Chapter 4

On the Uniqueness Problems in the Class $A$ of Meromorphic Functions
4.1 Introduction and Results

In this chapter, we study the value distribution and the uniqueness of meromorphic functions in class \( \mathcal{A} \). We obtain significant results which improve the result of Yang and Hua [42] in class \( \mathcal{A} \).

Let \( f(z) \) and \( g(z) \) are non-constant meromorphic functions and \( a \) be a finite complex number. We denote by \( \mathcal{N}_L(r,f) \) the counting function for the poles of both \( f \) and \( g \) about which \( f \) has larger multiplicity than \( g \), where multiplicity is not counted. Similarly, we have the notation for \( \mathcal{N}_L(r,g) \).

We say that \( f \) and \( g \) share a CM (counting multiplicities) if \( f-a \) and \( g-a \) have same zeros with same multiplicities and we say that \( f \) and \( g \) share a IM (ignoring multiplicities) if we do not consider the multiplicities.

We denote by \( \mathcal{A} \) the class of meromorphic functions \( f \) in \( \mathbb{C} \) which satisfy the condition \( \mathcal{N}(r,f) + \mathcal{N}(r,\frac{1}{f}) = S(r,f) \). Clearly all functions in \( \mathcal{A} \) are transcendental meromorphic functions.

Many results concerning the uniqueness and sharing values of meromorphic functions have been obtained. Authors like Q.L.Xiong [14], L.Yang [15], H.C.Xie [16], Gopalakrishna and Bhoosnurmath [17, 18], H.X.Yi [19, 20, 21] studied the uniqueness of meromorphic functions of the class \( \mathcal{A} \). But, so far much work has not been done in this direction. In this section, we prove two main results. In proving our results, we mainly use the result of Gopalakrishna and Bhoosnurmath [11]: \( T(r,P) \sim nT(r,f) \), where \( P \) is a homogeneous differential polynomial in \( f \) of degree \( n \), which plays a cardinal role. Using this technique, many of the above mentioned results are improved to a great extent.
In 1997, Yang and Hua [42] obtained following result.

**Theorem 4.1.1** Let $f$ and $g$ be two non-constant meromorphic functions, $n \geq 11$ an integer and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share the value $a$ CM, then either $f \equiv dg$ for some $(n+1)$ th root of unity $d$ or $g(z) = c_1e^{cz}$, $f(z) = c_2e^{-cz}$ where $c, c_1$ and $c_2$ are constants and satisfy $(c_1c_2)^{n+1}c^2 = -a^2$.

Theorem 4.1.1, motivate us to think that, whether there exists a similar result, if $f^n f'$ is replaced in Theorem 4.1.1 by $f^n f^{(k)}$. In this chapter, we prove significant results which improves as well as generalize Theorem 4.1.1 in class $\mathcal{A}$.

4.2 Definitions and Lemmas

We need following lemmas to prove our results.

**Definition 4.2.1** [26]. Any expression of the type

$$P(f) = \sum_{i=1}^{n} \alpha_i(z)f^{n_i_0}(f')^{n_i_1}(f'')^{n_i_2} \ldots \ldots (f^{(m)})^{n_i_m}$$

is called a differential polynomial in $f$ of degree $\overline{d}(P)$, lower degree $\underline{d}(P)$ and weight $\Gamma_P$ where for each $i = 1, 2, \ldots, n$, $n_i_0, n_i_1, \ldots, n_i_m$ are non-negative integers, $\alpha_i = \alpha_i(z)$ are meromorphic functions satisfying $T(r, \alpha_i) = S(r, f)$ and

$$\overline{d}(P) = \max \left\{ \sum_{j=0}^{m} n_{ij} : 1 \leq i \leq n \right\}, \quad \underline{d}(P) = \min \left\{ \sum_{j=0}^{m} n_{ij} : 1 \leq i \leq n \right\}$$

and

$$\Gamma_P = \max \left\{ \sum_{j=0}^{m} (j+1)n_{ij} : 1 \leq i \leq n \right\}.$$  

If $\overline{d}(P) = \underline{d}(P) = n$ (fixed integer), then $P(f)$ is called homogeneous differential polynomial of degree $n$.  

