Chapter 6

Some Results of Nevanlinna Theory in Complex Banach Space $E$
6.1 E-valued Borel Exceptional Values of Meromorphic Functions

6.2 Introduction, Definitions and results

In 1925, Nevanlinna’s first and second fundamental theorems were established by introducing characteristic function. Then H.J.W. Ziegler [35] extended the classical Nevanlinna’s theory of meromorphic functions to vector-valued meromorphic functions in finite dimensional spaces. In recent years, C.G. Hu and C.C. Yang made contribution to value distribution theory with range in an infinite dimensional Hilbert space. In 1998, C. Liu and C.G. Hu [31] extended the Nevanlinna’s first fundamental theorem to a vector-valued case with range in Hilbert space. In 1997, the Nevanlinna’s first and second fundamental theorem are extended to Banach spaces by C.G. Hu [27].

Basic Notions of Nevanlinna Theory in Banach Spaces:

Let $E$ be a complex Banach space with Schauder basis $\{e_j\}_{j=1}^{\infty}$. Let $f$ be an $E$-valued meromorphic function in $|z| \leq r$. For any $a \in E \cup \{\infty\}$, $n(r, a, f) = n(r, a)$ denotes the number of $a$-points of $f$ in $|z| \leq r$, counted with multiplicities and $n(r, \infty, f) = n(r, f)$ denote the number of poles of $f$ in $|z| \leq r$. Then we have the counting function of finite or infinite $a$-points as

$$N(r, a) \equiv N(r, a, f) = n(0, a) \log r + \int_0^r \frac{n(t, a) - n(0, a)}{t} dt$$

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\[ N(r, f) \equiv N(r, \infty, f) = n(0, f) \log r + \int_0^r \frac{n(t, f) - n(0, f)}{t} \, dt \]

and

\[ m(r, f) \equiv m(r, \infty, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \| f(re^{i\phi}) \| \, d\phi, \]

\[ m(r, a) \equiv m(r, a, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{\| f(re^{i\phi}) - a \|} \, d\phi, \quad (a \neq \infty), \]

\[ T(r, f) = m(r, f) + N(r, f), \]

where \( \log^+ x = \max\{ \log x, 0 \} \). The volume function associated with \( E \)-valued meromorphic function \( f \) is given by

\[ V(r, a, f) = \frac{1}{2\pi} \int_{C_r} \log \left| \frac{r}{\xi} \right| \Delta \log \| f(\xi) - a \| \, d\sigma \wedge d\tau. \]

In 2006, C.G.Hu and Qijian Hu [28] proved the following theorems.

Let \( D = C_r = \{ z : |z| < r \} \) and \( \mathbb{C} \) is the complex plane.

**Theorem 6.2.1** [28] *(the \( E \)-valued Nevanlinna’s first fundamental theorem)*

Let \( f(z) \) be an \( E \)-valued meromorphic mapping in \( C_R \). Then for \( 0 < r < R \), \( a \in E \), \( f(z) \neq a \),

\[ T(r, f) = V(r, a) + N(r, a) + m(r, a) + \log \| c_q(a) \| + \epsilon(r, a). \]

Here \( \epsilon(r, a) \) is a function such that

\[ |\epsilon(r, a)| \leq \log^+ \| a \| + \log 2, \epsilon(r, 0) \equiv 0, \]

and \( c_q(a) \in E \) is the co-efficient of the first term in the Laurent series at the point \( a \).

Since \( E \) is a complex Banach space with Schauder basis \( \{ e_j \}_{j=1}^{\infty} \), an \( E \)-valued meromorphic function \( f(z) \) in a domain \( D \subset \mathbb{C} \) can be written as

\[ f(z) = \sum_{j=1}^{\infty} f_j(z) e_j = (f_1(z), \ldots, f_j(z), \ldots) \]
where \( f_j(z) \) is a complex-valued meromorphic function in \( D \). Let \( E_n \) be an \( n \)-dimensional projective space of \( E \) with a basis \( \{e_j\}_{j=1}^n \). The projection operator \( P_n : E \to E_n \) is a realization of \( E_n \) associated to the basis.

**Definition 6.2.1** [28]. An \( E \)-valued holomorphic (or meromorphic) function \( f(z) \) in \( D \) is said to be of compact projection if and only if \( \|P_n f(z) - f(z)\| < \varepsilon \), as sufficiently large \( n \) in any fixed compact subset \( D_1 \subset D \).

**Theorem 6.2.2** [28] (the \( E \)-valued Nevanlinna’s second fundamental theorem)

Let \( f(z) \) be a non-constant \( E \)-valued meromorphic mapping of compact projection in \( C_R \) and \( a^{[k]} \in E \cup \{\infty\} \) \((k = 1, 2, \ldots, q)\) be \( q \geq 3 \) distinct finite or infinite points. Then

\[
\sum_{k=1}^{q} m(r, a^{[k]}) + G(r, f) \leq T(r, f) - N_1(r) + S(r)
\]

where \( N_1(r) = N(r, 0, f') + 2N(r, f) - N(r, f') \) and

\[
G(r, f) = \int_0^r \frac{dt}{2\pi} \int_{C_t} \Delta \log \|f'(\xi)\| d\sigma \wedge d\tau
\]

If \( R = +\infty \), then \( S(r) \) satisfies

\[
S(r) = O\{\log T(r, f)\} + O(\log r)
\]

as \( r \to +\infty \) without exception if \( f(z) \) has finite order and otherwise as \( r \to +\infty \) outside a set \( J \) of exceptional intervals of finite measure

\[
\int_J dr < +\infty.
\]

If \( 0 < R < +\infty \), then

\[
S(r) = O\{\log^+ T(r, f)\} + O\left\{\log \frac{1}{R - r}\right\}
\]
holds as \( r \to R \) without exception if \( f \) has finite order

\[
\rho = \lim_{r \to R} \frac{\log T(r, f)}{\log(1/R - r)},
\]

and otherwise as \( r \to R \) outside of a set \( J \) exceptional intervals such that

\[
\int_J \frac{1}{R - r} < +\infty.
\]

In all cases, the exceptional set \( J \) is independent of the choice of the finite points \( a^{[k]} \in E \) and of their number.

