Chapter 5

Uniqueness of Meromorphic Functions
5.1 Uniqueness of Meromorphic Functions Concerning Differential polynomials

5.2 Introduction, Definitions and Results

In this section, we present a different and very simple technique to handle various uniqueness problems involving three small entire functions. It also gives a new additional insight into such problems.

Let \( f(z) \) be a meromorphic function and \( a \) be an arbitrary complex number. Let \( k \) be a positive integer or \( \infty \). We denote by \( N_k(r, \frac{1}{f-a}) \) or \( N_k(r, a, f) \) the counting function for the zeros of \( f(z) - a \) with multiplicity \( \leq k \) and by \( \mathcal{N}_k(r, a, f) \) the counting function for the zeros of \( f(z) - a \) with multiplicity \( \leq k \), counting only once. Further we denote by \( N_2(r, \frac{1}{f-a}) \) the counting function for the zeros of \( f(z) - a \) where a simple zero is counted once and a multiple zero is counted twice. Similarly, we can define \( N_2(r, f) \).

We denote by \( \mathcal{N}_2(r, \frac{1}{f-a}) \) the counting function for the zeros of \( f(z) - a \) where a simple zero and multiple zero is counted only once. Similarly, we can define \( \mathcal{N}_2(r, f) \).

Let \( g(z) \) be non-constant meromorphic function. We denote by \( n_0^k(r, a) \) for the number of common zeros with multiplicity \( \leq k \) of \( f - a \) and \( g - a \) in \( |z| \leq r \), each zero being counted only once irrespective of its multiplicity. We denote by \( N_0^k(r, a) \) for the corresponding counting function. Thus

\[
N_0^k(r, a) = \int_0^r \frac{n_0^k(t, a) - n_0^k(0, a)}{t} + n_0^k(0, a)\log r.
\]

The content of this section, has been accepted for the publication in Tamkang Journal of Mathematics, Taiwan, China.
We also denote by

(i) $E(a, f)$ for the subset of $\mathbb{C}$ consisting of zeros of $f(z) - a$, each zero being counted according to its multiplicity.

(ii) $\overline{E}(a, f)$ for the subset of $\mathbb{C}$ consisting of zeros of $f(z) - a$, each zero being counted only once.

(iii) $E(a, k, f)$ (or $E_k(a, f)$) for the subset of $\mathbb{C}$ consisting of zeros of $f(z) - a$ with order of multiplicity $\leq k$, where each zero being counted according to its multiplicity.

(iv) $\overline{E}(a, k, f)$ (or $\overline{E}_k(a, f)$) for the subset of $\mathbb{C}$ consisting of zeros of $f(z) - a$ with order of multiplicity $\leq k$, where each zero being counted only once.

Clearly for $k = \infty$, $E(a, \infty, f) = E(a, f)$ and $\overline{E}(a, \infty, f) = \overline{E}(a, f)$.

We define

\[
\Theta_2(a, f) = 1 - \lim_{r \to \infty} \frac{N_2(r, a, f)}{T(r, f)} , \quad \delta_2(a, f) = 1 - \lim_{r \to \infty} \frac{N_2(r, a, f)}{T(r, f)}
\]

and in general

\[
\delta_m(a, f) = 1 - \lim_{r \to \infty} \frac{N_m(r, a, f)}{T(r, f)}
\]

**Definition 5.2.1** [26]. Any expression of the type

\[
P(f) = \sum_{i=1}^{n} \alpha_i(z) f^{n_i_0} (f')^{n_i_1} (f'')^{n_i_2} \ldots (f^{(m)})^{n_i_m},
\]

is called a differential polynomial in $f$ of degree $\bar{d}(P)$, lower degree $\underline{d}(P)$ and weight $\Gamma_P$ where for each $i = 1, 2 \ldots n$, $n_i_0, n_i_1, \ldots, n_i_m$ are non-negative integers, $\alpha_i = \alpha_i(z)$ are meromorphic functions satisfying $T(r, \alpha_i) = S(r, f)$ and

\[
\bar{d}(P) = \max \left\{ \sum_{j=0}^{m} n_{i_j} : 1 \leq i \leq n \right\} , \quad \underline{d}(P) = \min \left\{ \sum_{j=0}^{m} n_{i_j} : 1 \leq i \leq n \right\}
\]

and

\[
\Gamma_P = \max \left\{ \sum_{j=0}^{m} (j+1)n_{i_j} : 1 \leq i \leq n \right\}.
\]
In 1989, H.X.Yi [21] proved the following theorem.

**Theorem 5.2.1** [21]. Let $f_1(z)$ and $f_2(z)$ be non-constant meromorphic functions, $b_j (j = 1, 2, 3)$ be three distinct finite non-zero complex numbers, $k$ be a positive integer or $\infty$ and $n$ be a positive integer satisfying

$$E_k(b_j, f_1^{(n)}) = E_k(b_j, f_2^{(n)})$$

Furthermore, let

$$C_i = 3(k_1 + 1)\delta(0, f_i) + (2nk + 3n + k + 1)\Theta(\infty, f_i) - (2nk + 3n + 3k + 4) \quad (i = 1, 2).$$

If

$$\min \{C_1, C_2\} \geq 0,$$

$$\max \{C_1, C_2\} > 0$$

then $f_1(z) \equiv f_2(z)$.

In 2007, Anupama J. Patil [33] proved the following theorem which generalizes the above result to differential polynomials in $f$ and also improves the conditions in the above theorem.

**Theorem 5.2.2** [33]. Let $f_1(z)$ and $f_2(z)$ be two non-constant meromorphic functions and $P(f_1)$ and $P(f_2)$ be non-constant differential polynomials in $f_1$ and $f_2$ respectively. Let $\alpha_j(\neq 0, \infty)(j = 1, 2, 3)$ be three non-zero distinct entire small functions of $P(f_1)$ and $P(f_2)$, $k_1 \geq k_2 \geq k_3$ be positive integers or $\infty$ and $n$ be a positive integer satisfying

$$E_{k_i}(\alpha_j, P(f_1)) = E_{k_i}(\alpha_j, P(f_2)), \quad i, j = 1, 2, 3.$$

Furthermore, let

$$D_i = 3(k_1 + 1)(k_3 + 1)\delta_{m+1}(a, f_i) - (\tilde{d} + Qc_i)[6(k_1 + 1) + 4k_1(k_3 + 1)] \quad (i = 1, 2),$$

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where \( \overline{d} = \overline{d}(P(f_1)(z)) = \overline{d}(P(f_2)(z)) = \max \left\{ \sum_{j=0}^{m} n_{ij} ; 1 \leq i \leq n \right\} \),
\[
Q = \max \left\{ n_{i1} + 2n_{i2} + 3n_{i3} + \cdots + mn_{im} ; 1 \leq i \leq n \right\},
\]
m = order of the highest derivative of \( f \) occurring in \( P \) and \( c_i = 1 - \Theta(\infty, f_i) \). If
\[
\min \{ D_1, D_2 \} \geq 0 \quad (5.2.1)
\]
\[
\max \{ D_1, D_2 \} > 0 \quad (5.2.2)
\]
then \( P(f_1)(z) \equiv P(f_2)(z) \).

As a corollary to the above theorem, Anupama J.Patil [33] improves Theorem 5.2.1 by considering three non-zero small entire functions of \( f^{(k)} \) instead of three non-zero complex numbers.

**Corollary 5.2.1** [33]. Let \( f_1(z) \) and \( f_2(z) \) be non-constant meromorphic functions, \( b_j (j = 1, 2, 3) \) be three non-zero entire small functions of \( f_1^{(m)} \) and \( f_2^{(m)} \), \( k \) be a positive integer or \( \infty \), and \( m \) be a positive integer satisfying
\[
\overline{E}_k(b_j, f_1^{(m)}) = \overline{E}_k(b_j, f_2^{(m)})
\]
Furthermore, let
\[
C_i = 3(k+1)\delta(0, f_i) + (2mk + 3m + k + 1)\Theta(\infty, f_i) - (2mk + 3m + 3k + 4) \quad (i = 1, 2).
\]
If
\[
\min \{ C_1, C_2 \} \geq 0 , \quad \max \{ C_1, C_2 \} > 0
\]
then \( f_1(z) \equiv f_2(z) \).

**5.3 Lemmas**

In order to prove our results, we need the following lemmas.
Lemma 5.3.1 [23]. Let $f(z)$ be a non-constant meromorphic function, $a$ be an arbitrary complex number and $k$ be a positive integer. Then

\begin{align*}
(i) \quad & \mathcal{N} \left( r, \frac{1}{f-a} \right) \leq \frac{k}{k+1} \mathcal{N}_k \left( r, \frac{1}{f-a} \right) + \frac{1}{k+1} \mathcal{N} \left( r, \frac{1}{f-a} \right); \\
(ii) \quad & \mathcal{N} \left( r, \frac{1}{f-a} \right) \leq \frac{k}{k+1} \mathcal{N}_k \left( r, \frac{1}{f-a} \right) + \frac{1}{k+1} T(r, f) + O(1).
\end{align*}

Lemma 5.3.2 [37]. Suppose that $f(z)$ is a non-constant meromorphic function in the complex plane and $a \in \mathbb{C} \cup \{\infty\}$ is any complex number. Then

$$
\sum_a \Theta(a, f) \leq 2.
$$

Lemma 5.3.3 [25]. Let

$$
Q(\omega) = (n-1)^2(\omega^n - 1)(\omega^{n-2} - 1) - n(n-2)(\omega^{n-1} - 1)^2,
$$

then

$$
Q(\omega) = (\omega - 1)^4(\omega - \beta_1)(\omega - \beta_2)\ldots(\omega - \beta_{2n-6}).
$$

where $\beta_j \in C - [0, 1]$ ($j = 1, 2, \ldots, 2n - 6$), which are distinct respectively.

