0 \) to \( K_0(x) \) by a simple formula (see[2]or[3]).

In Chapter II, using the explicit construction of these valuations and the consequent results, we have generalised the Schönemann Irreducibility Criterion[40 ] (stated below) and hence the usual Eisenstein Irreducibility Criterion [9] to polynomials with coefficients in a valued field \((K,v)\), \( v \) being a valuation of any rank.
Schönemann’s Irreducibility Criterion: Let \( \mathbb{Z} \) denote the ring of integers and \( \mathbb{Q} \) the field of rational numbers. Suppose that \( p \) is a prime integer and that the polynomial \( F(x) \) has the form

\[
F(x) = [f(x)]^s + pM(x)
\]

where \( f(x) \in \mathbb{Z}[x] \) is a monic polynomial which is irreducible modulo \( p \), \( M(x) \in \mathbb{Z}[x] \) is a polynomial relatively prime to \( f(x) \) modulo \( p \) and the degree of \( M(x) \) is less than that of \( F(x) \). Then \( F(x) \) is irreducible in \( \mathbb{Q}[x] \).

Some results proved in Chapter II are to appear in “Mathematika”[15].

The well-known Hensel’s Lemma, which is the foundation stone of the theory of \( p \)-adic numbers asserts that the factoriability of polynomials over a complete, rank-1 valued field \( (K,v) \) is related to the factoriability of polynomials over the residue field of \( v \). In terms of Gaussian valuation \( v^X \) extending \( v \) to a simple transcendental extension \( K(x) \) of \( K \), the classical Hensel’s Lemma [10, Page 120(16.7)] can be stated as follows:

Hensel’s Lemma : Let \( (K,v) \) be a complete, rank-1 valued field with valuation ring \( R_v \) and residue field \( k_v \). If polynomials \( F(x), \ G_0(x), \ H_0(x) \) in \( R_v[x] \) are such that

(i) \( v^X(F(x)-G_0(x)H_0(x)) > 0 \),
(ii) the leading coefficient of \( G_0(x) \) has \( v \)-valuation zero, (iii) there are polynomials
A(x), B(x) belonging to the valuation ring of \( \nu^X \) satisfying 
\[ \nu^X(A(x)G_0(x) + B(x)H_0(x) - 1) > 0, \]
then there exist \( G(x), H(x) \) in \( K[x] \) such that (a) \( F(x) = G(x)H(x) \), (b) degree \( G(x) = degree G_0(x) \), (c) \( \nu^X(G(x) - G_0(x)) > 0 \), \( \nu^X(H(x) - H_0(x)) > 0 \).

In Chapter III, we have formulated and proved an analogue of Hensel's Lemma when \( \nu^X \) is replaced by any residually transcendental prolongation of \( \nu \) to \( K(x) \). A similar generalisation has been formulated and proved by Elena-Liliana Popescu [36] in 1993, when \((K,\nu)\) is a complete, discrete, rank-1 valued field. However, there is an error in her proof. We have given a counter example to show that her result is false.

Let \( \nu \) be a real henselian valuation of a field \( K \) with unique prolongation \( \overline{\nu} \) to a fixed algebraic closure \( \overline{K} \) of \( K \). For any \( \alpha \in \overline{K} \setminus K \), \( \alpha \) separable over \( K \), let \( \delta(\alpha), \omega(\alpha) \) (more precisely written as \( \delta_\nu(\alpha), \omega_\nu(\alpha) \)) denote the real numbers defined by
\[
\delta(\alpha) = \sup\{\overline{\nu}(\alpha - \beta) | \beta \in \overline{K}, [K(\beta):K] < [K(\alpha):K]\},
\]
\[
\omega(\alpha) = \max\{\overline{\nu}(\alpha - \alpha') | \alpha' \text{ runs over all } K\text{-conjugates of } \alpha, \alpha \neq \alpha'\}.
\]

Krasner's Principle[10, Page 122(16.8)] asserts that if \( \beta \in \overline{K} \) is such that \( \overline{\nu}(\alpha - \beta) > \omega(\alpha) \), then \( K(\alpha) \subseteq K(\beta) \). In Chapter IV, using residually transcendental extensions of valuations to simple transcendental extensions, we have generalised a Fundamental Principle proved by N.Popescu and A.Zaharescu [38].
for discrete, complete, rank-1 valued fields to henselian, rank-1 valued fields. More precisely, we have proved the following principle which is in consensus with the Krasner's Principle:

"If $\beta \in \bar{K}$ is such that $v(\alpha - \beta) > \delta(\alpha)$, then (i) $G(K(\alpha)) \subseteq G(K(\beta))$, $Res(K(\alpha)) \subseteq Res(K(\beta))$, where $G(L)$, $Res(L)$ stand for the value group and the residue field of the valuation obtained by restricting $\bar{v}$ to an algebraic extension $L$ of $K$; (ii) $[K(\alpha):K]$ divides $[K(\beta):K]$.

In Chapter IV, we also give some necessary and sufficient conditions for $\delta(\alpha)$ to be $\bar{v}(\alpha)$ or $\omega(\alpha)$ and establish an inequality relating $\delta(\alpha)$ and $\Delta(\alpha)$ where

$\Delta(\alpha) = \min{\bar{v}(\alpha-\alpha')|\alpha' \text{ runs over all } K\text{-conjugates of } \alpha, \alpha \neq \alpha'}$. 

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