CHAPTER I

SJÖGREN'S THEOREM ON DIMENSION SUBGROUPS AND THE
SCHUR MULTIPLICATOR

1. INTRODUCTION

Let $G$ be a group, $ZG$ its integral group ring and $\Delta(G)$ the augmentation ideal of $ZG$. Then $\gamma_n(G)$, the $n$-th term in the lower central series of $G$ is contained in $D_n(G) = G \cap (1 + \Delta^n(G))$, the $n$-th dimension subgroup of $G$ for all $n \geq 1$. A group $G$ is said to have dimension property if $D_n(G) = \gamma_n(G)$ for all $n \geq 1$. Free groups [16], divisible groups [27], free metabelian groups, free centre-by-metabelian groups [7], groups with lower central factors torsion-free [14], provide us with examples of groups having dimension property. It was at one time conjectured that every group $G$ has dimension property. In this direction it was proved that if $G$ is a group such that every prime power subquotient of $G$ has the dimension property, then so has $G$ itself [20]. This reduced the study to that of prime power groups. However, Rips [26] proved the dimension conjecture to be false by exhibiting a finite 2-group $G$ of order $2^{38}$ such that $D_4(G) / \gamma_4(G)$ is of order 2. Later Tahara [30] constructed other groups for which the fourth dimension subgroup differs...
from the corresponding term in the lower central series. These counter examples suggest two directions in the study of dimension subgroups. One is to look for groups satisfying $D_n(G) = \gamma_n(G)$ for all $n \geq 1$. The other is to find, for all groups $G$, a bound for the exponents of the quotients $D_n(G)/\gamma_n(G)$, $n \geq 1$, which are known to be periodic ([8], [14]). A remarkable result in the latter direction is due to Sjögren [29] who proved in 1979 that for all groups $G$, the quotient $D_n(G)/\gamma_n(G)$, $n \geq 3$ has exponent dividing an expression $\frac{(n-2)!}{(n-2)(n-2)}$.

where $b(k)$ = least common multiple of $1, 2, \ldots, k$.

Hartley [10], Cliff-Hartley [1], and more recently Gupta [4] have given alternative expositions of Sjögren's proof of the above theorem, which are considerably simpler than the original proof and which avoid the use of spectral sequences, trees and the law of Chen, Fox and Lyndon. We devote section 2 of this chapter to a brief outline of the exposition of Sjögren's theorem given by Gupta [4].

Let $G$ be a group and $M$ a divisible abelian group regarded as a trivial $G$-module. Let $H^2(G, M)$ denote the second cohomology group of $G$ with coefficients in $M$. A map $f : G \times G \to M$ is called (normalized) polynomial $2$-cocycle of degree $\leq n$ if
i) \( f(x,y) = 0 \) whenever \( x=1 \) or \( y=1 \).

ii) \( f(y,z) - f(xy,z) + f(x,yz) - f(x,y) = 0 \)
for all \( x,y,z \in G \).

iii) for every \( x \in G \), the linear extension of the map
\( f_x : G \rightarrow M \) defined by \( f_x(y) = f(x,y) \) to \( ZG \) vanishes on \( \triangle^{n+1}(G) \).

Let \( P_n H^2(G,M) \) be the subset of \( H^2(G,M) \) consisting of those elements of \( H^2(G,M) \) which have polynomial 2-cocycles of degree \( \leq n \) as representatives. Then \( P_n H^2(G,M) \) is a subgroup of \( H^2(G,M) \) and we have a filtration

\[
0 = P_0 H^2(G,M) \leq P_1 H^2(G,M) \leq \cdots \leq P_n H^2(G,M) \leq \cdots
\]

It is evident from the work of Passi and Passi - Verma (\[20\], \[21\], \[25\]) that the study of the filtration \( \{ P_n H^2(G,T) \} \) of the second cohomology group \( H^2(G,T) \) where \( T \) denotes the additive group of rationals mod one, is closely related to the study of the integral dimension subgroups. For example, it has been shown by Passi and Verma [25] that 'The integral dimension series of every nilpotent group terminates with identity in a finite number of steps if, and only if, for every nilpotent group \( G \), \( P_n H^2(G,T) = H^2(G,T) \) for some \( n \geq 1 \); and as a consequence of Sjögren's result that 'for every nilpotent group \( G \) of class \( \leq n \), there exist
constants $d_1, d_2, \ldots, d_n, \ldots$, such that

$$d_n H^\varphi(G, T) \leq \mathbb{P}_n H^\varphi(G, T).$$

In the third section of this chapter we review some results about the filtration $\{\mathbb{P}_n H^\varphi(G, T)\}$. 
2. Sjögren's Theorem

We start with a theorem of Gupta [4] which is a generalized version of a result due to Sjögren [29], and Hartley [10].