Lemma 5.3.4 [3]. Suppose $f(z)$ is a meromorphic function in the complex plane and $P(f) = a_0f^n + a_1f^{n-1} + \ldots + a_n$, where $a_n(\neq 0)$, $a_1, \ldots, a_n$ are are small meromorphic functions of $f(z)$. Then

$$
T(r, P(f)) = nT(r, f) + S(r, f).
$$

Lemma 5.3.5 [26]. Let $f$ be a meromorphic function and $P(f)$ be a differential polynomial in $f$. Then

$$
T(r, P(f)) = Q\mathcal{N}(r, f) + \overline{d}(P)T(r, f) + S(r, f),
$$

where

$$
\overline{d}(P) = \max \left\{ \sum_{j=0}^{n} n_i; 1 \leq i \leq n \right\},
$$

$$
Q = \max \left\{ n_{i_1} + 2n_{i_2} + 3n_{i_3} + \cdots + mn_{i_m}; 1 \leq i \leq n \right\}.
$$
Lemma 5.3.6 [26]. Let \( f \) be a meromorphic function and let \( P(f) \) be a non-constant differential polynomial in \( f \) such that every term in \( P(f) \) contains at least one of the derivatives of \( f \). Let \( m \) be the order of the highest derivative of \( f \) occurring in \( P(f) \).

If \( k \) is a positive integer \( \leq m \) such that every term in \( P(f) \) contains at least one of \( f', f'', \ldots, f^{(k)} \), then

\[
d_k T(r, f) \leq d_k N_{m+1}(r, a, f_1) + N \left( r, \frac{1}{P(f) - b} \right) + N \left( r, \frac{1}{P(f) - c} \right) + S(r, f)
\]

where \( a \in C, \ b(\neq 0), \ c(\neq 0) \) are distinct meromorphic functions satisfying

\[
T(r, b) = S(r, P), \ T(r, c) = S(r, P) \quad \text{and} \quad d_k = \min \left\{ \sum_{j=1}^{k} n_{ij} : 1 \leq i \leq n \right\}.
\]

The proof of Theorem 5.2.2 is in [33]. However, we give the proof for the sake of completeness.

**Proof of Theorem 5.2.2.** Let

\[
P(f_1)(z) = P_1(z) \\
N(f_2)(z) = P_2(z)
\]

From the Lemma 5.3.6, we have for the meromorphic function \( f_1 \), its differential polynomial \( P_1 \), any element \( a \in C \) and two distinct non-zero small functions \( \alpha, \beta \) of \( P(f_1) \),

\[
d_k T(r, f_1) \leq d_k N_{m+1}(r, a, f_1) + N \left( r, \frac{1}{P_1 - \alpha} \right) + N \left( r, \frac{1}{P_1 - \beta} \right) + S(r, f_1)
\]

Since \( d_k \geq 1 \),

\[
T(r, f_1) \leq N_{m+1}(r, a, f_1) + N \left( r, \frac{1}{P_1 - \alpha} \right) + N \left( r, \frac{1}{P_1 - \beta} \right) + S(r, f_1)
\]

where \( m \) is the order of the highest derivative of \( f \) occurring in \( P_1 \).

Since there are \( C^3_2 \) ways of selecting one element \( a \) and two elements from \( \alpha_1, \alpha_2, \alpha_3 \) .
For a given \( a \in C \) and three non-zero distinct entire small functions \( \alpha_1, \alpha_2, \alpha_3 \) of \( P_1 \) and \( P_2 \), we have

\[
T(r, f_1) \leq N_{m+1}(r, a, f_1) + N\left(r, \frac{1}{P_1 - \alpha_1}\right) + N\left(r, \frac{1}{P_1 - \alpha_2}\right) + S(r, f_1)
\]

\[
T(r, f_1) \leq N_{m+1}(r, a, f_1) + N\left(r, \frac{1}{P_1 - \alpha_2}\right) + N\left(r, \frac{1}{P_1 - \alpha_3}\right) + S(r, f_1)
\]

\[
T(r, f_1) \leq N_{m+1}(r, a, f_1) + N\left(r, \frac{1}{P_1 - \alpha_1}\right) + N\left(r, \frac{1}{P_1 - \alpha_3}\right) + S(r, f_1)
\]

Adding all the above equations, we get

\[
3T(r, f_1) \leq 3N_{m+1}(r, a, f_1) + 2 \sum_{i=1}^{3} N\left(r, \frac{1}{P_1 - \alpha_i}\right) + S(r, f_1)
\]

Obviously

\[
N_{m+1}(r, a, f_1) < (1 - \delta(a, f_1)) T(r, f_1) + S(r, f_1)
\]

So,

\[
3T(r, f_1) \leq 3 (1 - \delta_{m+1}(a, f_1)) T(r, f_1) + 2 \sum_{i=1}^{3} N\left(r, \frac{1}{P_1 - \alpha_i}\right) + S(r, f_1)
\]

Using Lemma 5.3.1, we have

\[
3\delta_{m+1}(a, f_1)T(r, f_1) \leq 2 \sum_{i=1}^{3} N\left(r, \frac{1}{P_1 - \alpha_i}\right) + S(r, f_1)
\]

\[
\leq 2 \sum_{i=1}^{3} \left[ N\left(r, \frac{1}{k_i + 1}\right) \left(r, \frac{1}{P_1 - \alpha_i}\right) + \frac{1}{k_i + 1} N\left(r, \frac{1}{P_1 - \alpha_i}\right)\right] + S(r, f_1)
\]

\[
\leq 2 \sum_{i=1}^{3} \frac{k_i}{k_i + 1} N\left(r, \frac{1}{P_1 - \alpha_i}\right) + 2 \sum_{i=1}^{3} \frac{1}{k_i + 1} N\left(r, \frac{1}{P_1 - \alpha_i}\right) + S(r, f_1)
\]

\[
\leq 2 \frac{k_1}{k_1 + 1} \sum_{i=1}^{3} N\left(r, \frac{1}{P_1 - \alpha_i}\right) + \frac{6}{k_3 + 1} T(r, P_1) + S(r, f_1)
\]
since \( \left( \frac{k_i}{k_{i+1}} \right) \) is an increasing sequence.

By Lemma 5.3.5, we have

\[
T(r, P) = QN(r, f_i) + \bar{d}(P)T(r, f_i) + S(r, f_i)
\]

\[
< Q(1 - \Theta(\infty, f_i)) T(r, f_i) + \bar{d}(P)T(r, f_i) + S(r, f_i)
\]

Therefore

\[
3\delta_{m+1}(a, f_i)T(r, f_i) \leq 2 \frac{k_1}{k_1 + 1} \sum_{i=1}^{3} N_{k_i} \left( r, \frac{1}{P_1 - \alpha_i} \right) + \frac{6(\bar{d} + Q(1 - \Theta(\infty, f_i)))}{k_3 + 1} T(r, f_i)
\]

\[
+ S(r, f_i)
\]

i.e.,

\[
d_i T(r, f_i) \leq 2k_1(k_3 + 1) \sum_{i=1}^{3} N_{k_i} \left( r, \frac{1}{P_1 - \alpha_i} \right) + S(r, f_i)
\]

where

\[
d_i = 3\delta_{m+1}(a, f_i)(k_1 + 1)(k_3 + 1) - 6(\bar{d} + Q(1 - \Theta(\infty, f_i)))(k_1 + 1), \quad (i = 1, 2)
\]

Similarly, we have

\[
d_2 T(r, f_2) \leq 2k_1(k_3 + 1) \sum_{i=1}^{3} N_{k_i} \left( r, \frac{1}{P_2 - \alpha_i} \right) + S(r, f_2)
\]

Adding the above two equations, we get

\[
d_1 T(r, f_1) + d_2 T(r, f_2) \leq 2k_1(k_3 + 1) \sum_{i=1}^{3} \left[ N_{k_i} \left( r, \frac{1}{P_1 - \alpha_i} \right) + N_{k_i} \left( r, \frac{1}{P_2 - \alpha_i} \right) \right]
\]

\[
+ S(r, f_1) + S(r, f_2)
\]

Since by hypothesis,

\[
E_{k_i} (\alpha_j, P(f_i)) = E_{k_i} (\alpha_j, P(f_j)) \quad (i, j = 1, 2, 3)
\]

so that

\[
N_{k_i} \left( r, \frac{1}{P_1 - \alpha_i} \right) = N_{k_i} \left( r, \frac{1}{P_2 - \alpha_i} \right) = N_{0}^{k_i} (r, \alpha_i)
\]

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Thus
\[ d_1 T(r, f_1) + d_2 T(r, f_2) \leq 2k_1(k_3 + 1) \sum_{i=1}^{3} \left[ N_{k_i} \left( r, \frac{1}{P_1 - \alpha_i} \right) + N_{k_i} \left( r, \frac{1}{P_2 - \alpha_i} \right) \right] + S(r, f_1) + S(r, f_2) \]

\[ d_1 T(r, f_1) + d_2 T(r, f_2) \leq 4k_1(k_3 + 1) \sum_{i=1}^{3} N_0^{k_i}(r, \alpha_i) + S(r, f_1) + S(r, f_2) \quad (5.3.1) \]

Suppose,
\[ P_1(z) \neq P_2(z) \quad (5.3.2) \]

Then under the assumption that \( P_1 \) and \( P_2 \) are distinct, it follows that for any \( \alpha \in S(P_1) \cap S(P_2) - \{0, \infty\} \), each common zero of \( P_1 - \alpha \) and \( P_2 - \alpha \) is a zero of \( P_1 - P_2 \).

Since \( \alpha_1, \alpha_2, \alpha_3 \) are distinct, we have
\[ \sum_{i=1}^{3} N_0^{k_i}(r, \alpha_i) \leq N \left( r, \frac{1}{P_1 - P_2} \right) \leq T(r, P_1 - P_2) \]
\[ \leq T(r, P_1) + T(r, P_2) + O(1) \]

\[ \sum_{i=1}^{3} N_0^{k_i}(r, \alpha_i) \leq (\bar{d} + Q(1 - \Theta(\infty, f_1)))T(r, f_1) + S(r, f_1) \]
\[ + (\bar{d} + Q(1 - \Theta(\infty, f_2)))T(r, f_2) + S(r, f_2) \quad (5.3.3) \]

From equations (5.3.1) and (5.3.3), we get
\[ D_1 T(r, f_1) + D_2 T(r, f_2) \leq S(r, f_1) + S(r, f_2) \quad (5.3.4) \]

where for \( i = 1, 2 \)
\[ D_i = d_i - 4k_1(k_3 + 1)(\bar{d} + Q(1 - \Theta(\infty, f_i))) \]
which is the $D_i$ defined in the statement of the theorem.

From conditions (5.2.1),(5.2.2) the above inequality (5.3.4) is not possible. Thus our assumption (5.3.2) is not true and hence we must have

$$P_1(z) \equiv P_2(z)$$

i.e., $$P(f_1)(z) \equiv P(f_2)(z).$$

This completes the proof the Theorem 5.2.2.