Other interesting forms of the \( E \)-valued Nevanlinna’s second fundamental theorem:

\[
(q - 1)T(r, f) + G(r, f) + N_1(r) \leq \sum_{k=1}^{q+1} \left[ V(r, a^{[k]}) + N(r, a^{[k]}) \right] + S(r)
\]

OR

\[
(q - 2)T(r, f) + G(r, f) \leq \sum_{k=1}^{q} \left[ V(r, a^{[k]}) + \overline{N}(r, a^{[k]}) \right] + S(r)
\]

with

\[
\overline{N}(r, a) = \pi(0, a) \log r + \int_0^r \frac{\pi(t, a) - \overline{\pi}(t, a)}{t} dt,
\]

where \( \pi(t, a) \) denotes the number of solutions of \( f(z) - a \) in \( |z| \leq t \), each solution counted only once.

Next we define the Borel exceptional values for the \( E \)-valued meromorphic functions.

Assume that \( E \) be a complex Banach space and \( \mathbb{C} \) is the complex plane.

Let \( D = C_r = \{ z : |z| < r \} \).

**Definition 6.2.2** Let \( f : D \to E \) be a meromorphic function and \( a \in E \cup \{ \infty \} \), if \( k \) is a positive integer, we denote by \( \overline{\pi}_k(r, a, f) \), the number of distinct zeros of order \( \leq k \) of \( f - a \) in \( |z| \leq r \) (each zero is counted only once irrespective of its multiplicity). \( \overline{N}_k(r, a, f) \)
is defined in terms of $\pi_k(r, a, f)$ as,

$$N_k(r, a, f) = \int_0^r \frac{\pi_k(r, a, f) - \pi(a, a, f)}{t} dt + \pi_k(0, a, f) log r.$$

**Definition 6.2.3** For an $E$-valued meromorphic function $f$ and $a \in E \cup \{\infty\}$, we define

$$\overline{\rho}_k(a, f) = \lim_{r \to \infty} \frac{\log^+ [V(r, a, f) + \overline{N}_k(r, a, f)]}{\log r}$$

$$\overline{\rho}(a, f) = \lim_{r \to \infty} \frac{\log^+ [V(r, a, f) + N(r, a, f)]}{\log r}$$

$$\rho(a, f) = \lim_{r \to \infty} \frac{\log^+ [V(r, a, f) + N(r, a, f)]}{\log r}$$

where

$$V(r, a, f) = \frac{1}{2\pi} \int_{C_r} \log \left| \frac{r}{\xi} \right| \Delta \log \|f(\xi) - a\| \, d\sigma \wedge d\tau.$$

**Definition 6.2.4** If $f$ is an $E$-valued meromorphic function of order $\rho$ $(0 \leq \rho \leq \infty)$, $a \in E \cup \{\infty\}$ and $k$ is a positive integer, we say that $a$ is an

i) $E$-valued evB (exceptional value in the sense of Borel) for $f$ for distinct zeros of order $\leq k$ if $\overline{\rho}_k(a, f) < \rho$.

ii) $E$-valued evB for $f$ for distinct zeros if $\overline{\rho}(a, f) < \rho$.

iii) $E$-valued evB for $f$ (for the whole aggregate of zeros) if $\rho(a, f) < \rho$.

Thus we call $a$ is an $E$-valued evB for $f$ for simple zeros if $\overline{\rho}_1(a, f) < \rho$ and an $E$-valued evB for $f$ for distinct simple and double zeros if $\overline{\rho}_2(a, f) < \rho$.

**Definition 6.2.5** Let $f$ be a non-constant $E$-valued meromorphic function, a point $a \in E \cup \{\infty\}$ is called $E$-valued Picard exceptional value (evP) if $V(r, a) + N(r, a) = O\{\log r\}$.

We call $a$ is an $E$-valued evP for $f$ for zeros of order $\leq k$ if

$$V(r, a) + \overline{N}_k(r, a, f) = O\{\log r\}.$$
**Definition 6.2.6** The order $\rho$ of an $E$-valued meromorphic function $f$ is defined by
\[
\rho = \lim_{r \to \infty} \frac{\log T(r,f)}{\log r}
\]
and the lower order $\lambda$ of $f$ is defined by
\[
\lambda = \lim_{r \to \infty} \frac{\log T(r,f)}{\log r}.
\]

In [29], H.S.Gopalkrishna and S.S.Bhoosnurmath obtained much stronger results than those of Valiron for meromorphic functions of all orders (finite or infinite) with the usual definitions of order and evB only and proved the following result.

**Theorem 6.2.3** [29]. Let $f$ be a meromorphic function of order order $\rho$, $0 \leq \rho \leq \infty$. If there exist distinct elements $a_1, a_2, \ldots, a_p$, $b_1, b_2, \ldots, b_q$, $c_1, c_2, \ldots, c_s$ in $\mathbb{C}$ such that $a_1, a_2, \ldots, a_p$ are evB for $f$ for distinct zeros of order $\leq k$, $b_1, b_2, \ldots, b_q$ are evB for $f$ for distinct zeros of order $\leq l$ and $c_1, c_2, \ldots, c_s$ are evB for $f$ for distinct zeros of order $\leq m$ where $k, l$ and $m$ are positive integers, then
\[
\frac{pk}{k+1} + \frac{ql}{l+1} + \frac{sm}{m+1} \leq 2
\]

It is natural to consider whether there exists a similar result, if meromorphic function $f$ is replaced by $E$-valued meromorphic function $f$. In this section, we extend the above theorem to $E$-valued meromorphic function.