2.1 Theorem [4]

Let $H = H_1 \triangleright H_2 \triangleright \cdots$, and $K = K_1 \triangleright K_2 \triangleright \cdots$, be series of normal subgroups of any group $F$ and let $D_k : 1 \leq k < \ell$ be a family of subgroups of $F$ such that

a) $D_k,k+1 = H_k \cdot K_{k+1}$;

b) $H_k \leq D_k,\ell$;

c) $D_{k+1} \leq D_k,\ell$ for all $k < \ell$, and

d) for each $2 \leq k+m \leq n+1$, $k,m,n \geq 1$, there exist positive integers $a(k)$, depending only on $k$ and $n$ such that

$$ (K_{k+m} \cap D_{k,k+m+1})^{a(k)} \leq D_{k+1,k+m+1} \cdot H_k $$

Then

$$ (D_1,n+2)^{a(1,n+1)} \leq H_1 \cdot K_{n+2}, \text{ where} $$

$$ a(1,n+1) = a(1)^1 \ldots a(n)^n. $$

In fact, it has been proved by induction on $m$, $1 \leq m \leq n$, that
\begin{equation}
\left( d_{k,k+m+1} \right) \leq H_k \cdot K_{k+m+1} \quad \text{where}
\end{equation}
\begin{equation}
a(k,m+1) = a(k)^{\left( \begin{array}{c} 1 \\ m \end{array} \right)} \cdots a(k+m-1)^{\left( \begin{array}{c} m \\ m \end{array} \right)}
\end{equation}

The theorem then follows as a special case on taking \( k = 1, m = n \).

Let \( G \) be a group given by a positive countable presentation of the form
\begin{equation}
G = \langle x_1, x_2, \ldots; r_1, r_2, \ldots \rangle,
\end{equation}
where each \( r_i \) is a product of positive powers of \( x_j \)'s.
Then \( G \cong F/R \) where \( F = \langle x_1, x_2, \ldots \rangle \) is free of finite or countable rank and \( R = \langle r_1, r_2, \ldots \rangle \) is the normal closure of the relators \( r_1, r_2, \ldots \) in \( F \).

Define \( r(1) = 3^F \triangle (R) \), and, more generally
\begin{equation}
r(k) = \sum_{i+j=k-1} \triangle^{i(F)} \triangle (R) \triangle^j (F), \quad k \geq 1, \quad i, j \geq 0.
\end{equation}

Also set \( R(1) = R \), and, more generally,
\begin{equation}
R(k) = \left[ R, F, F, \ldots, F \right] \quad \text{for } k \geq 1.
\end{equation}

\( k-1 \) times

Let \( D(\ell, r(k)) = F \cap (1 + r(k) + \triangle^\ell (F)) \),
Then to prove that $D^c_n (G) \leq \sqrt[n+1]{}(G)$ for some $c > 0$, it is sufficient to prove that

$$(D(n+1, \mathbb{R}(1))^c \leq \sqrt[n+1]{}(F).$$

For $w \in D_{n+1} (\mathbb{P}/\mathbb{R})$, $w \in \mathbb{F}$, implies that

$$w-1 \in \mathbb{Z}_w (R-1) + \Delta^{n+1} (\mathbb{F})$$

$$= \mathbb{F}(1) + \Delta^{n+1} (\mathbb{F})$$

which implies further that

$$w \in D(n+1, \mathbb{F}(1)) \text{ and hence}$$

$$w^c \in \sqrt[n+1]{}(\mathbb{F}) \text{ which implies that}$$

$$(w \mathbb{R})^c \in \sqrt[n+1]{}(\mathbb{F}/\mathbb{R}).$$

The following two lemmas are the key-results in the proof of Sjögren's theorem.

2.3. **Lemma [4]**

Let $w \in \mathbb{F}_n (\mathbb{F})$, $n \geq 2$ such that $w-1 \in \mathbb{F}(k) + \triangle^{n+1} (\mathbb{F})$ for some $2 \leq k \leq n$. Then

$$w b(k) - 1 \equiv f_k - 1 \mod (\mathbb{F}(k+1) + \Delta^{n+1} (\mathbb{F})) \text{ for some}$$

$f_k \in \mathbb{R}(k)$, where $b(k) = \text{least common multiple of } 1, 2, \ldots, k$. 
2.4 LEMMA [4].

\[ D(k+1, r(k)) = P \cap (1+\frac{r}{r}(k) + \Delta^{k+1}(\mathcal{F})) = R(k) \cdot \sqrt[n+1]{\mathcal{F}} \]

for all \( k \geq 2 \).