In this section, we study the uniqueness problems on meromorphic functions concerning differential polynomials that share three entire small functions as an application of Theorem 5.2.2. Here our techniques employed are much different and relatively simple and lead to several significant results. They also throw new light on such topics.

### 5.4 Statement and Proofs of Main Results

**Theorem 5.4.1** Let $f$ and $g$ be two non-constant meromorphic functions and $\alpha_j(\not= 0, \infty) \ (j = 1, 2, 3)$ be three non-zero distinct entire small functions, $k$ be a positive integer or $\infty$ satisfying

$$E_k(\alpha_j, f^n(f^p - 1)f') = E_k(\alpha_j, g^n(g^p - 1)g'), \quad j = 1, 2, 3,$$

where $n$ and $p$ are positive integers, then either $f \equiv g$ or

$$g = \left[ \frac{(n + p + 1)(h^{n+1} - 1)}{(n + 1)(h^{n+p+1} - 1)} \right]^{1/p}, \quad f = \left[ \frac{(n + p + 1)h^p(h^{n+1} - 1)}{(n + 1)(h^{n+p+1} - 1)} \right]^{1/p},$$

where $h$ is a non-constant meromorphic function.

**Proof.** First, we need to show that $f^n(f^p - 1)f' \equiv g^n(g^p - 1)g'$.

Consider $k_1 = k_2 = k_3 = k$ and $a = 0$. We have
\[ \forall \{ f^n(f^p - 1)f' \} = \forall \{ g^n(g^p - 1)g' \} = n + p + 1 \quad Q = 1 \quad \text{and} \quad m = 1. \]

Therefore

\[
\min\{D_f, D_g\} \geq 0 \quad \text{and} \quad \max\{D_f, D_g\} > 0,
\]

where

\[
D_f = 3(k + 1)^2 \delta_2(0, f) - (n + p + 2)(k + 1)(6 + 4k) + (k + 1)(6 + 4k)\Theta(\infty, f),
\]

\[
D_g = 3(k + 1)^2 \delta_2(0, g) - (n + p + 2)(k + 1)(6 + 4k) + (k + 1)(6 + 4k)\Theta(\infty, g)
\]

By Theorem 5.2.2, we get

\[
f^n(f^p - 1)f' \equiv g^n(g^p - 1)g' \quad (F^*)' \equiv (G^*)'
\]

then \( F^* \equiv G^* + c \quad c \quad \text{is a constant.} \quad (5.4.1)\)

where

\[
F^* = \frac{f^{n+p+1}}{n + p + 1} - \frac{f^{n+1}}{n + 1}, \quad G^* = \frac{g^{n+p+1}}{n + p + 1} - \frac{g^{n+1}}{n + 1}
\]

By Lemma 5.3.4, we have

\[
T(r, F^*) = (n + p + 1)T(r, f) + S(r, f)
\]

Note that

\[
\mathcal{N}_2 \left( r, \frac{1}{F^*} \right) = \mathcal{N}_2 \left( r, \frac{1}{f} \right) + \mathcal{N}_2 \left( r, \frac{1}{\frac{f^p}{n + p + 1} - \frac{1}{n + 1}} \right)
\]

\[
\leq \mathcal{N}_2 \left( r, \frac{1}{f} \right) + \mathcal{T} \left( r, \frac{1}{\frac{f^p}{n + p + 1} - \frac{1}{n + 1}} \right)
\]

\[
= \mathcal{N}_2 \left( r, \frac{1}{f} \right) + pT(r, f) + S(r, f).
\]
So,
\[
\frac{N_2\left(r, \frac{1}{T}\right)}{T(r, F^*)} \leq \frac{N_2\left(r, \frac{1}{T}\right)}{(n + p + 1)T(r, f) + S(r, f)} + pT(r, f)
\]

Therefore,
\[
\lim_{r \to \infty} \frac{N_2\left(r, \frac{1}{T}\right)}{T(r, F^*)} \leq \lim_{r \to \infty} \frac{N_2\left(r, \frac{1}{T}\right)}{n + p + 1T(r, f) + S(r, f)} + \frac{p}{n + p + 1}
\]

\[
1 - \Theta_2(0, F^*) \leq \frac{1 - \Theta_2(0, f)}{n + p + 1} + \frac{p}{n + p + 1}
\]

i.e.,
\[
\Theta_2(0, F^*) \geq \frac{n}{n + p + 1} + \frac{\Theta_2(0, f)}{n + p + 1}
\]

Similarly, we have
\[
\Theta_2(0, G^*) \geq \frac{n}{n + p + 1} + \frac{\Theta_2(0, f)}{n + p + 1}
\]

Note that \(N_2(r, F^*) = N_2(r, f)\). So,
\[
\frac{N_2(r, F^*)}{T(r, F^*)} = \frac{N_2(r, f)}{(n + p + 1)T(r, f) + S(r, f)}
\]

\[
\lim_{r \to \infty} \frac{N_2(r, F^*)}{T(r, F^*)} = \lim_{r \to \infty} \frac{N_2(r, f)}{(n + p + 1)T(r, f) + S(r, f)}
\]

\[
1 - \Theta_2(\infty, F^*) = \frac{1 - \Theta_2(\infty, f)}{n + p + 1}
\]

\[
\Theta_2(\infty, F^*) = \frac{n + p}{n + p + 1} + \frac{\Theta_2(\infty, f)}{n + p + 1}
\]

And, by the definition of \(\Theta_2\) and (5.4.1), we have
\[
\Theta_2(c, F^*) = 1 - \lim_{r \to \infty} \frac{N_2\left(r, \frac{1}{T(r, F^*)}\right)}{T(r, F^*)} = 1 - \lim_{r \to \infty} \frac{N_2\left(r, \frac{1}{T(r, F^*)}\right)}{T(r, F^*)}
\]

Since \(F^* \equiv G^* + c\), \(c\) a constant, so \(T(r, F^*) = T(r, G^*)\).

Therefore,
\[
\Theta_2(c, F^*) = 1 - \lim_{r \to \infty} \frac{N_2\left(r, \frac{1}{T(r, F^*)}\right)}{T(r, F^*)} = \Theta_2(0, G^*)
\]

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We now show that \( c = 0 \) in (5.4.1). Suppose that \( c \neq 0 \), then

\[
\Theta_2(0, F^*) + \Theta_2(\infty, F^*) + \Theta_2(c, F^*)
\]

\[
\geq \frac{n}{n+p+1} + \frac{\Theta_2(0, f)}{n+p+1} + \frac{n+p}{n+p+1} + \frac{\Theta_2(\infty, f)}{n+p+1} + \frac{n}{n+p+1} + \frac{\Theta_2(0, g)}{n+p+1}
\]

\[
\geq \frac{n}{n+p+1} + \frac{\delta_2(0, f)}{n+p+1} + \frac{n+p}{n+p+1} + \frac{\Theta(\infty, f)}{n+p+1} + \frac{n}{n+p+1} + \frac{\delta_2(0, g)}{n+p+1}, \quad (5.4.2)
\]

because \( \overline{N}_2(r, f) = \overline{N}(r, f) \) and hence

\[
\lim_{r \to \infty} \frac{\overline{N}_2(r, f)}{T(r, f)} = \lim_{r \to \infty} \frac{\overline{N}(r, f)}{T(r, f)}
\]

\[\implies \Theta_2(\infty, f) = \Theta(\infty, f).\]

Since \( \min \{D_f, D_g\} \geq 0 \), we have

\[
3(k+1)^2 \delta_2(0, f) - (n+p+2)(k+1)(6+4k) + (k+1)(6+4k) \Theta(\infty, f) \geq 0 \]

\[
3(k+1)^2 \delta_2(0, g) - (n+p+2)(k+1)(6+4k) + (k+1)(6+4k) \Theta(\infty, g) \geq 0.
\]

Therefore

\[
\delta_2(0, f) \geq \frac{(n+p+2)(6+4k)}{3(k+1)} - \frac{(6+4k)}{3(k+1)} \Theta(\infty, f) \quad (5.4.3)
\]

\[
\delta_2(0, g) \geq \frac{(n+p+2)(6+4k)}{3(k+1)} - \frac{(6+4k)}{3(k+1)} \Theta(\infty, g) \quad (5.4.4)
\]

Substituting (5.4.3) and (5.4.4) in (5.4.2), we get

\[
\Theta_2(0, F^*) + \Theta_2(\infty, F^*) + \Theta_2(c, F^*)
\]

\[
\geq \frac{3n+p}{n+p+1} + \frac{1}{n+p+1} \left\{ \frac{(n+p+2)(6+4k)}{3(k+1)} - \frac{(6+4k)}{3(k+1)} \Theta(\infty, f) \right\}
\]

\[
+ \frac{\Theta(\infty, f)}{n+p+1} + \frac{1}{n+p+1} \left\{ \frac{(n+p+2)(6+4k)}{3(k+1)} - \frac{(6+4k)}{3(k+1)} \Theta(\infty, g) \right\}
\]

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\[
\begin{align*}
\frac{3n + p}{n + p + 1} + \frac{2(n + p + 2)(6 + 4k)}{3(k + 1)(n + p + 1)} & - \frac{(k + 3)}{3(k + 1)(n + p + 1)} \Theta(\infty, f) \\
- \frac{(6 + 4k)}{3(k + 1)(n + p + 1)} \Theta(\infty, g) \\
\geq \frac{3n + p}{n + p + 1} + \frac{2(n + p + 2)(6 + 4k)}{3(k + 1)(n + p + 1)} & - \frac{(k + 3)}{3(k + 1)(n + p + 1)} - \frac{(6 + 4k)}{3(k + 1)(n + p + 1)} \\
= \frac{17kn + 11kp + 21n + 15p + 11k + 15}{3(k + 1)(n + p + 1)} > 2
\end{align*}
\]

Because, let
\[
H_n = \frac{17kn + 11kp + 21n + 15p + 11k + 15}{3(k + 1)(n + p + 1)} , \quad k > 0, \quad p > 0.
\]

Then
\[
H'_n = \frac{6kp + 6p + 6k + 6}{3(k + 1)(n + p + 1)^2} > 0 \quad \text{for} \quad k > 0 , \quad p > 0.
\]

Thus $H_n$ is an increasing function and
\[
H_n \quad \text{at} \quad \{p = 1, n = 1\} = \frac{26k + 62}{9k + 9} \quad \text{and} \quad \lim_{k \to \infty} \left\{ \frac{26k + 62}{9k + 9} \right\} = \frac{26}{9} = 2.888...
\]

This shows that $H_n$ always exceeds the value $2$, which contradicts Lemma 5.3.2.