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Lemma 4.2.1 [11]. If $P$ is a homogeneous differential polynomial in $f$ of degree $n \geq 1$, then

$$ m \left( r, \frac{P}{f^n} \right) = S(r, f). $$

The following lemmas play a cardinal role in proving our theorems which are due to Gopalakrishna and Bhoosnurmath [11]. For the sake of completeness we give the proof of the following lemma :

Lemma 4.2.2 [11] Let $f$ be a meromorphic function of finite order and $P$ a homogeneous differential polynomial in $f$ of degree $n$. Let

$$ \alpha = \lim_{r \to \infty} \frac{N(r, f) + N(r, 1/f)}{T(r, f)}. $$

Then

$$ n(1 - m\alpha) \leq \lim_{r \to \infty} \frac{T(r, P)}{T(r, f)} \leq \lim_{r \to \infty} \frac{T(r, P)}{T(r, f)} \leq n(1 + m\alpha) \quad (4.2.1) $$

where $f^{(m)}$ is the highest derivative of $f$ occurring in $P$, provided that $P$ does not reduce to constant.

Proof. Since a zero or a pole of $f$, which is not a pole of any co-efficient $\alpha(z)$ of $P$, is a pole of $\frac{P}{f^n}$ of degree $mn$ at most, we have

$$ N \left( r, \frac{P}{f^n} \right) \leq mn \left[ N(r, f) + N \left( r, \frac{1}{f} \right) \right] + S(r, f) \quad (4.2.2) $$

Now,

$$ T \left( r, \frac{P}{f^n} \right) = m \left( r, \frac{P}{f^n} \right) + N \left( r, \frac{P}{f^n} \right) $$

$$ \leq mn \left[ N(r, f) + N \left( r, \frac{1}{f} \right) \right] + S(r, f) \quad , \quad (4.2.3) $$
by (4.2.2) and Lemma 4.2.1.

Therefore, by using (4.2.3)

\[ T(r, P) \leq T \left( r, \frac{P}{f^n} \right) + T(r, f^n) \]

\[ \leq mn \left[ N(r, f) + N \left( r, \frac{1}{f} \right) \right] + nT(r, f) + S(r, f) \quad (4.2.4) \]

The right inequality in (4.2.1) follows from (4.2.4).

On the other hand, using (4.2.3) we have

\[ nT(r, f) = T(r, f^n) \]

\[ \leq T \left( r, \frac{f^n}{P} \right) + T(r, P) \]

\[ = T(r, P) + T \left( r, \frac{P}{f^n} \right) + O(1) \]

\[ \leq T(r, P) + mn \left[ N(r, f) + N \left( r, \frac{1}{f} \right) \right] + S(r, f) \]

which gives the left inequality in (4.2.1).

This completes the proof of Lemma.

An immediate consequence of Lemma 4.2.2 is given below.

**Lemma 4.2.3** [11] Let \( f \) be a meromorphic function of finite order and \( P \) a homogeneous differential polynomial in \( f \) of degree \( n \). If \( \Theta(0, f) = \Theta(\infty, f) = 1 \), then

\[ T(r, P) \sim nT(r, f). \]

**Lemma 4.2.4** [12]. Let \( f_j \ (j = 1, 2, 3) \) be meromorphic functions that satisfy

\[ \sum_{j=1}^{3} f_j = 1 \]
Assume that \( f_1 \) is not a constant, and

\[
\sum_{j=1}^{3} N_2 \left( r, \frac{1}{f_j} \right) + \sum_{j=1}^{3} \mathcal{N}(r, f_j) < (\lambda + o(1)) T(r), \quad r \in I,
\]

where \( \lambda < 1 \), \( T(r) = \max \{ T(r, f_1), T(r, f_2), T(r, f_3) \} \), \( N_2 \left( r, \frac{1}{f_j} \right) \) is the counting function of zeros of \( f_j \) \((j = 1, 2, 3)\), where a multiple zero is counted two times and a simple zero is counted once. Then \( f_2 = 1 \) or \( f_3 = 1 \).

**Lemma 4.2.5** [38]. Let \( f \) be a non-constant meromorphic function. Then

\[
N \left( r, \frac{1}{f^{(k)}} \right) \leq N(r, \frac{1}{f}) + k \mathcal{N}(r, f) + S(r, f)
\]

where \( k \) is a positive integer.