Sjögren's main theorem follows easily from Theorem 2.1, Lemma 2.3 and Lemma 2.4.

2.5 THEOREM (Sjögren [29]).

For all groups \( G \), \( D_{n+1}(G) / \sqrt[n+1]{G} \), \( n \geq 2 \) has exponent dividing \( c(n-1) = b(1)^{(n-1)} \cdots b(n-1)^{(n-1)} \) where

\[ b(k) = \text{least common multiple of } 1, 2, \ldots, k. \]

Proof:

Let \( G \) be any group given by positive countable presentation of the type (2.2). Let \( n \geq 2 \) be fixed. Then, as remarked earlier, to prove the theorem it is sufficient to prove that

\[ (D(n+1), P(1)) \leq R \sqrt[n+1]{\mathcal{F}} \]

In Theorem 2.1, take \( H_k = R(k) \), \( k \geq 1 \):

\[ k_1 = \sqrt[G]{\mathcal{F}} \quad, \quad l \geq 1 \quad \text{and} \quad D_{k,l} = D(\mathcal{F}, P(k)) = P \cap (1+\frac{P}{P}(k) + \Delta(\mathcal{F})), 1 \leq k < l. \]

Then conditions (b) and (c) of Theorem 2.1 are easy to prove. Condition (a) follows from Lemma 2.4 whereas Lemma 2.3 implies condition (a) of Theorem 2.1 with \( a(k) = b(k) \).
It follows therefore that the exponent of
\[ D_{1,n+1}/H_{H_{1}K_{n+1}} \text{ divides } c(n-1) = b(1)^{(n-1)} \cdots b(n-1)^{(n-1)}. \]

Hence \( D(n+1, \varphi(1)) / R \) has exponent dividing \( c(n-1) \) as desired.

Sjögren's theorem implies that, for any group \( G, D_{3}(G) = \gamma_{3}(G) \) and \( D_{4}(G) / \gamma_{4}(G) \) has exponent dividing 2, which is best possible in view of the counter examples given by Rips [26] and Tahara [30]. However, the exponent of \( D_{5}(G) / \gamma_{5}(G) \) as given by Sjögren's theorem is not best possible as Tahara [32] has shown that the exponent of \( D_{5}(G) / \gamma_{5}(G) \), for any group \( G \), divides 6, whereas the constant given by Sjögren's theorem in this case is equal to \( c(3) = 48 \).

It is still an open problem to decide whether the exponent 6 for \( D_{5}(G) / \gamma_{5}(G) \), for any group \( G \) is best possible or not. We are yet not aware of any group \( G \) such that the quotient \( D_{n}(G) / \gamma_{n}(G) \) contains a non-trivial element of odd order for some \( n \). Tahara has conjectured [32] that for any group \( G \), \( D_{n}(G) / \gamma_{n}(G) \) has exponent dividing \((n-2)!\) for all \( n \geq 2 \).

The exponent of \( D_{n}(G) / \gamma_{n}(G) \) as given by Sjögren is coprime to \( p \) if \( n \leq p+1 \) for any prime \( n \). Therefore, we have the following result, as a corollary to Sjögren's theorem, which improves an earlier result of Moran [18].
2.6 **COROLLARY [29]**

If $G$ is a $p$-group, then

$$D_n(G) = \gamma_n(G) \text{ for all } n \leq p+1$$

We will apply corollary 2.6 to identify the normal subgroups $G \cap (1 + \Delta^n_q(G) + \Delta^q(G) \Delta^q(\gamma(G)))$ in Chapter V.

In the case of a metabelian $p$-group $G$, Gupta - Tahara [6] have shown that for $p$ odd, $D_n(G) = \gamma_n(G)$ for $n \leq p+2$, and recently, as an extension of this result, Gupta [5] has proved that if $G$ is a finite metabelian $p$-group, then $D_n(G) = \gamma_n(G)$ for all $n \leq 2p-1$. For $p=2$, this result is best possible in view of counter examples of Rips and Tahara. For $p > 2$ it is an open problem to decide whether $D_{2p}(G) = \gamma_{2p}(G)$ for a metabelian $p$-group $G$ or not.
3. THE SCHUR MULTIPLICATOR

In this section we review a few results of Passi and Passi-Vermani about the filtration \( \{ P_n H^2(G,T) \}_{n \geq 0} \) of the Schur multiplicator \( H^2(G,T) \), where \( T \) denotes the additive group of rationals mod integers. Our aim is to bring out the importance of the study of the filtration \( \{ P_n H^2(G,T) \}_{n \geq 0} \) in the study of dimension subgroups.