### 6.3 Lemmas

The $E$-valued Picard’s theorem and the $E$-valued Picard-Borel’s theorem are the important applications of $E$-valued Nevanlinna’s second fundamental theorem. These theorems are given by C.G.Hu [27]. We now give another significant application of the $E$-valued Nevanlinna’s second fundamental theorem.
Lemma 6.3.1 Let $f(z)$ be a non-constant $E$-valued meromorphic function with compact projection is of finite order in $\mathbb{C}$. If, for a positive number $\lambda$, the integral

$$\int_{r_0}^{+\infty} \frac{V(r,a) + N(r,a)}{r^{\lambda+1}} dr, \quad (r_0 > 0) \quad (6.3.1)$$

converges for three different $a \in E \cup \{\infty\}$, then the integral

$$\int_{r_0}^{+\infty} \frac{T(r,f) + G(r,f)}{r^{\lambda+1}} dr$$

converges, so that $f(z)$ is at most of convergence class of of order $\lambda$, and (6.3.1) converges for every $a \in E \cup \{\infty\}$.

Proof. By the $E$-valued Nevanlinna second fundamental theorem with $q = 3$,

$$T(r,f) + G(r,f) \leq \sum_{i=1}^{3} \left[ V(r,a^{[i]}) + N(r,a^{[i]}) \right] - N_1(r) + S(r)$$

Since $N_1(r)$ is a non-negative number and can be neglected and $f$ is of finite order, so $S(r) = O(\log r)$. Therefore,

$$T(r,f) + G(r,f) \leq \sum_{i=1}^{3} \left[ V(r,a^{[i]}) + N(r,a^{[i]}) \right] + O(\log r)$$

$$\int_{r_0}^{r} \frac{T(r,f) + G(r,f)}{r^{\lambda+1}} dr \leq \sum_{i=1}^{3} \int_{r_0}^{r} \frac{V(r,a^{[i]}) + N(r,a^{[i]})}{r^{\lambda+1}} dr + \int_{r_0}^{r} \frac{O(\log r)}{r^{\lambda+1}} dr, \quad (r > r_0)$$

letting $r \to +\infty$, we get

$$\int_{r_0}^{+\infty} \frac{T(r,f) + G(r,f)}{r^{\lambda+1}} dr \leq \sum_{i=1}^{3} \int_{r_0}^{+\infty} \frac{V(r,a^{[i]}) + N(r,a^{[i]})}{r^{\lambda+1}} dr + \int_{r_0}^{+\infty} \frac{O(\log r)}{r^{\lambda+1}} dr$$

using (6.3.1) and observing $\int_{r_0}^{+\infty} \frac{O(\log r)}{r^{\lambda+1}} dr < +\infty$ we get

$$\int_{r_0}^{+\infty} \frac{T(r,f) + G(r,f)}{r^{\lambda+1}} dr < \infty$$

Hence the integral $\int_{r_0}^{+\infty} \frac{T(r,f) + G(r,f)}{r^{\lambda+1}} dr$ converges.

This completes the proof of the Lemma.
6.4 Statement and Proof of Main Result

**Theorem 6.4.1** Let \( f(z) \) be an \( E \)-valued meromorphic function of order \( \rho \), \( 0 \leq \rho \leq \infty \) in \( \mathbb{C} \). If there exist distinct elements \( a^{[1]}, a^{[2]}, \ldots, a^{[p]}; b^{[1]}, b^{[2]}, \ldots, b^{[q]}; c^{[1]}, c^{[2]}, \ldots, c^{[s]} \) in \( E \cup \{\infty\} \) such that \( a^{[1]}, a^{[2]}, \ldots, a^{[p]} \) are \( E \)-valued evB for \( f \) for distinct zeros of order \( \leq k \), \( b^{[1]}, b^{[2]}, \ldots, b^{[q]} \) are \( E \)-valued evB for \( f \) for distinct zeros of order \( \leq l \) and \( c^{[1]}, c^{[2]}, \ldots, c^{[s]} \) are \( E \)-valued evB for \( f \) for distinct zeros of order \( \leq m \) where \( k, l \) and \( m \) are positive integers, then

\[
\frac{pk}{k+1} + \frac{ql}{l+1} + \frac{sm}{m+1} \leq 2 \tag{6.4.1}
\]

**Proof.** By the \( E \)-valued Nevanlinna’s second fundamental theorem, we have

\[(q-2)T(r,f) + G(r,f) \leq \sum_{i=1}^{p} [V(r,a^{[i]}) + \mathcal{N}(r,a^{[i]})] + S(r,f) \tag{6.4.2}\]

Given that \( a^{[1]}, a^{[2]}, \ldots, a^{[p]}; b^{[1]}, b^{[2]}, \ldots, b^{[q]}; c^{[1]}, c^{[2]}, \ldots, c^{[s]} \) are distinct elements in \( E \cup \{\infty\} \). So (6.4.2) can be written as,

\[(p + q + s - 2)T(r,f) + G(r,f) \leq \sum_{i=1}^{p} [V(r,a^{[i]}) + \mathcal{N}(r,a^{[i]})] + \sum_{j=1}^{q} [V(r,b^{[j]}) + \mathcal{N}(r,b^{[j]})] + \sum_{t=1}^{s} [V(r,c^{[t]}) + \mathcal{N}(r,c^{[t]})] + S(r,f) \tag{6.4.3}\]