**Lemma 4.2.6** [38]. Let \( F \) and \( G \) be two distinct non-constant meromorphic functions, and let \( c \) be a complex number such that \( c \neq 0, 1 \). If \( F \) and \( G \) share 1 and \( c \) IM, and if \( \mathcal{N}(r, \frac{1}{F}) + \mathcal{N}(r, F) = S(r, F) \) and \( \mathcal{N}(r, \frac{1}{G}) + \mathcal{N}(r, G) = S(r, G) \), then \( F \) and \( G \) share 0, 1, \( c, \infty \) CM.

**Lemma 4.2.7** [13]. If \( f \) and \( g \) are distinct non-constant meromorphic functions that share four values \( a_1, a_2, a_3, a_4 \) CM, then \( f \) is Mobius transformation of \( g \); two of the shared values, say \( a_1 \) and \( a_2 \) are Picard exceptional values and the cross ratio \((a_1, a_2, a_3, a_4) = -1.

**Lemma 4.2.8** [38]. If \( f(z) \in A \) and \( k \) is a positive integer, then

\[
T \left( r, \frac{f^{(k)}}{f} \right) = S(r, f).
\]

**Lemma 4.2.9** [36]. Let \( f \) be a non-constant meromorphic functions and \( a_1, a_2, a_3 \) be three distinct small meromorphic functions of \( f \), then

\[
T(r, f) \leq \sum_{j=1}^{3} \mathcal{N} \left( r, \frac{1}{f - a_j} \right) + S(r, f).
\]
Lemma 4.2.10 [36]. Suppose that $f$ is a non-constant meromorphic function, $k \geq 2$ is an integer. If
\[ N(r, f) + N(r, \frac{1}{f}) + N\left(r, \frac{1}{f^{(k)}}\right) = S\left(r, \frac{f'}{f}\right), \]
then $f = e^{ax+b}$, where $a \neq 0$, $b$ are constants.

Following lemmas play a prominent role in proving our results.

Lemma 4.2.11 Let $f, g \in \mathcal{A}$, $n \geq 2$ and $k$ be a positive integer. If $f^{n}f^{(k)}$ and $g^{n}g^{(k)}$ share 1 CM, then
\[ T(r, g) \leq \frac{n+1}{n-1} T(r, f) + S(r, g). \]

Proof. Let $G = g^{n}g^{(k)}$. Then it is a differential polynomial of degree $(n+1)$.

By Lemma 4.2.3, we have
\[ (n+1)T(r, g) \sim T(r, G) \quad (4.2.5) \]

Applying Lemma 4.2.9 to $T(r, G)$, we get
\[ (n+1)T(r, g) \leq \overline{N}(r, G) + \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) + S(r, G) \]
\[ = \overline{N}(r, g^{n}g^{(k)}) + \overline{N}\left(r, \frac{1}{g^{n}g^{(k)}}\right) + \overline{N}\left(r, \frac{1}{g^{n}g^{(k)}-1}\right) + S(r, g^{n}g^{(k)}) \]

Noting that
\[ \overline{N}(r, g^{n}g^{(k)}) \leq \overline{N}(r, g^{n}) + N(r, g^{(k)}) \]
\[ \leq \overline{N}(r, g) + N(r, g) + k\overline{N}(r, g) \]
\[ = N(r, g) + (k+1)\overline{N}(r, g), \]
and $S(r, G) = S(r, g)$, (by (4.2.5))
So,
\[(n + 1)T(r, g) \leq N(r, g) + (k + 1)\overline{N}(r, g) + \overline{N}\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g^{(k)}}\right)\]
\[+ \overline{N}\left(r, \frac{1}{g^{n}g^{(k)} - 1}\right) + S(r, g)\]

Since \(f^n f^{(k)}\) and \(g^n g^{(k)}\) share 1 CM, it implies that \(f^n f^{(k)} - 1\) and \(g^n g^{(k)} - 1\) have same zeros with the same multiplicities, using this with Lemma 4.2.5, we obtain that
\[(n + 1)T(r, g) \leq N(r, g) + (k + 1)\overline{N}(r, g) + \overline{N}\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g}\right) + k\overline{N}(r, g)\]
\[+ \overline{N}\left(r, \frac{1}{f^n f^{(k)} - 1}\right) + S(r, g)\]  \hspace{1cm} (4.2.6)