Hence $c = 0$. Therefore
\[
F^* \equiv G^*
\]

\[
i.e, \quad \frac{f^{n+p+1}}{n + p + 1} - \frac{f^{n+1}}{n + 1} = \frac{g^{n+p+1}}{n + p + 1} - \frac{g^{n+1}}{n + 1}
\]

\[
f^{n+1}\left\{ \frac{f^p}{n + p + 1} - \frac{1}{n + 1} \right\} = g^{n+1}\left\{ \frac{g^p}{n + p + 1} - \frac{1}{n + 1} \right\}
\]
Now, let $h = \frac{f}{g}$. If $h \equiv 1$, then $f \equiv g$.

Suppose $h \not\equiv 1$, then

$$\left(\frac{f}{g}\right)^{n+1}\left\{\frac{fp}{n+p+1} - \frac{1}{n+1}\right\} = \frac{g^p}{n+p+1} - \frac{1}{n+1}$$

$$h^{n+1}\left\{\frac{(hg)^p}{n+p+1} - \frac{1}{n+1}\right\} = \frac{g^p}{n+p+1} - \frac{1}{n+1}$$

$$h^{n+1}\left\{\frac{(hg)^p(n+1) - (n+p+1)}{(n+1)(n+p+1)}\right\} = \frac{g^p(n+1) - (n+p+1)}{(n+1)(n+p+1)}$$

$$h^{n+p+1}g^p(n+1) - h^{n+1}(n+p+1) = g^p(n+1) - (n+p+1)$$

$$h^{n+p+1}g^p(n+1) - g^p(n+1) = h^{n+1}(n+p+1) - (n+p+1)$$

$$g^p(n+1)\left[h^{n+p+1} - 1\right] = (n+p+1)\left(h^{n+1} - 1\right)$$

$$g^p = \frac{(n+p+1)(h^{n+1} - 1)}{(n+1)(h^{n+p+1} - 1)}$$

Similarly

$$f^p = \frac{(n+p+1)h^p(h^{n+1} - 1)}{(n+1)(h^{n+p+1} - 1)}$$

Therefore,

$$f = \left[\frac{(n+p+1)h^p(h^{n+1} - 1)}{(n+1)(h^{n+p+1} - 1)}\right]^{1/p}, \quad g = \left[\frac{(n+p+1)(h^{n+1} - 1)}{(n+1)(h^{n+p+1} - 1)}\right]^{1/p},$$

This completes the proof of the Theorem.

Letting $k \to \infty$ and $p = 1$ in Theorem 5.4.1, we have the following result.

**Corollary 5.4.1** Let $f$ and $g$ be two non-constant meromorphic functions and $\alpha_j(\not\equiv 0, \infty)$ $(j = 1, 2, 3)$ be three non-zero distinct entire small functions satisfying

$$\overline{E}(\alpha_j, f^n(f-1)f') = \overline{E}(\alpha_j, g^n(g-1)g')$$, \quad $j = 1, 2, 3$
where \( n \) is a positive integer, then either \( f \equiv g \) or

\[
g = \frac{(n + 2)(h^{n+1} - 1)}{(n + 1)(h^{n+2} - 1)}, \quad f = \frac{(n + 2)h(h^{n+1} - 1)}{(n + 1)(h^{n+2} - 1)}
\]

where \( h \) is a non-constant meromorphic function.

**Theorem 5.4.2** Let \( f \) and \( g \) be two non-constant meromorphic functions and 
\( \alpha_j(\neq 0, \infty) \ (j = 1, 2, 3) \) be three non-zero distinct entire small functions, \( k \) be a positive integer or \( \infty \) and \( p \) is a positive integer satisfying

\[
E_k(\alpha_j, f^n(f - 1)^pg') = E_k(\alpha_j, g^n(g - 1)^pg'), \quad j = 1, 2, 3,
\]

(i) if \( p = 1 \) and \( n \) is a positive integer, then either \( f \equiv g \) or

\[
g = \frac{(n + 2)(h^{n+1} - 1)}{(n + 1)(h^{n+2} - 1)}, \quad f = \frac{(n + 2)h(h^{n+1} - 1)}{(n + 1)(h^{n+2} - 1)}
\]

where \( h \) is a non-constant meromorphic function.

(ii) if \( p = 2 \) and \( n \geq 3 \), then \( f \equiv g \).

(iii) if \( p > 2 \) and \( n \) is a positive integer, then

\[
f^{n+1} \sum_{l=0}^{p} \frac{(-1)^l}{n+p-l+1} {C_p^l} f^{p-l} \equiv g^{n+1} \sum_{l=0}^{p} \frac{(-1)^l}{n+p-l+1} {C_p^l} g^{p-l}.
\]

**Proof.** First, we show that \( f^n(f - 1)^pg' \equiv g^n(g - 1)^pg' \).

where

\[
(f - 1)^p = f^p - pf^{p-1} + \frac{p(p-1)}{2} f^{p-2} - \ldots + (-1)^p
\]

So,

\[
f^n(f - 1)^pg' = f^{n+p}g' - p f^{n+p-1}g' + \frac{p(p-1)}{2} f^{n+p-2}g' - \ldots + (-1)^p f^n f'
\]

\[
g^n(g - 1)^pg' = g^{n+p}g' - pg^{n+p-1}g' + \frac{p(p-1)}{2} g^{n+p-2}g' - \ldots + (-1)^pg^n g'
\]
Consider \( k_1 = k_2 = k_3 = k \) and \( a = 0 \). We have,
\[
\overline{d} \{ f^n(f-1)^p f' \} = \overline{d} \{ g^n(g-1)^p g' \} = n + p + 1, \quad Q = 1 \text{ and } m = 1.
\]
Therefore
\[
\min \{ D_f, D_g \} \geq 0, \quad \max \{ D_f, D_g \} > 0,
\]
where
\[
D_f = 3(k+1)^2 \delta_2(0,f) - (n+p+2)(k+1)(6+4k) + (k+1)(6+4k)\Theta(\infty,f)
\]
\[
D_g = 3(k+1)^2 \delta_2(0,g) - (n+p+2)(k+1)(6+4k) + (k+1)(6+4k)\Theta(\infty,g).
\]
By Theorem 5.2.2, we obtain
\[
f^n(f-1)^p f' \equiv g^n(g-1)^p g'
\]
\[
(F^*)' \equiv (G^*)'
\]
Then \( F^* \equiv G^* + c \), \( c \) is a constant \( (5.4.5) \)

where
\[
F^* = \frac{f^{n+p+1}}{n+p+1} - p \frac{f^{n+p}}{n+p} + \frac{p(p-1)}{2} \frac{f^{n+p-1}}{n+p-1} - \frac{p(p-1)(p-2)}{6} \frac{f^{n+p-2}}{n+p-2} + \cdots + (-1)^p \frac{f^{n+1}}{n+1}
\]
\[
G^* = \frac{g^{n+p+1}}{n+p+1} - p \frac{g^{n+p}}{n+p} + \frac{p(p-1)}{2} \frac{g^{n+p-1}}{n+p-1} - \frac{p(p-1)(p-2)}{6} \frac{g^{n+p-2}}{n+p-2} + \cdots + (-1)^p \frac{g^{n+1}}{n+1}
\]

By Lemma 5.3.4, we have
\[
T(r,F^*) = (n+p+1)T(r,f) + S(r,f)
\]
\[
T(r,G^*) = (n+p+1)T(r,g) + S(r,g)
\]
Using Lemma 5.3.4, we note that

\[
\overline{N}_2 \left( r, \frac{1}{F^*} \right) = \overline{N}_2 \left( r, \frac{1}{f} \right) + \overline{N}_2 \left( r, \frac{f^p}{n+p+1} - \frac{fp^{p-1}}{n+p} + \ldots + \frac{(-1)^p}{n+1} \right)
\]

\[
\leq \overline{N}_2 \left( r, \frac{1}{f} \right) + T \left( r, \frac{f^p}{n+p+1} - \frac{fp^{p-1}}{n+p} + \ldots + \frac{(-1)^p}{n+1} \right)
\]

\[
= \overline{N}_2 \left( r, \frac{1}{f} \right) + pT(r, f) + S(r, f)
\]

So,

\[
\frac{\overline{N}_2 \left( r, \frac{1}{F^*} \right)}{T(r, F^*)} \leq \frac{\overline{N}_2 \left( r, \frac{1}{f} \right)}{(n+p+1)T(r, f) + S(r, f)} + \frac{pT(r, f)}{(n+p+1)T(r, f) + S(r, f)}
\]

Therefore,

\[
\lim_{r \to \infty} \frac{\overline{N}_2 \left( r, \frac{1}{F^*} \right)}{T(r, F^*)} \leq \lim_{r \to \infty} \frac{\overline{N}_2 \left( r, \frac{1}{f} \right)}{(n+p+1)T(r, f) + S(r, f)} + \frac{p}{n+p+1}
\]

\[
1 - \Theta_2(0, F^*) \leq \frac{1 - \Theta_2(0, f)}{n+p+1} + \frac{p}{n+p+1}
\]

\[
i.e., \quad \Theta_2(0, F^*) \geq \frac{n}{n+p+1} + \frac{\Theta_2(0, f)}{n+p+1}
\]

Similarly, we have

\[
\Theta_2(0, G^*) \geq \frac{n}{n+p+1} + \frac{\Theta_2(0, g)}{n+p+1}
\]

Note that \( \overline{N}_2(r, F^*) = \overline{N}_2(r, f) \). So,

\[
\frac{\overline{N}_2(r, F^*)}{T(r, F^*)} = \frac{\overline{N}_2(r, f)}{(n+p+1)T(r, f) + S(r, f)}
\]

\[
\lim_{r \to \infty} \frac{\overline{N}_2(r, F^*)}{T(r, F^*)} = \lim_{r \to \infty} \frac{\overline{N}_2(r, f)}{(n+p+1)T(r, f) + S(r, f)}
\]

\[
1 - \Theta_2(\infty, F^*) = \frac{1 - \Theta_2(\infty, f)}{n+p+1}
\]
\[ \Theta_2(\infty, F^*) = \frac{n + p}{n + p + 1} + \frac{\Theta_2(\infty, f)}{n + p + 1} \]

And, by the definition of \( \Theta_2 \) and (5.4.5), we have

\[ \Theta_2(c, F^*) = 1 - \lim_{r \to \infty} \frac{N_2 \left(r, \frac{1}{F^*-c} \right)}{T(r, F^*)} \]

Since \( F^* \equiv G^* + c \), \( c \) a constant, which implies that

\[ T(r, F^*) = T(r, G^*). \]

Therefore

\[ \Theta_2(c, F^*) = 1 - \lim_{r \to \infty} \frac{N_2 \left(r, \frac{1}{G^*} \right)}{T(r, G^*)} = \Theta_2(0, G^*) \]

We now show that \( c = 0 \). Proceeding as in the proof of Theorem 5.4.1, we obtain

\[ F^* \equiv G^* \]

(i) If \( p = 1 \), then

\[ F^* = \frac{f^{n+2}}{n + 2} - \frac{f^{n+1}}{n + 1}, \quad G^* = \frac{g^{n+2}}{n + 2} - \frac{g^{n+1}}{n + 1} \]

Then, we can write

\[ f^{n+1} \left\{ \frac{f}{n + 2} - \frac{1}{n + 1} \right\} = g^{n+1} \left\{ \frac{g}{n + 2} - \frac{1}{n + 1} \right\} \]

Now, let \( h = \frac{f}{g} \). If \( h \equiv 1 \), then \( f \equiv g \).