By hypothesis, we have

\[\overline{\rho}_k(a^{[i]},f) < \rho \quad \text{for} \quad i = 1, 2, \ldots, p,\]

\[\overline{\rho}_l(b^{[j]},f) < \rho \quad \text{for} \quad j = 1, 2, \ldots, q,\]

\[\overline{\rho}_m(c^{[t]},f) < \rho \quad \text{for} \quad t = 1, 2, \ldots, s.\]

We choose the positive number \( \lambda < \rho \) such that

\[\overline{\rho}_k(a^{[i]},f) < \lambda \quad \text{for} \quad i = 1, 2, \ldots, p,\]

\[\overline{\rho}_l(b^{[j]},f) < \lambda \quad \text{for} \quad j = 1, 2, \ldots, q,\]

\[\overline{\rho}_m(c^{[t]},f) < \lambda \quad \text{for} \quad t = 1, 2, \ldots, s.\]
\( \overline{\rho}_i(b^{[j]}, f) < \lambda \) for \( j = 1, 2, \ldots, q \),
\( \overline{\rho}_m(c^{[t]}, f) < \lambda \) for \( t = 1, 2, \ldots, s \).

By the Definition 6.2.3, we have
\[
\overline{\rho}_k(a^{[i]}, f) = \lim_{r \to \infty} \frac{\log^+ [V(r, a^{[i]}) + \overline{N}_k(r, a^{[i]})]}{\log r} < \lambda ,
\]
\[
\overline{\rho}_l(b^{[j]}, f) = \lim_{r \to \infty} \frac{\log^+ [V(r, b^{[j]}) + \overline{N}_l(r, b^{[j]})]}{\log r} < \lambda ,
\]
\[
\overline{\rho}_m(c^{[t]}, f) = \lim_{r \to \infty} \frac{\log^+ [V(r, c^{[t]}) + \overline{N}_m(r, c^{[t]})]}{\log r} < \lambda
\]
for \( i = 1, 2, \ldots, p \), \( j = 1, 2, \ldots, q \) and \( t = 1, 2, \ldots, s \). Then
\[
V(r, a^{[i]}) + \overline{N}_k(r, a^{[i]}) = O(r^{\lambda}) , \quad i = 1, 2, \ldots, p
\]
\[
V(r, b^{[j]}) + \overline{N}_l(r, b^{[j]}) = O(r^{\lambda}) , \quad j = 1, 2, \ldots, q
\]
\[
V(r, c^{[t]}) + \overline{N}_m(r, c^{[t]}) = O(r^{\lambda}) , \quad t = 1, 2, \ldots, s
\]
it follows that
\[
\int_{r_0}^{+\infty} \frac{V(r, a^{[i]}) + \overline{N}_k(r, a^{[i]})}{r^{1+\lambda}} dr < \infty , \int_{r_0}^{+\infty} \frac{V(r, b^{[j]}) + \overline{N}_l(r, b^{[j]})}{r^{1+\lambda}} dr < \infty,
\]
\[
\int_{r_0}^{+\infty} \frac{V(r, c^{[t]}) + \overline{N}_m(r, c^{[t]})}{r^{1+\lambda}} dr < \infty \tag{6.4.4}
\]
for \( i = 1, 2, \ldots, p \), \( j = 1, 2, \ldots, q \) and \( t = 1, 2, \ldots, s \).

If \( a \in E \cup \{ \infty \} \) and \( d \) is a positive integer, we have
\[
\overline{N}(r, a, f) \leq \frac{1}{d+1} \{ d\overline{N}_d(r, a, f) + N(r, a, f) \} \tag{6.4.5}
\]
Using (6.4.5) in (6.4.3), we get
\[
(p+q+s-2)T(r, f) + G(r, f) \leq \sum_{i=1}^{p} V(r, a^{[i]}) + \frac{k}{k+1} \sum_{i=1}^{p} \overline{N}_k(r, a^{[i]}) + \frac{1}{k+1} \sum_{i=1}^{p} N(r, a^{[i]})
\]
\[+ \sum_{j=1}^{q} V(r, b^{[j]}) + \frac{l}{l+1} \sum_{j=1}^{q} \overline{N}_l(r, b^{[j]}) + \frac{1}{l+1} \sum_{j=1}^{q} N(r, b^{[j]})
\]
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\[ + \sum_{t=1}^{s} V(r, c^{[t]}) + \frac{m}{m+1} \sum_{t=1}^{s} \mathcal{N}_m(r, c^{[t]}) + \frac{1}{m+1} \sum_{t=1}^{s} N(r, c^{[t]}) + S(r, f) \]

By Theorem 6.2.1, we have

\[ N(r, a, f) \leq T \left( r, \frac{1}{f-a} \right) = T(r, f) - V(r, a) + O(1) \]

So,

\[ (p + q + s - 2) T(r, f) + G(r, f) \leq \sum_{i=1}^{p} V(r, a^{[i]}) + \frac{k}{k+1} \sum_{i=1}^{p} \mathcal{N}_k(r, a^{[i]}) + \]

\[ \frac{1}{k+1} \sum_{i=1}^{p} \left[ T(r, f) - V(r, a^{[i]}) + O(1) \right] + \sum_{j=1}^{q} V(r, b^{[j]}) + \frac{l}{l+1} \sum_{j=1}^{q} \mathcal{N}_l(r, b^{[j]}) + \]

\[ + \frac{1}{l+1} \sum_{j=1}^{q} \left[ T(r, f) - V(r, b^{[j]}) + O(1) \right] + \sum_{t=1}^{s} V(r, c^{[t]}) + \frac{m}{m+1} \sum_{t=1}^{s} \mathcal{N}_m(r, c^{[t]}) + S(r, f) \]