Passi [20] proved that for all Abelian groups \( G \), \( P_1 H^2(G,T) = H^2(G,T) \) and that \( P_2 H^2(G,T) = H^2(G,T) \) for all finite p-groups of class 2, \( p \neq 2 \); and used these results to establish the following:

3.1 THEOREM ([20], [28])

For every group \( \Pi \),
\[
\chi_3(\Pi) = \Pi \cap (1 + \Delta^2(\Pi)) \Delta (\mathcal{Z}(\Pi)) ,
\]
where \( \mathcal{Z}(\Pi) \) denotes the centre of \( \Pi \).

3.2 THEOREM [21]

If \( \Pi \) is a p-group, \( p \neq 2 \), then
\[
\chi_4(\Pi) = \Pi \cap (1 + \Delta^4(\Pi)) \Delta (\chi_3(\Pi))
\]
That \( D_3(G) = \chi_3(G) \) for any group \( G \) and that \( D_4(G) = \chi_4(G) \) for a finite p-group \( G \), \( p \neq 2 \), clearly follow from Theorem 3.1 and Theorem 3.2 respectively, as corollaries.
The following result brings out the relationship between the filtration \( \{ P_n H^2(G, M) \} \) and subgroups determined by certain two sided ideals in group rings.

3.3 **PROPOSITION [25].**

Let \( G \) be a group, \( A \) a divisible abelian group regarded as a trivial \( G \)-module, \( \zeta \in H^2(G, A) \) and \( 1 \to A \to \Pi \to G \to 1 \) a central extension corresponding to \( \zeta \). Then \( \zeta \in P_n H^2(G, A) \) if, and only if,

\[
A \cap (1 + \Delta^{n+2}(\Pi) + \Delta(\Pi) \Delta(A)) = (1).
\]

Passi-Vermahi [25] have established the equivalence of the following statements:

3.4 For every nilpotent group \( G \), there exists an integer \( n \geq 1 \) such that \( D_n(G) = (1) \).

3.5 For every nilpotent group \( G \), there exists an integer \( n \geq 1 \) such that \( P_n H^2(G, T) = H^2(G, T) \).

It is yet not known whether, for every nilpotent group \( G \), there always exists an integer \( n \) such that \( P_n H^2(G, T) = H^2(G, T) \). This problem is the principal motivation for our work. If \( G \) is a finite \( p \)-group, then it is known that such an integer exists ([20], Theorem 6.1). The existence of such an integer has now been established by Passi-Vermahi [25] for torsion-free nilpotent groups and
finitely generated nilpotent groups. However, even in the case of a finite $p$-group $G$, the problem remains whether an integer depending only on the nilpotency class of $G$ exists.

Finally we state another result of Passi-Vermani [25] which shows the equivalence of Sjögren's theorem to an analogous property of the filtration $\{P_n^\alpha(G, T)\}$.

3.6 Theorem [25]

The following statements are equivalent:

(i) There exist constants $c_1, c_2, \ldots, c_n, \ldots$ such that for every group $G$, $D_n^\alpha(G) \leq \gamma_n(G)$ for all $n > 1$.

(ii) There exist constants $d_1, d_2, \ldots, d_n, \ldots$ such that, for every nilpotent group $G$ of class $\leq n$, $d_n H^\alpha(G, T) \leq P_n H^\alpha(G, T)$.

In fact, it is shown that if (i) holds, then (ii) holds with $d_n = c_{n+2}$. Sjögren's theorem states that (i) holds with $c_{n+2} = c(n) = b(1)^{\binom{n}{1}} \cdots b(n)^{\binom{n}{n}}$

$b(k) = \text{least common multiple of } 1, 2, \ldots, k$.

Therefore by the above theorem, for every nilpotent group $G$ of class $\leq n$, we have

$c(n) H^\alpha(G, T) \leq P_n H^\alpha(G, T)$.

It will be of interest to establish (ii) independently and then deduce Sjögren's theorem as a consequence.