Suppose \( h \not\equiv 1 \), then we easily obtain

\[ g = \frac{(n + 2)(h^{n+1} - 1)}{(n + 1)(h^{n+2} - 1)}, \quad f = \frac{(n + 2)h(h^{n+1} - 1)}{(n + 1)(h^{n+2} - 1)}. \]

(ii) If \( p = 2 \), then

\[ F^* \equiv G^* \]

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\[ \frac{f^{n+3}}{n+3} - 2 \frac{f^{n+2}}{n+2} + \frac{f^{n+1}}{n+1} = \frac{g^{n+3}}{n+3} - 2 \frac{g^{n+2}}{n+2} + \frac{g^{n+1}}{n+1} \]  

(5.4.6)

Set \( h = \frac{f}{g} \). Substitute \( f = hg \) in (5.4.6), we obtain

\[(n + 2)(n + 1)g^2(h^{n+3} - 1) - 2(n + 3)(n + 1)g(h^{n+2} - 1)\]

\[+ (n + 2)(n + 3)(h^{n+1} - 1) = 0 \]  

(5.4.7)

If \( h \) is not a constant, then

\[[(n + 2)(n + 1)g(h^{n+3} - 1) - (n + 3)(n + 1)(h^{n+2} - 1)]^2\]

\[= (n + 1)^2(n + 2)^2g^2(h^{n+3} - 1)^2 - 2(n + 1)(n + 2)(n + 3)(n + 1)g(h^{n+3} - 1)(h^{n+2} - 1)\]

\[+ (n + 3)^2(n + 1)^2(h^{n+2} - 1)^2\]

\[= (n + 2)(n + 1)(h^{n+3} - 1) \left[(n + 2)(n + 1)g^2(h^{n+3} - 1) - 2(n + 3)(n + 1)g(h^{n+2} - 1)\right]\]

\[+ (n + 3)^2(n + 1)^2(h^{n+2} - 1)^2\]

\[= (n + 2)(n + 1)(h^{n+3} - 1) \left\{- (n + 2)(n + 3)(h^{n+1} - 1)\right\} + (n + 3)^2(n + 1)^2(h^{n+2} - 1)^2\]

\[= -(n + 3)(n + 1)\left\{(n + 2)^2(h^{n+3} - 1)(h^{n+1} - 1) - (n + 3)(n + 1)(h^{n+1} - 1)^2\right\}\]

using Lemma 5.3.3, we get

\[[(n + 2)(n + 1)g(h^{n+3} - 1) - (n + 3)(n + 1)(h^{n+2} - 1)]^2 = -(n + 3)(n + 1)Q(h),\]

where \( Q(h) = (h - 1)^4(h - \beta_1)(h - \beta_2)\ldots(h - \beta_{2n}) \), \( \beta_j \in C - \{0, 1\} \) \( (j = 1, 2, \ldots, 2n) \), which are pairwise distinct.

This implies that every zero of \( (h - \beta_j) \) \( (j = 1, 2, \ldots, 2n) \) has a multiplicity of at least 2. By the second fundamental theorem we obtain \( n \leq 2 \), which is again a contradiction. Therefore, \( h \) is a constant. We have from (5.4.7) that \( h^{n+1} - 1 = 0 \) and \( h^{n+2} - 1 = 0 \), which imply \( h = 1 \) and hence \( f \equiv g \).
(iii) If \( p \geq 2 \), we get

\[
\frac{f^{n+p+1}}{n+p+1} - p \frac{f^{n+p}}{n+p} + \frac{p(p-1)}{2} \frac{f^{n+p-1}}{n+p-1} - \frac{p(p-1)(p-2)}{6} \frac{f^{n+p-2}}{n+p-2} + \cdots + (-1)^p \frac{f^{n+1}}{n+1}
\]

\[
= \frac{g^{n+p+1}}{n+p+1} - p \frac{g^{n+p}}{n+p} + \frac{p(p-1)}{2} \frac{g^{n+p-1}}{n+p-1} - \frac{p(p-1)(p-2)}{6} \frac{g^{n+p-2}}{n+p-2} + \cdots + (-1)^p \frac{g^{n+1}}{n+1}
\]

above equality can be represented as

\[
f^{n+1} \sum_{l=0}^{p} \frac{(-1)^l C_p^l}{n+p-l+1} f^{p-l} = g^{n+1} \sum_{l=0}^{p} \frac{(-1)^l C_p^l}{n+p-l+1} g^{p-l}.
\]

This completes the proof of the Theorem.

**Theorem 5.4.3** Let \( f \) and \( g \) be two non-constant meromorphic functions and
\[\alpha_j(\neq 0, \infty) \ (j = 1, 2, 3) \] be three non-zero distinct entire small functions, \( k \) be a positive integer or \( \infty \) satisfying

\[E_k(\alpha_j, f^n f') = E_k(\alpha_j, g^n g'), \quad j = 1, 2, 3\]

where \( n \) is positive integer, then either \( f \equiv g \) or \( h^{n+1} - 1 = 0 \), where \( h \) is a non-constant meromorphic function.

**Proof.** First we show that \( f^n f' \equiv g^n g' \).

Consider \( k_1 = k_2 = k_3 = k \) and \( a = 0 \). We have

\[d(f^n f') = d(g^n g') = n + 1 \quad , \quad Q = 1 \quad \text{and} \quad m = 1.
\]

Therefore

\[\min \{D_f, D_g\} \geq 0 \quad , \quad \max \{D_f, D_g\} > 0 \quad ,
\]
where
\[ D_f = 3(k + 1)^2 \delta_2(0, f) - (n + 2)(k + 1)(6 + 4k) + (k + 1)(6 + 4k)\Theta(\infty, f) \]
\[ D_g = 3(k + 1)^2 \delta_2(0, g) - (n + 2)(k + 1)(6 + 4k) + (k + 1)(6 + 4k)\Theta(\infty, g) \]

By Theorem 5.2.2, we get
\[ f^n f' \equiv g^n g' \]
\[ (F^*)' \equiv (G^*)' \]

Then \( F^* \equiv G^* + c \), \( c \) is a constant. (5.4.8)

where
\[ F^* = \frac{f^{n+1}}{n+1} \quad G^* = \frac{g^{n+1}}{n+1} \]

By Lemma 5.3.4, we have
\[ T(r, F^*) = (n + 1)T(r, f) + S(r, f). \]

Note that
\[ \overline{N_2} \left( r, \frac{1}{F^*} \right) = \overline{N_2} \left( r, \frac{1}{f} \right) \]
\[ \overline{N_2} (r, F^*) = \overline{N_2} (r, f) \]

Similarly, as in the proof of Theorem 5.4.1, we obtain
\[ \Theta_2(0, F^*) = \frac{n}{n + 1} + \frac{\Theta_2(0, f)}{n + 1} \]
\[ \Theta_2(\infty, F^*) = \frac{n}{n + 1} + \frac{\Theta_2(\infty, f)}{n + 1} \]
\[ \Theta_2(c, F^*) = \Theta_2(0, G^*) \]

and
\[ \Theta_2(0, G^*) = \frac{n}{n + 1} + \frac{\Theta_2(0, g)}{n + 1} \]
We now show that \( c = 0 \) in (5.4.8). Suppose that \( c \neq 0 \), then

\[
\Theta_2(0, F^*) + \Theta_2(\infty, F^*) + \Theta_2(c, F^*)
\]

\[
= \frac{n}{n+1} + \frac{\Theta_2(0, f)}{n+1} + \frac{n}{n+1} + \frac{\Theta_2(\infty, f)}{n+1} + \frac{n}{n+1} + \frac{\Theta_2(0, g)}{n+1}
\]

\[
= \frac{3n}{n+1} + \frac{\delta_2(0, f)}{n+1} + \frac{\Theta(\infty, f)}{n+1} + \frac{\delta_2(0, g)}{n+1}
\]  

(5.4.9)

Since \( \min \{D_f, D_g\} \geq 0 \), we have

\[
D_f = 3(k+1)^2\delta_2(0, f) - (n+2)(k+1)(6+4k) + (k+1)(6+4k)\Theta(\infty, f) \geq 0
\]

\[
D_f = 3(k+1)^2\delta_2(0, f) - (n+2)(k+1)(6+4k) + (k+1)(6+4k)\Theta(\infty, f) \geq 0
\]

Therefore

\[
\delta_2(0, f) \geq \frac{(n+2)(6+4k)}{3(k+1)} - \frac{(6+4k)}{3(k+1)}\Theta(\infty, f)
\]  

(5.4.10)

\[
\delta_2(0, g) \geq \frac{(n+2)(6+4k)}{3(k+1)} - \frac{(6+4k)}{3(k+1)}\Theta(\infty, g)
\]  

(5.4.11)