Therefore

\[ \left( p + q + s - 2 - \frac{p}{k+1} - \frac{q}{l+1} - \frac{s}{m+1} \right) T(r, f) + G(r, f) \leq \left( 1 - \frac{1}{k+1} \right) \sum_{i=1}^{p} V(r, a^{[i]}) + \]

\[ \frac{k}{k+1} \sum_{i=1}^{p} \mathcal{N}_k(r, a^{[i]}) + \left( 1 - \frac{1}{l+1} \right) \sum_{j=1}^{q} V(r, b^{[j]}) + \frac{l}{l+1} \sum_{j=1}^{q} \mathcal{N}_l(r, b^{[j]}) + \]

\[ + \left( 1 - \frac{m}{m+1} \right) \sum_{t=1}^{s} V(r, c^{[t]}) + \frac{m}{m+1} \sum_{t=1}^{s} \mathcal{N}_m(r, c^{[t]}) + S(r, f) \]

\[ \left( \frac{pk}{k+1} + \frac{ql}{l+1} + \frac{sm}{m+1} - 2 \right) T(r, f) + G(r, f) \leq \frac{k}{k+1} \sum_{i=1}^{p} V(r, a^{[i]}) + \frac{k}{k+1} \sum_{i=1}^{p} \mathcal{N}_k(r, a^{[i]}) + \]

\[ + \frac{l}{l+1} \sum_{j=1}^{q} V(r, b^{[j]}) + \frac{l}{l+1} \sum_{j=1}^{q} \mathcal{N}_l(r, b^{[j]}) + \frac{m}{m+1} \sum_{t=1}^{s} V(r, c^{[t]}) + \]

\[ \frac{m}{m+1} \sum_{t=1}^{s} \mathcal{N}_m(r, c^{[t]}) + S(r, f) \quad (6.4.6) \]

We have

\[ \int_{r_o}^{r} S(x, f) dx = o \left( \int_{r_o}^{r} T(x, f) \frac{dx}{x^{1+\lambda}} \right) \quad (r > r_o) \]
Clearly it implies that
\[ \int_{r_0}^r \frac{S(x,f)}{x^{1+\lambda}} \, dx = o \left( \int_{r_0}^r \frac{T(x,f) + G(x,f)}{x^{1+\lambda}} \, dx \right), \quad (r > r_0) \]

Hence (6.4.6) yields that
\[ \left( \frac{pk}{k+1} + \frac{ql}{l+1} + \frac{sm}{m+1} - 2 + o(1) \right) \int_{r_0}^r \frac{T(x,f) + G(x,f)}{x^{1+\lambda}} \, dx \leq \]
\[ \frac{k}{k+1} \sum_{i=1}^p \int_{r_0}^r \frac{V(x,a[i]) + \overline{N}_k(x,a[i])}{x^{1+\lambda}} \, dx + \frac{l}{l+1} \sum_{j=1}^q \int_{r_0}^r \frac{V(x,b[j]) + \overline{N}_l(x,b[j])}{x^{1+\lambda}} \, dx + \]
\[ \frac{m}{m+1} \sum_{t=1}^s \int_{r_0}^r \frac{V(x,c[t]) + \overline{N}_m(x,c[t])}{x^{1+\lambda}} \, dx \]

letting \( r \to +\infty \), we get
\[ \left( \frac{pk}{k+1} + \frac{ql}{l+1} + \frac{sm}{m+1} - 2 + o(1) \right) \int_{r_0}^{+\infty} \frac{T(x,f) + G(x,f)}{x^{1+\lambda}} \, dx \leq \]
\[ \frac{k}{k+1} \sum_{i=1}^p \int_{r_0}^{+\infty} \frac{V(x,a[i]) + \overline{N}_k(x,a[i])}{x^{1+\lambda}} \, dx + \frac{l}{l+1} \sum_{j=1}^q \int_{r_0}^{+\infty} \frac{V(x,b[j]) + \overline{N}_l(x,b[j])}{x^{1+\lambda}} \, dx + \]
\[ \frac{m}{m+1} \sum_{t=1}^s \int_{r_0}^{+\infty} \frac{V(x,c[t]) + \overline{N}_m(x,c[t])}{x^{1+\lambda}} \, dx \]

Thus, if \( \frac{pk}{k+1} + \frac{ql}{l+1} + \frac{sm}{m+1} > 2 \), then using (6.4.4), we get
\[ \int_{r_0}^{+\infty} \frac{T(r,f) + G(r,f)}{r^{1+\lambda}} \, dr < \infty \]

which would imply \( \lim_{r \to \infty} \frac{T(r,f)}{r^\lambda} = 0 \), so that the order of \( f \leq \lambda \). But, we have \( \lambda < \rho \),

which is contradiction. So we should have
\[ \frac{pk}{k+1} + \frac{ql}{l+1} + \frac{sm}{m+1} \leq 2. \]

This completes the proof of Theorem.
6.5 On a Result of Wittich and Milloux in Complex Banach Spaces

6.6 Introduction, Definitions and Results

In this section, results analogous to Wittich [40] and Milloux [41] are extended for the vector-valued case with domain in the complex plane \( \mathbb{C} \) and range in a complex Banach space \( E \).

Assume that \( E \) is a complex Banach space with a Schauder basis \( \{e_j\}_{j=1}^{\infty} \) and \( \mathbb{C} \) is the complex plane. Let \( D = C_r = \{z : |z| < r\} \).