By hypothesis, we have
\[
\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) = S(r, f) \quad \text{and} \quad \overline{N}(r, g) + \overline{N}\left(r, \frac{1}{g}\right) = S(r, g).
\]

Using Nevanlinna’s first fundamental theorem and Lemma 4.2.3, we have
\[
\overline{N}(r, \frac{1}{f^n f^{(k)} - 1}) \leq T(r, \frac{1}{f^n f^{(k)} - 1}) \leq T(r, f^n f^{(k)}) + O(1)
\]
\[\sim (n + 1)T(r, f) + O(1)
\]

So,
\[
\overline{N}(r, \frac{1}{f^n f^{(k)} - 1}) \leq (n + 1)T(r, f) + O(1) \quad \text{(4.2.7)}
\]

Using (4.2.7), (4.2.6) becomes
\[(n + 1)T(r, g) \leq N(r, g) + N\left(r, \frac{1}{g}\right) + (n + 1)T(r, f) + S(r, g)\]
\[\leq 2T(r, g) + (n + 1)T(r, f) + S(r, g)\]
\[(n - 1)T(r, g) \leq (n + 1)T(r, f) + S(r, g)\]
\[T(r, g) \leq \left(\frac{n + 1}{n - 1}\right) T(r, f) + S(r, g)\]

This completes the proof of Lemma.
Lemma 4.2.12 Let $f, g \in A$, $n \geq 2$ and $k$ be a positive integer. If $f^n f^{(k)}$ and $g^n g^{(k)}$ share 1 CM, then $S(r, f) = S(r, g)$.

Proof. Proceeding as in the proof of Lemma 4.2.11, we have

$$T(r,g) \leq \left(\frac{n+1}{n-1}\right) T(r, f) + S(r,g)$$

Similarly, we have

$$T(r,f) \leq \left(\frac{n+1}{n-1}\right) T(r, g) + S(r,f)$$

using above two inequalities we easily obtain

$$S(r,f) = S(r,g).$$

This completes the proof of Lemma.

Using the method in [22], we prove the following lemma.

Lemma 4.2.13 Let $f, g \in A$, $n \geq 2$ and $k$ be a positive integer. If $f^n f^{(k)} g^n g^{(k)} = 1$, then $f = c_3 e^{p z}$ and $g = c_4 e^{-p z}$ where $c_3, c_4$ and $p$ are constants such that

$$(-1)^k (c_3 c_4)^{n+1} p^{2k} = 1.$$ 

Proof. Let $F = f^n f^{(k)}$ and $G = g^n g^{(k)}$ (4.2.8)

By Lemma 4.2.3, we have

$$T(r,F) \sim (n+1)T(r, f) , \quad T(r,G) \sim (n+1)T(r, g)$$

Clearly $S(r,F) = S(r, f)$ and $S(r,G) = S(r, g)$.

By Lemma 4.2.12, we have

$$S(r, f) = S(r, g).$$
Thus

\[ S(r, F) = S(r, f) = S(r, g) = S(r, G). \]  \hfill (4.2.9)

By hypothesis, we have

\[ f^n f^{(k)} g^n g^{(k)} = 1 \quad \text{or} \quad FG = 1 \]  \hfill (4.2.10)

From (4.2.10) and \( f \) and \( g \) are transcendental functions, it follows that

\[ N\left(r, \frac{1}{f}\right) = 0 \quad \text{and} \quad N\left(r, \frac{1}{g}\right) = 0 \]  \hfill (4.2.11)

By hypothesis, we have

\[
\overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) = S(r, f) \quad \text{and} \quad \overline{N}(r, g) + \overline{N}(r, \frac{1}{g}) = S(r, g)
\]  \hfill (4.2.12)

(4.2.10) can be expressed as

\[ f^n f^{(k)} = \frac{1}{g^n g^{(k)}} \]

so we deduce that

\[ N(r, f^n f^{(k)}) = N\left(r, \frac{1}{g^n g^{(k)}}\right) \]  \hfill (4.2.13)