Substituting (5.4.10) and (5.4.11) in (5.4.9), we get

\[
\Theta_2(0, F^*) + \Theta_2(\infty, F^*) + \Theta_2(c, F^*)
\]

\[
\geq \frac{3n}{n+1} + \frac{1}{n+1} \left\{ \frac{(n+2)(6+4k)}{3(k+1)} - \frac{(6+4k)}{3(k+1)}\Theta(\infty, f) \right\} + \frac{\Theta(\infty, f)}{n+1}
\]

\[
+ \frac{1}{n+1} \left\{ \frac{(n+2)(6+4k)}{3(k+1)} - \frac{(6+4k)}{3(k+1)}\Theta(\infty, g) \right\}
\]

\[
= \frac{3n}{n+1} + \frac{(2n+4)(6+4k)}{3(k+1)(n+1)} - \frac{(k+3)}{3(k+1)(n+1)}\Theta(\infty, f) - \frac{(6+4k)}{3(k+1)(n+1)}\Theta(\infty, g)
\]

\[
\geq \frac{3n}{n+1} + \frac{(2n+4)(6+4k)}{3(k+1)(n+1)} - \frac{(k+3)}{3(k+1)(n+1)} - \frac{(6+4k)}{3(k+1)(n+1)}
\]
Because, let

\[ Q_n = \frac{17nk + 21n + 11k + 15}{3(k+1)(n+1)} \geq 4.666.. \]

Then

\[ Q'_n = \frac{18k^2 + 36k + 18}{9(k+1)^2(n+1)^2} > 0 \text{ for } k > 0 \]

Thus \( Q_n \) is an increasing function and

\[ Q_n \text{ at } n = 1 = \frac{28k + 36}{6k + 6}, \quad \lim_{k \to \infty} \left\{ \frac{28k + 36}{6k + 6} \right\} = \frac{28}{6} = 4.666... \]

This shows that \( Q_n \) always exceeds the value 4.666...which contradicts Lemma 5.3.2.

Hence \( c = 0 \). Therefore

\[ F^* \equiv G^* \]

\[ \frac{f^{n+1}}{n+1} = \frac{g^{n+1}}{n+1} \]

Let \( h = \frac{f}{g} \). If \( h \equiv 1 \) then \( f \equiv g \).

Suppose \( h \not\equiv 1 \), then \( h^{n+1} - 1 = 0 \).

This completes the proof of Theorem.

Letting \( k \to \infty \) in the Theorem 5.4.3, we have the following result.

**Corollary 5.4.2** Let \( f \) and \( g \) be two non-constant meromorphic functions and \( \alpha_j(\neq 0, \infty) \ (j = 1, 2, 3) \) be three non-zero distinct entire small functions satisfying

\[ E(\alpha_j, f^n f') = E(\alpha_j, g^n g'), \quad j = 1, 2, 3 \]

where \( n \) is a positive integer, then either \( f \equiv g \) or \( h^{n+1} - 1 = 0 \), where \( h \) is a non-constant meromorphic function.
5.5 Uniqueness of Meromorphic Functions that Share One Small Function with its Differential Polynomials

5.6 Introduction and Results

In this section, we study the uniqueness theorems that share one small function with its differential polynomials. We prove some important results which improve the result due to Anupama Patil [33], H.X.Yi [21] and others.

In 2002, Fang and Fang [5] posed following question:

**Question A.** Whether there exists a differential polynomial $P$ such that for any pair of non-constant meromorphic functions $f$ and $g$ can we get $f \equiv g$ if $P(f)$ and $P(g)$ share one value CM?

In the same paper, Fang and Fang [5] gave positive answer to Question A and proved the following theorem.

**Theorem 5.6.1** Let $f$ and $g$ be two non-constant meromorphic functions, $k(\geq 3)$, $n(\geq 13)$ be two positive integers. If $E_k(1, f^n(f - 1)^2f') = E_k(1, g^n(g - 1)^2g')$, then $f \equiv g$.

In connection to this, in 2009, Chen, Lin and Zhang [39] proved the following result.

**Theorem 5.6.2** Let $f$ and $g$ be two transcendental meromorphic functions, let $m,n,k$ be positive integers and let $\alpha \not\equiv 0, \infty$ be a meromorphic function such that

$T(r,\alpha) = o(T(r,f))$, $T(r,\alpha) = o(T(r,g))$. If $E_k(\alpha, f^n(f - 1)^m f') = E_k(\alpha, g^n(g - 1)^m g')$ and $n > \max \{(m + 4)k^* + 6, 3m + 5\}$ then
(i) When \( m \geq 3 \),

\[
f^{n+1} \sum_{k=0}^{m} \frac{(-1)^k C_m^m}{n+m-k+1} f^{m-k} \equiv g^{n+1} \sum_{k=0}^{m} \frac{(-1)^k C_m^m}{n+m-k+1} g^{m-k},
\]

(ii) When \( m = 2 \), \( f \equiv g \);

(iii) When \( m = 1 \), either \( f \equiv g \) or

\[
g = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})} \quad \text{and} \quad f = \frac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})}
\]

where \( k^* = \max\{1, \frac{3}{k}\} \) and \( h \) is a non-constant meromorphic function.

Naturally, one can pose the following question:

**Question B.** Whether there exists a differential polynomial \( P \) such that for any pair of non-constant meromorphic functions \( f \) and \( g \) can we get \( f \equiv g \) if \( P(f) \) and \( P(g) \) share one value IM?

In this section, we improve Theorem 5.2.2 by considering one small meromorphic function instead of three non-zero distinct entire small functions. As application of this, we obtain some significant results which improve the result due to H.X.Yi (see Theorem 5.2.1) and Theorem 5.6.1 and 5.6.2 with some conditions. It is interesting to note that, the restriction on \( n \) disappears. Our results also give a positive answer to Question B.

### 5.7 Lemmas

We need the following lemma to prove our results.

**Lemma 5.7.1** [33]. Let \( f \) be a meromorphic function and let \( P(f) \) be a non-constant differential polynomial in \( f \) such that every term in \( P(f) \) contains at least one of the derivatives of \( f \). Let \( m \) be the order of the highest derivative of \( f \) occurring in \( P(f) \).
If \( k \) is a positive integer \( \leq m \) such that every term in \( P(f) \) contains at least one of \( f', f'', \ldots, f^{(k)} \), then

\[
d_k T(r, f) \leq N(r, f) + d_k N_{k+1}(r, a, f) + N\left(r, \frac{1}{P(f) - b}\right) + S(r, f),
\]

where \( a \in C, b(\neq 0) \) is a meromorphic function satisfying \( T(r, b) = S(r, P) \) and

\[
d_k = \min\left\{ \sum_{j=1}^{k} n_{ij} : 1 \leq i \leq n \right\}.
\]

### 5.8 Statement and Proofs of Main Results

**Theorem 5.8.1** Let \( f \) and \( g \) be two non-constant meromorphic functions and \( P(f) \) and \( P(g) \) be non-constant differential polynomials in \( f \) and \( g \) respectively. Let \( \alpha(z)(\neq 0, \infty) \) be a small meromorphic function of \( P(f) \) and \( P(g) \), \( k \) be a positive integer or \( \infty \) and \( n \) be a positive integer satisfying

\[
E_k(\alpha, P(f)) = E_k(\alpha, P(g)).
\]

Furthermore, let

\[
D_f = (k + 1)\delta_{m+1}(a, f) + (k + 1)\Theta(\infty, f) - (k + 1) - (2k + 1)(\bar{d} + Q(1 - \Theta(\infty, f))) \quad \text{and}
\]

\[
D_g = (k + 1)\delta_{m+1}(a, g) + (k + 1)\Theta(\infty, g) - (k + 1) - (2k + 1)(\bar{d} + Q(1 - \Theta(\infty, g)))
\]

where \( \bar{d} = \bar{d}(P(f)) = \bar{d}(P(g)) = \max\left\{ \sum_{j=0}^{m} n_{ij} : 1 \leq i \leq n \right\}, \)

\( Q = \max\{n_{i_1} + n_{i_2} + 3n_{i_3} + \cdots + mn_{i_m} : 1 \leq i \leq n\} \), \( m = \) order of the highest derivative of \( f \) occurring in \( P \). If

\[
\min\{D_f, D_g\} \geq 0, \quad (5.8.1)
\]

\[
\max\{D_f, D_g\} > 0 \quad (5.8.2)
\]

then \( P(f) \equiv P(g) \).
Proof. From the Lemma 5.7.1, we have for the meromorphic function \( f \), its differential polynomial \( P(f) \), any element \( a \in \mathbb{C} \) and \( \alpha(z) \) be a small meromorphic function of \( P(f) \),

\[
d_k T(r, f) \leq \overline{N}(r, f) + d_k N_{m+1}(r, a, f) + \overline{N}\left(r, \frac{1}{P(f) - \alpha}\right) + S(r, f)
\]

Since \( d_k \geq 1, \)

\[
T(r, f) \leq \overline{N}(r, f) + N_{m+1}(r, a, f) + \overline{N}\left(r, \frac{1}{P(f) - \alpha}\right) + S(r, f)
\]

\[
T(r, f) < (1 - \Theta(\infty, f)) T(r, f) + (1 - \delta_{m+1}(a, f)) T(r, f) + \overline{N}\left(r, \frac{1}{P(f) - \alpha}\right) + S(r, f)
\]

\[
\Rightarrow \quad (\delta_{m+1}(a, f) + \Theta(\infty, f) - 1) T(r, f) < \overline{N}\left(r, \frac{1}{P(f) - \alpha}\right) + S(r, f)
\]

By Lemma 5.3.1 and 5.3.5, we get

\[
(\delta_{m+1}(a, f) + \Theta(\infty, f) - 1) T(r, f) \leq \frac{k}{k+1} \overline{N}_k\left(r, \frac{1}{P(f) - \alpha}\right) + \frac{1}{k+1} (\overline{d}(P) + Q(1 - \Theta(\infty, f))) T(r, f) + S(r, f)
\]

\[
i.e. \quad d_f T(r, f) \leq k \overline{N}_k\left(r, \frac{1}{P(f) - \alpha}\right) + S(r, f) \tag{5.8.3}
\]

where \( d_f = (\delta_{m+1}(a, f) + \Theta(\infty, f) - 1)(k+1) - (\overline{d}(P) + Q(1 - \Theta(\infty, f))) \)