Firstly we define \( E \)-valued Differential Polynomials.

**Definition 6.6.1** Let \( E \) be a complex Banach space with a Schauder basis \( \{e_j\}_{j=1}^{\infty} \) and \( f(z) \) be an \( E \)-valued transcendental meromorphic function. Let

\[
F_k = a(z)(f)^{l_0}(f^{(1)})^{l_1} \cdots (f^{(m)})^{l_m}
\]

and

\[
P = P(f) = \sum_{k=1}^{N} F_k
\]

where \( f^{(1)}, f^{(2)}, \ldots, f^{(m)} \) are the successive derivatives of \( f \) and \( l_0, l_1, \ldots, l_m \) are non-negative integers and \( a(z) \) is a small function of \( f(z) \) i.e. the growth of \( T(r,a(z)) \) is slower than \( T(r,f) \)

i.e.,

\[
\frac{T(r,a(z))}{T(r,f)} \to 0 \quad \text{as} \quad r \to \infty,
\]

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then $P(f)$ is called $E$-valued differential polynomial in $f(z)$.

If $l_0 + l_1 + \ldots + l_m = n$ (a finite positive integer) in every term of $P$, then $P$ is called $E$-valued homogeneous differential polynomial in $f$ of degree $n$.

In general, if $\max (l_0 + l_1 + \ldots + l_m) = n$, where the maximum is taken over the terms of $f$, then $P$ is said to be an $E$-valued non-homogeneous differential polynomial in $f$ of degree at most $n$.

6.7 Lemmas

Lemma 6.7.1 [28]. If $f(z)$ is an $E$-valued meromorphic function with the property of compact projection in $D$. Then

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f) \quad \text{where} \quad k = 1, 2, \ldots$$

Lemma 6.7.2 If $P$ is an $E$-valued differential polynomial in $f$ of degree $n \geq 1$, then

$$m\left(r, \frac{P}{f^n}\right) = S(r, f)$$

In this section, We now obtain an extended result analogous to Wittich [40] for $E$-valued homogeneous differential polynomial and an extended theorem of Milloux [41] for $E$-valued meromorphic function in complex Banach space.

6.8 Statement and Proofs of Main Results

Theorem 6.8.1 Let $f(z)$ be an transcendental $E$-valued meromorphic function in $C_+$ and $P$ be an $E$-valued differential polynomial in $f$ of degree $n$. Suppose that $P$ does not involve $f$ i.e. $P$ is an $E$-valued differential polynomial of degree $n$ in $f^{(1)}, f^{(2)}, \ldots$ with
co-efficients of the form $a(z)$. If $P$ is not a constant and $(a_1^{[\nu]}, a_2^{[\nu]}, \ldots, a_j^{[\nu]}, \ldots) = a^{[\nu]} \in E$ ($\gamma = 1, 2, \ldots p$ and $p \geq 2$) are distinct finite points then

$$n \sum_{\mu=1}^{p} m(r, a^{[\nu]}, f) + N \left( r, \frac{1}{P} \right) + V(r, 0, P) \leq T(r, P) + S(r, f)$$

where

$$V(r, 0, P) = \frac{1}{2\pi} \int_{C_r} \log |r| \Delta \log \|P(\xi)\| \, d\sigma \wedge d\tau$$

$$\xi = \sigma + i\tau, d\xi = d\sigma \wedge d\tau$$

Proof.

Let

$$F(z) = \sum_{\mu=1}^{p} \frac{1}{\|f(z) - a^{[\nu]}\|^n}$$

(6.8.1)

where $a^{[\nu]} = (a_1^{[\nu]}, \ldots, a_j^{[\nu]}, \ldots) \in E, p \geq 2$

Substitue $\delta = \min_{i \neq j} \|a^{[i]} - a^{[j]}\|$ and let $\mu \in \{1, 2, \ldots p\}$ be fixed. Suppose for some $\mu$,

$$\|f(z) - a^{[\mu]}\| < \frac{\delta}{2p} \leq \frac{\delta}{4}$$

(6.8.2)

Then for $\mu \neq \nu$, we obtain

$$\|f(z) - a^{[\nu]}\| = \|f(z) - a^{[\mu]} + a^{[\mu]} - a^{[\nu]}\|$$

$$\geq \|a^{[\mu]} - a^{[\nu]}\| - \|f(z) - a^{[\mu]}\|$$

$$> \delta - \frac{\delta}{4} = \frac{3\delta}{4}$$

So the set of points on $\partial C_r$ which is determined by (6.8.2) is either empty or any two of such sets for different $\mu$ have empty intersection.

In any case, on $|z| = r$, we have

$$\frac{1}{2\pi} \int_{0}^{2\pi} \log^+ F(r e^{i\phi}) d\phi \geq \frac{1}{2\pi} \sum_{\mu=1}^{p} \int \|f - a^{[\nu]}\| < \frac{\delta}{4p} \log^+ F(r e^{i\phi}) d\phi$$

$$m(r, F) \geq \frac{1}{2\pi} \sum_{\mu=1}^{p} \int \|f - a^{[\nu]}\| < \frac{\delta}{4p} \log^+ \frac{1}{\|f - a^{[\nu]}\|^n} d\phi$$