Using (4.2.12), we get

\[
N(r, f^n f^{(k)}) = N(r, f^n) + N(r, f^{(k)})
\]

\[
= nN(r, f) + N(r, f) + k\overline{N}(r, f)
\]

\[
= (n + 1)N(r, f) + k\overline{N}(r, f)
\]

\[
= (n + 1)N(r, f) + S(r, f)
\]

Using this with Lemma 4.2.5 and (4.2.9), (4.2.11) and (4.2.12), (4.2.13) can be written as

\[
(n + 1)N(r, f) + S(r, f) \leq N\left(r, \frac{1}{g^n}\right) + N\left(r, \frac{1}{g^{(k)}}\right)
\]

\[
\leq (n + 1)N(r, \frac{1}{g}) + k\overline{N}(r, g) + S(r, g)
\]

\[
= S(r, g)
\]
which implies that

\[ N(r, f) = S(r, f) \]  

(4.2.14)

Similarly,

\[ N(r, g) = S(r, g) \]  

(4.2.15)

By (4.2.11), (4.2.12) and Lemma 4.2.5, we have

\[
\begin{align*}
N \left( r, \frac{1}{F} \right) &= N \left( r, \frac{1}{f^{n} f^{(k)}} \right) \\
&\leq N \left( r, \frac{1}{f^{(k)}} \right) + N \left( r, \frac{1}{f} \right) \\
&\leq N \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{f^{(k)}} \right) + kN(r, f) + S(r, f) \\
&= S(r, f)
\end{align*}
\]

Therefore

\[ N \left( r, \frac{1}{F} \right) = S(r, F) \]  

(4.2.16)

Similarly

\[ N \left( r, \frac{1}{G} \right) = S(r, G) \]  

(4.2.17)

Moreover by using (4.2.12) and (4.2.14), we have

\[
\begin{align*}
N(r, F) &= N(r, f^{n} f^{(k)}) \\
&\leq N(r, f) + N(r, f^{(k)}) \\
&\leq N(r, f) + N(r, f) + kN(r, f) \\
&= S(r, f)
\end{align*}
\]

Therefore

\[ N(r, F) = S(r, F) \]  

(4.2.18)

Similarly

\[ N(r, G) = S(r, G) \]  

(4.2.19)
It follows from (4.2.16) - (4.2.19) that

\[ N \left( r, \frac{1}{F} \right) + \overline{N}(r, F) = S(r, F), \quad N \left( r, \frac{1}{G} \right) + \overline{N}(r, G) = S(r, G) \]  \tag{4.2.20}

In view of (4.2.10), we know that \( F \) and \( G \) share 1 and \(-1\) IM, together this with (4.2.20) and Lemma 4.2.6 implies that \( F \) and \( G \) share \( 1, -1, 0, \infty \) CM, thus by Lemma 4.2.7 we get that 0 and \( \infty \) are Picard values of \( F \) and \( G \). Thus we deduce from (4.2.8) that both \( f \) and \( g \) are transcendental entire functions. By (4.2.11) we have

\[ f(z) = e^{\alpha(z)}, \quad g(z) = e^{\beta(z)} \]  \tag{4.2.21}

where \( \alpha(z) \) and \( \beta(z) \) are non-constant entire functions.

Then

\[ T \left( r, \frac{f'}{f} \right) = T \left( r, \frac{e^{\alpha} \alpha'}{e^{\alpha}} \right) = T(r, \alpha') \]

We claim that \( \alpha(z) + \beta(z) = c \), \( c \) is a constant.

From (4.2.21), we know that either \( \alpha \) and \( \beta \) are transcendental functions or both \( \alpha \) and \( \beta \) polynomials.

From (4.2.10), we have

\[
N \left( r, \frac{1}{f^{(k)}} \right) = N(r, g^n g^{(k)} f^n) \\
\leq nN(r, g) + N(r, g^{(k)}) + nN(r, f) \\
= 0
\]

From this and (4.2.10), we get

\[ N(r, f) + N \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{f^{(k)}} \right) = 0 \]
If \( k \geq 2 \), suppose that \( \alpha \) is transcendental entire function. From Lemma 4.2.10, we have \( f = e^{\alpha(z)} = e^{az+b} \), it implies that \( \alpha(z) = az + b \), a polynomial, which is a contradiction. Thus \( \alpha \) and \( \beta \) polynomials.