Similarly, we have

\[
d_g T(r, g) \leq k \overline{N}_k\left(r, \frac{1}{P(g) - \alpha}\right) + S(r, g) \tag{5.8.4}
\]

where \( d_g = (\delta_{m+1}(a, g) + \Theta(\infty, g) - 1)(k+1) - (\overline{d}(P) + Q(1 - \Theta(\infty, g))) \)

Adding (5.8.3) and (5.8.4) gives,

\[
d_f T(r, f) + d_g T(r, g) \leq k \left\{ \overline{N}_k\left(r, \frac{1}{P(f) - \alpha}\right) + \overline{N}_k\left(r, \frac{1}{P(g) - \alpha}\right) \right\} + S(r, g) + S(r, f) \tag{5.8.5}
\]
Since by hypothesis,
\[
\mathcal{E}_{k}(\alpha, P(f)) = \mathcal{E}_{k}(\alpha, P(g))
\]
so that
\[
N_k \left( r, \frac{1}{P(f) - \alpha} \right) = \mathcal{N}_k \left( r, \frac{1}{P(g) - \alpha} \right) = N_0^k(r, \alpha)
\]
Thus (5.8.5) becomes
\[
d_f T(r, f) + d_g T(r, g) \leq 2k N_0^k(r, \alpha) + S(r, g) + S(r, f) \tag{5.8.6}
\]
Suppose
\[
P(f) \not\equiv P(g) \tag{5.8.7}
\]
Assuming that $P(f)$ and $P(g)$ are distinct, it follows that for any $\alpha \in S(P(f)) \cap S(P(g)) - \{0, \infty\}$, each common zero of $P(f) - \alpha$ and $P(g) - \alpha$ is a zero of $P(f) - P(g)$, we have
\[
N_0^k(r, \alpha) \leq N \left( r, \frac{1}{P(f) - P(g)} \right) \leq T \left( r, \frac{1}{P(f) - P(g)} \right) \\
\leq T(r, P(f)) + T(r, P(g)) + O(1) \\
\leq (\overline{d}(P) + Q(1 - \Theta(\infty, f))) T(r, f) + S(r, f) \\
+ (\overline{d}(P) + Q(1 - \Theta(\infty, g))) T(r, g) + S(r, g) \tag{5.8.8}
\]
substituting (5.8.8) in (5.8.6), we obtain
\[
D_f T(r, f) + D_g T(r, g) \leq S(r, f) + S(r, g) \tag{5.8.9}
\]
where
\[
D_f = d_f - 2k (\overline{d}(P) + Q(1 - \Theta(\infty, f))) \\
and \quad D_g = d_g - 2k (\overline{d}(P) + Q(1 - \Theta(\infty, g)))
\]
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which is the $D_f$ and $D_g$ defined in the statement of the theorem.

From the conditions (5.8.1) and (5.8.2), the above inequality (5.8.9) is not possible. Thus our assumption (5.8.7) is false and hence we must have

$$P(f) \equiv P(g).$$

This completes the proof of Theorem.

As application of Theorem 5.8.1, we have the following results.

**Theorem 5.8.2** Let $f$ and $g$ be two non-constant meromorphic functions and $\alpha(z)(\neq 0, \infty)$ be a small meromorphic function. Let $Q(z) = a_p z^p + a_{p-1} z^{p-1} + \ldots + a_1 z + a_0$, where $a_0 \neq 0$, $a_1, \ldots, a_{p-1}, a_p \neq 0$ are complex constants and $k$ be a positive integer or $\infty$ and $p$ is a positive integer satisfying

$$E_k(\alpha(z), f^n Q(f) f') = E_k(\alpha(z), g^n Q(g) g').$$

Furthermore, let

$$D_f = (k + 1)\delta_2(0, f) + (3k + 2)\Theta(\infty, f) - (2kn + 2kp + 5k + n + p + 3)$$

and

$$D_g = (k + 1)\delta_2(0, g) + (3k + 2)\Theta(\infty, g) - (2kn + 2kp + 5k + n + p + 3)$$

If

$$\min \{D_f, D_g\} \geq 0 \quad , \quad \max \{D_f, D_g\} > 0$$

then either $f \equiv t g$ for a constant $t$ such that $t^d = 1$, where

$d = (n + p + 1, \ldots, n + p + 1 - i, \ldots, n + 1), a_{p-i} \neq 0$ for some $i = 0, 1, \ldots, p$ or $f$ and $g$ can be expressed as

$$f^{n+1} \sum_{i=0}^p \frac{a_{p-i}}{n + p - i + 1} f^{p-i} = g^{n+1} \sum_{i=0}^p \frac{a_{p-i}}{n + p - i + 1} g^{p-i}$$
Proof. Let $F = f^nQ(f)f'$, $G = g^nQ(g)g'$

and

$$F^* = a_p \frac{f^{n+p+1}}{n+p+1} + a_{p-1} \frac{f^{n+p}}{n+p} + a_{p-2} \frac{f^{n+p-1}}{n+p-1} + ... + a_1 \frac{f^{n+2}}{n+2} + a_0 \frac{f^{n+1}}{n+1} \quad (5.8.10)$$

$$G^* = a_p \frac{g^{n+p+1}}{n+p+1} + a_{p-1} \frac{g^{n+p}}{n+p} + a_{p-2} \frac{g^{n+p-1}}{n+p-1} + ... + a_1 \frac{g^{n+2}}{n+2} + a_0 \frac{g^{n+1}}{n+1} \quad (5.8.11)$$

Consider $a = 0$. We have,

$$\bar{d}\{f^nQ(f)f'\} = \bar{d}\{g^nQ(g)g'\} = n + p + 1, \quad Q = 1, \quad m = 1$$

Now the constants in Theorem 5.8.1 becomes,

$$D_f = (k + 1)\delta_2(0, f) + (k + 1)\Theta(\infty, f) - (k + 1) - (2k + 1)[(n + p + 1) + (1 - \Theta(\infty, f))]$$

$$= (k + 1)\delta_2(0, f) + (3k + 2)\Theta(\infty, f) - (k + 1) - (2kn + 2kp + 5k + n + p + 3)$$

Similarly

$$D_g = (k + 1)\delta_2(0, f) + (3k + 2)\Theta(\infty, f) - (k + 1) - (2kn + 2kp + 5k + n + p + 3)$$

By hypothesis,

$$\min\{D_f, D_g\} \geq 0 \quad , \quad \max\{D_f, D_g\} > 0$$

Thus applying Theorem 5.8.1 to $F$ and $G$, we obtain

$$f^nQ(f)f' \equiv g^nQ(g)g'$$

$$(F^*)' \equiv (G^*)'$$

then $F^* \equiv G^* + c$, where $c$ is a constant. \hfill (5.8.12)

By Lemma 5.3.4, we have

$$T(r, F^*) = (n + p + 1)T(r, f) + S(r, f)$$
\[ T(r, G^*) = (n + p + 1)T(r, g) + S(r, g) \]

By (5.8.10) and Lemma 5.3.4, we note that

\[
\begin{align*}
\mathcal{N}_2 \left( \frac{1}{F_*} \right) & = \mathcal{N}_2 \left( \frac{1}{f} \right) + \mathcal{N}_2 \left( r, \frac{1}{a_p n + p + 1 + a_{p-1} \frac{f_p}{n + p} + \ldots + a_1 \frac{f_1}{n + 2} + \frac{a_0}{n + 1}} \right) \\
& \leq \mathcal{N}_2 \left( \frac{1}{f} \right) + T \left( r, \frac{1}{a_p n + p + 1 + a_{p-1} \frac{f_p}{n + p} + \ldots + a_1 \frac{f_1}{n + 2} + \frac{a_0}{n + 1}} \right) \\
& = \mathcal{N}_2 \left( \frac{1}{f} \right) + pT(r, f) + S(r, f)
\end{align*}
\]

So,

\[
\frac{\mathcal{N}_2 \left( \frac{1}{F_*} \right)}{T(r, F^*)} \leq \frac{\mathcal{N}_2 \left( \frac{1}{f} \right)}{(n + p + 1)T(r, f) + S(r, f)} + \frac{pT(r, f)}{(n + p + 1)T(r, f) + S(r, f)}
\]

\[
= \frac{\mathcal{N}_2(r, 1/f)/T(r, f)}{n + p + 1} + \frac{p}{n + p + 1}
\]

taking \( \lim \sup \) as \( r \to \infty \), we get

\[
\lim_{r \to \infty} \frac{\mathcal{N}_2(r, 1/f)}{T(r, F^*)} \leq \lim_{r \to \infty} \frac{\mathcal{N}_2(r, 1/f)/T(r, f)}{n + p + 1} + \frac{p}{n + p + 1}
\]

\[
1 - \Theta_2(0, F^*) \leq \frac{1 - \Theta_2(0, f)}{n + P + 1} + \frac{p}{n + p + 1}
\]

or

\[
\Theta_2(0, F^*) \geq \frac{n}{n + p + 1} + \frac{\Theta_2(0, f)}{n + p + 1}
\]

Similarly,

\[
\Theta_2(0, G^*) \geq \frac{n}{n + p + 1} + \frac{\Theta_2(0, g)}{n + p + 1}
\]

we know that

\[
\mathcal{N}_2(r, F^*) = \mathcal{N}_2(r, f)
\]
so

\[
\lim_{r \to \infty} \frac{N_2(r, \frac{1}{r})}{T(r, F^*)} = \lim_{r \to \infty} \frac{N_2(r, F^*)/T(r, F^*)}{n + p + 1}
\]

\[1 - \Theta_2(\infty, F^*) = \frac{1 - \Theta_2(\infty, f)}{n + p + 1}\]

i.e. \[\Theta_2(\infty, F^*) = \frac{n + p}{n + p + 1} + \frac{\Theta_2(\infty, f)}{n + p + 1}\]

By the definition of \(\Theta_2\) and (5.8.12), we easily obtain that

\[\Theta_2(c, F^*) = \Theta_2(0, G^*)\]

We now show that \(c = 0\) in (5.8.12). Suppose that \(c \neq 0\), then

\[\Theta_2(0, F^*) + \Theta_2(\infty, F^*) + \Theta_2(0, F^*)\]

\[\geq \frac{n}{n + p + 1} + \frac{\Theta_2(0, f)}{n + p + 1} + \frac{n + p}{n + p + 1} + \frac{\Theta_2(\infty, f)}{n + p + 1} + \frac{n}{n + p + 1} + \frac{\Theta_2(0, g)}{n + p + 1}\]

\[= \frac{3n + p}{n + p + 1} + \frac{\Theta_2(0, f)}{n + p + 1} + \frac{\Theta_2(0, g)}{n + p + 1} + \frac{\Theta_2(\infty, f)}{n + p + 1}\]

\[
geq \frac{3n + p}{n + p + 1} + \frac{\delta_2(0, f)}{n + p + 1} + \frac{\delta_2(0, g)}{n + p + 1} + \frac{\Theta(\infty, f)}{n + p + 1} \tag{5.8.13}
\]

because \(\overline{N}_2(r, f) = \overline{N}(r, f) \Rightarrow \Theta_2(\infty, f) = \Theta(\infty, f)\).