(6.8.3)
Since
\[
\frac{1}{2\pi} \int_{\|f-a^{[\mu]}\| < \frac{d}{2p}} \log^+ \frac{1}{\|f-a^{[\mu]}\|^n} d\phi
\]
\[
= \frac{1}{2\pi} \int_{0}^{2\pi} \log^+ \frac{1}{\|f-a^{[\mu]}\|^n} d\phi - \frac{1}{2\pi} \int_{\|f-a^{[\mu]}\| \geq \frac{d}{2p}} \log^+ \frac{1}{\|f-a^{[\mu]}\|^n} d\phi
\]
\[
\geq m \left( r, \frac{1}{(f-a^{[\mu]})^n} \right) - \log^+ \left( \frac{2p}{\delta} \right)^n \left( \frac{1}{2\pi} \int_{0}^{2\pi} d\phi \right)
\]
\[
= m \left( r, \frac{1}{(f-a^{[\mu]})^n} \right) - n \log^+ \left( \frac{2p}{\delta} \right)
\]

Therefore (6.8.3) becomes
\[
m(r, F) \geq \sum_{\mu=1}^{p} \left\{ m \left( r, \frac{1}{(f-a^{[\mu]})^n} \right) - n \log^+ \left( \frac{2p}{\delta} \right) \right\}
\]
\[
m(r, F) \geq \sum_{\mu=1}^{p} m \left( r, \frac{1}{(f-a^{[\mu]})^n} \right) - np \log^+ \left( \frac{2p}{\delta} \right)
\]
\[
m(r, F) + O(1) \geq \sum_{\mu=1}^{p} m \left( r, \frac{1}{(f-a^{[\mu]})^n} \right) = \sum_{\mu=1}^{p} m \left( r, \frac{1}{(f-a^{[\mu]})^n} \right)
\]

Thus
\[
n \sum_{\mu=1}^{p} m \left( r, \frac{1}{(f-a^{[\mu]})^n} \right) = n \sum_{\mu=1}^{p} \left\{ m(r, a^{[\mu]}, f) \right\}
\]
\[
\leq m(r, F) + O(1)
\]
\[
\leq m(r, PF) + m \left( r, \frac{1}{P} \right) + O(1)
\]

Therefore
\[
n \sum_{\mu=1}^{p} m(r, a^{[\mu]}, f) \leq m \left( r, \frac{1}{P} \right) + \sum_{\mu=1}^{p} m \left( r, \frac{P}{(f-a^{[\mu]})^n} \right) + O(1) \quad (6.8.4)
\]

Now, for 1 \leq \mu \leq p , \ P \ is an E-valued homogeneous differential polynomial of degree n in \ (f-a^{[\mu]}) , since the successive derivatives of \ (f-a^{[\mu]}) \ are clearly those of \ f \ and so , by Lemma 6.7.2 , we have
\[
m \left( r, \frac{P}{(f-a^{[\mu]})^n} \right) = S \left( r, (f-a^{[\mu]}) \right) = S(r, f) \quad \text{for} \quad \mu = 1, 2, \ldots, p.
\]
Therefore (6.8.4) becomes

\[ n \sum_{\mu=1}^{p} m(r, a^{[\mu]}, f) \leq m \left( r, \frac{1}{P} \right) + S(r, f) \]

\[ n \sum_{\mu=1}^{p} m(r, a^{[\mu]}, f) + N \left( r, \frac{1}{P} \right) \leq m \left( r, \frac{1}{P} \right) + N \left( r, \frac{1}{P} \right) + S(r, f) \]

\[ = T \left( r, \frac{1}{P} \right) + S(r, f) \]

Using Lemma 6.2.1, we get

\[ n \sum_{\mu=1}^{p} m(r, a^{[\mu]}, f) + N \left( r, \frac{1}{P} \right) \leq T(r, P) - V(r, 0, P) + S(r, f) \]

Hence

\[ n \sum_{\mu=1}^{p} m(r, a^{[\mu]}, f) + N \left( r, \frac{1}{P} \right) + V(r, 0, P) \leq T(r, P) + S(r, f). \]

This completes the proof of the Theorem 6.8.1.

We now prove an extended theorem of Milloux for \( E \)-valued meromorphic functions in complex Banach space.

**Theorem 6.8.2** Let \( f(z) \) be an \( E \)-valued meromorphic function in \( C_r \) and \( P \) is an \( E \)-valued homogeneous differential polynomial in \( f \) of degree \( n \). If \( P \) is not a constant and \( a \in E - \{0\} \), then

\[ nT(r, f) + V(r, 0, P) + G(r, P) \leq \sum_{k=1}^{2} V(r, a^{[k]}, P) + \overline{N}(r, f) + nN \left( r, \frac{1}{f} \right) \]

\[ + \overline{N} \left( r, \frac{1}{P - a} \right) - N_0 \left( r, \frac{1}{P'} \right) + V(r, 0, f^n) + S(r, f) \]

where in \( N_0 \left( r, \frac{1}{P'} \right) \), only zeros of \( P' \) which are not zeros of \( P - a \) are to be considered and

\[ G(r, P) = \int_{0}^{r} \frac{dt}{2\pi} \int_{C_r} \Delta \log \|P'(\xi)\| d\sigma \wedge d\tau \]
Proof. We have E-valued Nevanlinna’s second fundamental theorem,

\[(q - 1)T(r, f) + G(r, f) + N_1(r) \leq \sum_{k=1}^{q} [V(r, a^{[k]}) + N(r, a^{[k]})] + N(r, f) + S(r, f)\]

where \(N_1(r) = N\left(r, \frac{1}{P}\right) + 2N(r, f) - N(r, f').\)

For \(q = 2\), we have

\[T(r, f) + G(r, f) + N_1(r) \leq \sum_{k=1}^{2} [V(r, a^{[k]})] + N(r, f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f - a}\right) + S(r, f)\]