We deduce from (4.2.21) that

\[
f^{(k)} = \left[ (\alpha')^k + P_{k-1}(\alpha') \right] e^{\alpha} , \quad g^{(k)} = \left[ (\beta')^k + Q_{k-1}(\beta') \right] e^{\beta}
\]

where \( P_{k-1}(\alpha') \) and \( Q_{k-1}(\beta') \) are differential polynomials in \( \alpha' \) and \( \beta' \) of degree at most \((k-1)\) respectively.

Thus by (4.2.10) we obtain that

\[
\left[ (\alpha')^k + P_{k-1}(\alpha') \right] \left[ (\beta')^k + Q_{k-1}(\beta') \right] e^{(n+1)(\alpha + \beta)} = 1 \tag{4.2.22}
\]

we deduce from (4.2.22) that \( \alpha(z) + \beta(z) = c \), \( c \) is a constant.

If \( k = 1 \), from (4.2.21) we get

\[
\alpha' \beta' e^{(n+1)(\alpha + \beta)} = 1 \tag{4.2.23}
\]

Let \( \alpha + \beta = \gamma \). If \( \alpha \) and \( \beta \) are transcendental entire functions, then \( \gamma \) is not a constant and (4.2.23) implies that

\[
\alpha' (\gamma' - \alpha') e^{(n+1)\gamma} = 1 \tag{4.2.24}
\]

Since

\[
T(r, \gamma') = m(r, \gamma')
\]

\[
= m \left( r, \frac{e^{(n+1)\gamma \gamma'}}{e^{(n+1)\gamma}} \right)
\]

\[
= m \left( r, \frac{(e^{(n+1)\gamma})'}{e^{(n+1)\gamma}} \right) = S(r, e^{(n+1)\gamma})
\]

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Thus (4.2.24) implies that

\[
T(r, e^{(n+1)\gamma}) = T\left(r, \frac{1}{\alpha'(\gamma' - \alpha')}\right)
\leq T(r, \alpha'(\gamma' - \alpha')) + O(1)
\leq 2T(r, \alpha') + S(r, e^{(n+1)\gamma})
\]

which implies that

\[
T(r, e^{(n+1)\gamma}) = O(T(r, \alpha')).
\]

Thus

\[
T(r, \gamma') = S(r, \alpha')
\]

In view of (4.2.24) and by Lemma 4.2.9, we get

\[
T(r, \alpha') \leq N(r, \alpha') + N\left(r, \frac{1}{\alpha}\right) + N\left(r, \frac{1}{\alpha' - \gamma}\right) + S(r, \alpha')
\]

Since \( \alpha \) and \( \beta \) are transcendental entire functions and in view of (4.2.24), we obtain

\[
T(r, \alpha') \leq S(r, \alpha')
\]

and this implies that \( \alpha' \) is a constant, which is a contradiction. Thus \( \alpha \) and \( \beta \) are both polynomials and \( \alpha(z) + \beta(z) = c \), for a constant \( c \).

Hence from (4.2.22) we get

\[
(\alpha')^{2k} = 1 + \widetilde{P}_{2k-1}(\alpha') \tag{4.2.25}
\]

where \( \widetilde{P}_{2k-1}(\alpha') \) is a differential polynomial in \( \alpha' \) of degree at most \( 2k - 1 \).

From (4.2.25) we have

\[
2kT(r, \alpha') = T(r, (\alpha')^{2k}) = m(r, (\alpha')^{2k})
\leq m\left(r, \widetilde{P}_{2k-1}(\alpha')\right) + O(1)
\]
\[
\leq m \left( r, \frac{\tilde{P}_{2k-1}(\alpha')}{(\alpha')^{2k-1}} \cdot (\alpha')^{2k-1} \right) + O(1)
\]
\[
\leq m \left( r, \frac{\tilde{P}_{2k-1}(\alpha')}{(\alpha')^{2k-1}} \right) + m \left( r, (\alpha')^{2k-1} \right) + O(1)
\]
\[
\leq (2k-1)T(r, \alpha') + S(r, \alpha')
\]
Therefore
\[
T(r, \alpha') \leq S(r, \alpha'),
\]
which implies that \( \alpha' \) is a constant.