Since \(\min \{D_f, D_g\} \geq 0\), we have

\[(k + 1)\delta_2(0, f) + (3k + 2)\Theta(\infty, f) - (2kn + 2kp + 5k + n + p + 3) \geq 0\]

and \[(k + 1)\delta_2(0, g) + (3k + 2)\Theta(\infty, g) - (2kn + 2kp + 5k + n + p + 3) \geq 0\]

Therefore

\[\delta_2(0, f) \geq \frac{2kn + 2kp + 5k + n + p + 3}{k + 1} - \frac{3k + 2}{k + 1} \Theta(\infty, f) \tag{5.8.14}\]
\[
\delta_2(0, g) \geq \frac{2kn + 2kp + 5k + n + p + 3}{k + 1} - \frac{3k + 2}{k + 1} \Theta(\infty, g) \quad (5.8.15)
\]

From (5.8.13)-(5.8.15), we get

\[
\Theta_2(0, F^*) + \Theta_2(\infty, F^*) + \Theta_2(c, F^*)
\]

\[
\geq \frac{3n + p}{n + p + 1} + \frac{4kn + 4kp + 10k + 2n + 2p + 6}{(n + p + 1)(k + 1)} - \frac{2k + 1}{(k + 1)(n + p + 1)} \Theta(\infty, f)
\]

\[
- \frac{3k + 2}{(k + 1)(n + p + 1)} \Theta(\infty, g)
\]

\[
\geq \frac{3n + p}{n + p + 1} + \frac{4kn + 4kp + 10k + 2n + 2p + 6}{(n + p + 1)(k + 1)} - \frac{2k + 1}{(k + 1)(n + p + 1)}
\]

\[
- \frac{3k + 2}{(k + 1)(n + p + 1)}
\]

\[
= \frac{7nk + 5kp + 5n + 5k + 3p + 2}{(k + 1)(n + p + 1)} > 2,
\]

Because, let

\[
H_k = \frac{7nk + 5kp + 5n + 5k + 3p + 2}{(k + 1)(n + p + 1)}, \quad for \quad n > 0, \quad p > 0
\]

Then

\[
H'_k = \frac{2n + 2p + 3}{(k + 1)^2(n + p + 1)} > 0, \quad for \quad n > 0, \quad p > 0.
\]

Thus \( H_k \) is an increasing function and

\[
H_k \quad at \quad \{n = 1, p = 1\} = \frac{17k + 10}{3k + 3} \quad and
\]

\[
\lim_{k \to \infty} \left\{ \frac{17k + 10}{3k + 3} \right\} = \frac{17}{3} = 5.666...
\]

which implies that \( H_k \) always exceeds the value 2, which contradicts Lemma 5.3.2.

Hence \( c = 0 \). Therefore

\[
F^* \equiv G^*
\]

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\[
\begin{align*}
&\frac{a_p f_{n+p+1}}{n+p+1} + a_{p-1} \frac{f_{n+p}}{n+p} + a_{p-2} \frac{f_{n+p-1}}{n+p-1} + \ldots + a_1 \frac{f_{n+2}}{n+2} + a_0 \frac{f_{n+1}}{n+1} \\
&= \frac{a_p g_{n+p+1}}{n+p+1} + a_{p-1} \frac{g_{n+p}}{n+p} + a_{p-2} \frac{g_{n+p-1}}{n+p-1} + \ldots + a_1 \frac{g_{n+2}}{n+2} + a_0 \frac{g_{n+1}}{n+1} \quad (5.8.16)
\end{align*}
\]

Let \( h = \frac{f}{g} \). If \( h \) is a constant, then substituting \( f = gh \) in (5.8.16), we deduce that

\[
\frac{a_p g_{n+p+1}(h_{n+p+1}-1)}{n+p+1} + \frac{a_{p-1} g_{n+p}(h_{n+p}-1)}{n+p} + \ldots + \frac{a_1 g_{n+1}(h_{n+1}-1)}{n+1} = 0
\]

which implies that \( h^d = 1 \), where \( d = \{n+p+1, n+p, \ldots, n+p+1-i, \ldots, n+1\} \), \( a_{p-i} \neq 0 \) for some \( i = 0, 1, 2, \ldots, p \). Thus \( f \equiv tg \) for a constant \( t^d = 1 \), where

\( d = \{n+p+1, n+p, \ldots, n+p+1-i, \ldots, n+1\} \), \( a_{p-i} \neq 0 \) for some \( i = 0, 1, 2, \ldots, p \).

If \( h \) is not a constant, then \( f \) and \( g \) can be expressed as

\[
f_{n+1} + \sum_{i=0}^{p} \frac{a_{p-i}}{n+p-i+1} f^{p-i} = g_{n+1} + \sum_{i=0}^{p} \frac{a_{p-i}}{n+p-i+1} g^{p-i}
\]

This completes the proof of Theorem.

As immediate consequence of Theorem 5.8.2, we have the following Corollaries.

**Corollary 5.8.1** Let \( f \) and \( g \) be two non-constant meromorphic functions and \( \alpha(z) \) \((\neq 0, \infty)\) be a small meromorphic function, \( k \) be a positive integer or \( \infty \) and \( p \) is a positive integer satisfying

\[
E_k(\alpha(z), f^n(f-1)^p f') = E_k(\alpha(z), g^n(g-1)^p g')
\]

Furthermore, let

\[
D_f = (k+1)\delta_2(0, f) + (3k+2)\Theta(\infty, f) - (2kn+2kp+5k+n+p+3)
\]

and

\[
D_g = (k+1)\delta_2(0, g) + (3k+2)\Theta(\infty, g) - (2kn+2kp+5k+n+p+3)
\]

If

\[
\min \{D_f, D_g\} \geq 0, \quad \max \{D_f, D_g\} > 0
\]

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(i) if $p = 1$ and $n$ is a positive integer, then either $f(z) \equiv g(z)$ or

$$g = \frac{(n + 2)(h^{n+1} - 1)}{(n + 1)(h^{n+2} - 1)}, \quad f = \frac{(n + 2)h(h^{n+1} - 1)}{(n + 1)(h^{n+2} - 1)},$$

where $h$ is a non-constant meromorphic function.

(ii) if $p = 2$ and $n \geq 3$, then $f(z) \equiv g(z)$.

(iii) if $p > 2$ and $n$ is a positive integer, then

$$f^{n+1} \sum_{l=0}^{p} \frac{(-1)^l C^l_p}{n+p-l+1} f^{p-l} \equiv g^{n+1} \sum_{l=0}^{p} \frac{(-1)^l C^l_p}{n+p-l+1} g^{p-l}.$$

**Corollary 5.8.2** Let $f$ and $g$ be two non-constant meromorphic functions and $\alpha(z)$

$(\not\equiv 0, \infty)$ be a small meromorphic function, $k$ be a positive integer or $\infty$ and $n$ and $p$

are positive integers satisfying

$$E_k(\alpha(z), f^{n}(f^p - 1)f') = E_k(\alpha(z), g^{n}(g^p - 1)g'),$$

Furthermore, let

$$D_f = (k + 1)\delta_2(0, f) + (3k + 2)\Theta(\infty, f) - (2kn + 2kp + 5k + n + p + 3)$$

and

$$D_g = (k + 1)\delta_2(0, g) + (3k + 2)\Theta(\infty, g) - (2kn + 2kp + 5k + n + p + 3)$$

If

$$\min\{D_f, D_g\} \geq 0, \quad \max\{D_f, D_g\} > 0$$

then either $f(z) \equiv g(z)$ or

$$g = \left[\frac{(n + p + 1)(h^{n+1} - 1)}{(n + 1)(h^{n+p+1} - 1)}\right]^{1/p}, \quad f = \left[\frac{(n + p + 1)h^p(h^{n+1} - 1)}{(n + 1)(h^{n+p+1} - 1)}\right]^{1/p}$$

where $h$ is a non-constant meromorphic function.

**Theorem 5.8.3** Let $f$ and $g$ be two non-constant meromorphic functions and $\alpha(z)$

$(\not\equiv 0, \infty)$ be a small meromorphic function, $k$ be a positive integer or $\infty$ and $n$ is positive
integer satisfying
\[ E_k(z, f^n f') = E_k(z, g^n g') \]

If
\[ \min \{ D_f, D_g \} \geq 0, \quad \max \{ D_f, D_g \} > 0 \]

where
\[ D_f = (k + 1)\delta_2(0, f) + (3k + 2)\Theta(\infty, f) - (2kn + 5k + n + 3) \]
and
\[ D_g = (k + 1)\delta_2(0, g) + (3k + 2)\Theta(\infty, g) - (2kn + 5k + n + 3) \]

then either \( f \equiv g \) or \( f \equiv tg \) for \( t^{n+1} = 1 \).

**Theorem 5.8.4** Let \( f \) and \( g \) be two non-constant meromorphic functions, \( \alpha(z)(\not\equiv 0, \infty) \) be a small meromorphic function, \( k \) be a positive integer or \( \infty \) and \( n \) is positive integer satisfying
\[ E_k(z, f^n) = E_k(z, g^n) \]

If
\[ \min \{ D_f, D_g \} \geq 0, \quad \max \{ D_f, D_g \} > 0 \]

where
\[ D_f = (k + 1)\delta_2(0, f) + (3k + 2)\Theta(\infty, f) - (5k + 3) \]
and
\[ D_g = (k + 1)\delta_2(0, g) + (3k + 2)\Theta(\infty, g) - (5k + 3) \]

then \( f(z) \equiv g(z) \).

Using the same argument as in the proof of Theorem 5.8.2, we can prove the Theorem 5.8.3 and 5.8.4.

**Note:** Theorem 5.8.4 is an improvement of the result due to H.X.Yi [21] (see Theorem 5.2.1).