Now \(P\) is an E-valued differential polynomial in \(f\) then above equation can be written as

\[T(r, P) + G(r, P) + N_1(r, P) \leq \sum_{k=1}^{2} [V(r, a^{[k]}, P)] + N(r, P) + N\left(r, \frac{1}{P}\right)
+N\left(r, \frac{1}{P - a}\right) + S(r, P),\]

where \(N_1(r, P) = N\left(r, \frac{1}{P}\right) + 2N(r, P) - N(r, P').\)

Hence

\[T(r, P) + G(r, P) \leq \sum_{k=1}^{2} [V(r, a^{[k]}, P)] + N\left(r, \frac{1}{P}\right) - N(r, P)
+N\left(r, \frac{1}{P - a}\right) - N\left(r, \frac{1}{P'}\right) + N(r, P') + S(r, P) \quad (6.8.5)\]

But

\[T(r, P) = T\left(r, \frac{1}{P}\right) + V(r, 0, P) + O(1)\]
\[= m\left(r, \frac{1}{P}\right) + N\left(r, \frac{1}{P}\right) + V(r, 0, P) + O(1)\]

Therefore, using above equality, (6.8.5) can be written as

\[m\left(r, \frac{1}{P}\right) + N\left(r, \frac{1}{P}\right) + V(r, 0, P) + G(r, P) + O(1)\]
\[\leq \sum_{k=1}^{2} [V(r, a^{[k]}, P)] + N\left(r, \frac{1}{P}\right) - N(r, P) + N\left(r, \frac{1}{P - a}\right)
-N\left(r, \frac{1}{P'}\right) + N(r, P') + S(r, P).\]
We know that $N(r, P) = N(r, P') - N(r, P)$.

Therefore

$$m \left( r, \frac{1}{P} \right) + V(r, 0, P) + G(r, P) \leq \sum_{k=1}^{2} \left[ V(r, a^{[k]}, P) \right] + \overline{N}(r, P) + N \left( r, \frac{1}{P - a} \right)$$

$$- N \left( r, \frac{1}{P'} \right) + S(r, P)$$

Consider

$$N \left( r, \frac{1}{P - a} \right) - N \left( r, \frac{1}{P'} \right)$$

At a zero of $P - a$ of order $l$, $P'$ has a zero of order $l - 1$ so that

$$N \left( r, \frac{1}{P - a} \right) - N \left( r, \frac{1}{P'} \right) = \overline{N} \left( r, \frac{1}{P - a} \right) - N_0 \left( r, \frac{1}{P'} \right)$$

So

$$m \left( r, \frac{1}{P} \right) + V(r, 0, P) + G(r, P) \leq \sum_{k=1}^{2} \left[ V(r, a^{[k]}, P) \right] + \overline{N}(r, P) + \overline{N} \left( r, \frac{1}{P - a} \right)$$

$$- N_0 \left( r, \frac{1}{P'} \right) + S(r, P)$$

where in $N_0 \left( r, \frac{1}{P'} \right)$, only zeros of $P'$ which are not zeros of $P - a$ are to be considered.

Since the poles of $P$ can occur only at the poles of $f$ or the co-efficient $a(z)$ so that

$$\overline{N}(r, P) \leq \overline{N}(r, f) + S(r, f) \quad \text{and} \quad \sum \overline{N}(r, a(z)) \leq \sum T(r, a(z)) = S(r, f)$$

where summation is taken over all the co-efficient $a(z)$ of $P$.

Therefore

$$m \left( r, \frac{1}{P} \right) + V(r, 0, P) + G(r, P) \leq \sum_{k=1}^{2} \left[ V(r, a^{[k]}, P) \right] + \overline{N}(r, f) + \overline{N} \left( r, \frac{1}{P - a} \right)$$

$$- N_0 \left( r, \frac{1}{P'} \right) + S(r, f) + S(r, P) \quad (6.8.6)$$
By Lemma 6.2.1, we have

\[ T(r, f^n) = T\left( r, \frac{1}{f^n} \right) + V(r, 0, f^n) + O(1) \]

Therefore

\[ nT(r, f) = T(r, f^n) = T\left( r, \frac{1}{f^n} \right) + V(r, 0, f^n) + O(1) \]

\[ = m\left( r, \frac{1}{f^n} \right) + N\left( r, \frac{1}{f^n} \right) + V(r, 0, f^n) + O(1) \]

\[ nT(r, f) \leq m\left( r, \frac{1}{P} \right) + m\left( r, \frac{P}{f^n} \right) + nN\left( r, \frac{1}{f} \right) + V(r, 0, f^n) + O(1) \]

By Lemma 6.7.2, we get

\[ nT(r, f) = m\left( r, \frac{1}{P} \right) + nN\left( r, \frac{1}{f} \right) + V(r, 0, f^n) + S(r, f) \]

using (6.8.6), we get

\[ nT(r, f) \leq \sum_{k=1}^{2} \left[ V(r, a[k], P) \right] + \overline{N}(r, f) + \overline{N}\left( r, \frac{1}{P-a} \right) \]

\[ -N_0\left( r, \frac{1}{P} \right) - V(r, 0, P) - G(r, P) + nN\left( r, \frac{1}{f} \right) \]

\[ + V(r, 0, f^n) + S(r, f) + S(r, P) \]

or

\[ nT(r, f) + V(r, 0, P) + G(r, P) \leq \sum_{k=1}^{2} \left[ V(r, a[k], P) \right] + \overline{N}(r, f) + \overline{N}\left( r, \frac{1}{P-a} \right) \]

\[ -N_0\left( r, \frac{1}{P} \right) + nN\left( r, \frac{1}{f} \right) + V(r, 0, f^n) \]

\[ + S(r, f) + S(r, P) \]

We easily show that \( S(r, P) = S(r, f) \). Hence the result.