Thus \( \alpha = pz + c_1, \beta = -pz + c_2 \). By (4.2.21), we represent \( f \) and \( g \) as
\[
f = c_3 e^{pz}, \quad g = c_4 e^{-pz},
\]
where \( c_3, c_4 \) and \( p \) are constants such that \((-1)^k (c_3c_4)^{n+1} p^{2k} = 1\).

This completes the proof of Lemma.

In this chapter, we prove the value distribution and the uniqueness of meromorphic functions of the type \( f^n f^{(k)} \), where \( n \) and \( k \) are positive integers, in the class \( A \). It is interesting to note that for functions in class \( A \), the condition on \( n \) is greatly improved.

4.3 Statement and Proofs of Main Results

**Theorem 4.3.1** If \( f, g \in A, n \geq 2 \) and \( k \) be a positive integer. Then \( f^n f^{(k)} = 1 \) has infinitely many zeros.

**Proof.** Let \( F = f^n f^{(k)} \). By Lemma 4.2.3 and 4.2.9, we have
\[
(n + 1)T(r, f) \sim T(r, f^n f^{(k)})
\]
\[
\leq N(r, f^n f^{(k)}) + N\left(r, \frac{1}{f^n f^{(k)}}\right) + N\left(r, \frac{1}{f^n f^{(k)} - 1}\right)
\]
Noting that

\[ N(r, f^n f^{(k)}) \leq N(r, f^n) + N(r, f^{(k)}) \]
\[ \leq N(r, f) + N(r, f) + kN(r, f) \]
\[ \leq N(r, f) + (k + 1)N(r, f) \]

\[ \overline{N}(r, \frac{1}{f^{n f^{(k)}}}) \leq \overline{N}(r, \frac{1}{f^n}) + N\left(r, \frac{1}{f^{(k)}}\right) \]
\[ = \overline{N}(r, \frac{1}{f}) + N\left(r, \frac{1}{f^{(k)}}\right) \]

and

\[ (n + 1)T(r, f) \sim T(r, f^n f^{(k)}) \]
\[ S(r, f^n f^{(k)}) = S(r, f) \]

Substituting above inequalities in (4.3.1), we obtain

\[(n + 1)T(r, f) \leq N(r, f) + (k + 1)N(r, f) + \overline{N}(r, \frac{1}{f}) + N\left(r, \frac{1}{f^{(k)}}\right) + \overline{N}\left(r, \frac{1}{f^{n f^{(k)}} - 1}\right) + S(r, f) \]

using Lemma 4.2.5, we get

\[(n + 1)T(r, f) \leq N(r, f) + (k + 1)N(r, f) + \overline{N}(r, \frac{1}{f}) + N\left(r, \frac{1}{f^{(k)}}\right) + kN(r, f) + \overline{N}\left(r, \frac{1}{f^{n f^{(k)}} - 1}\right) + S(r, f) \]

By hypothesis, we have

\[ \overline{N}(r, f) = S(r, f), \quad \overline{N}\left(r, \frac{1}{f}\right) = S(r, f). \]

Therefore (4.3.2) becomes,

\[(n + 1)T(r, f) \leq N(r, f) + N\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^{n f^{(k)}} - 1}\right) + S(r, f) \]
\[ \leq 2T(r, f) + \overline{N}\left(r, \frac{1}{f^{n f^{(k)}} - 1}\right) + S(r, f) \]

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\[i.e., \quad (n - 1)T(r, f) \leq \overline{N}\left(r, \frac{1}{f^{n}f^{(k)} - 1}\right) + S(r, f)\]

which implies that \(f^{n}f^{(k)} - 1\) has infinitely many zeros for \(n \geq 2\).

This completes the proof of Theorem.

**Theorem 4.3.2** Let \(f, g \in \mathcal{A}, \ n \geq 5\) and \(k\) be a positive integer. If \(f^{n}f^{(k)}\) and \(g^{n}g^{(k)}\) share 1 CM, then either \(f \equiv tg\) for a constant \(t\) such that \(t^{n+1} = 1\) or \(f = c_{3}e^{pz}\), \(g = c_{4}e^{-pz}\) where \(c_{3}, c_{4}\) and \(p\) are constants such that \((-1)^{k}(c_{3}c_{4})^{n+1}p^{2k} = 1\).

**Proof.** By hypothesis, \(f^{n}f^{(k)}\) and \(g^{n}g^{(k)}\) share 1 CM.

Let

\[H(z) = \frac{f^{n}f^{(k)} - 1}{g^{n}g^{(k)} - 1}\]  \hspace{1cm} (4.3.3)

Then \(H(z)\) is a meromorphic function satisfying \(T(r, H) = O(T(r, f) + T(r, g))\), by the first fundamental theorem and Lemma 4.2.3.

From (4.3.3), we see that the zeros and poles of \(H(z)\) are multiple and satisfy

\[\overline{N}(r, H) \leq \overline{N}_{L}(r, f) \quad , \quad \overline{N}\left(r, \frac{1}{H}\right) \leq \overline{N}_{L}(r, g)\]  \hspace{1cm} (4.3.4)

Let

\[f_{1} = f^{n}f^{(k)} \quad , \quad f_{2} = -Hg^{n}g^{(k)} \quad , \quad f_{3} = H\]  \hspace{1cm} (4.3.5)

Then using (4.3.3), we easily see that

\[f_{1} + f_{2} + f_{3} = f^{n}f^{(k)} - Hg^{n}g^{(k)} + H = f^{n}f^{(k)} - H\left(g^{n}g^{(k)} - 1\right) = f^{n}f^{(k)} - \left(\frac{f^{n}f^{(k)} - 1}{g^{n}g^{(k)} - 1}\right)(g^{n}g^{(k)} - 1) = 1.\]
Assuming that $f_1$ is non-constant and by Lemma 4.2.4, we have

\[
\sum_{j=1}^{3} N_2 \left( r, \frac{1}{f_j} \right) + \sum_{j=1}^{3} N(r, f_j) = N_2 \left( r, \frac{1}{f_1} \right) + N_2 \left( r, \frac{1}{f_2} \right) + N_2 \left( r, \frac{1}{f_3} \right) + N(r, f_1) + N(r, f_2) + N(r, f_3)
\]

\[
\leq N_2 \left( r, \frac{1}{f_1} \right) + N_2 \left( r, \frac{1}{g_1} \right) + N_2 \left( r, \frac{1}{H} \right) + N(r, f_1) + N(r, f_2) + N(r, f_3)
\]

\[
+ N(r, f_2) + N(r, f_3) + N(r, H) \quad (4.3.6)
\]

Noting that

\[
N(r, f^{(k)}) \leq N(r, f) + (k+1)N(r, f)
\]

\[
N(r, g^{(k)}) \leq N(r, g) + (k+1)N(r, g)
\]

Using this with (4.3.4) and Lemma 4.2.5, (4.3.6) becomes

\[
\sum_{j=1}^{3} N_2 \left( r, \frac{1}{f_j} \right) + \sum_{j=1}^{3} N(r, f_j) \leq 2N_2 \left( r, \frac{1}{f} \right) + N_2 \left( r, \frac{1}{f(k)} \right) + 2N_2 \left( r, \frac{1}{g(k)} \right) + N_2 \left( r, \frac{1}{g} \right) + N(r, f) + N(r, g) + (k+1)N(r, f) + N(r, g) + N(r, H)
\]

\[
\leq 2N_2 \left( r, \frac{1}{f} \right) + N_2 \left( r, \frac{1}{f} \right) + kN(r, f) + 2N_2 \left( r, \frac{1}{g} \right) + N_2 \left( r, \frac{1}{g} \right) + kN(r, g) + 2N_2 \left( r, \frac{1}{f} \right) + N(r, f) + (k+1)N(r, f) + N(r, g) + (k+1)N(r, g) + N_2 \left( r, \frac{1}{f} \right) + N_2 \left( r, \frac{1}{f} \right) + N_2 \left( r, \frac{1}{f} \right) + N_2 \left( r, \frac{1}{f} \right)
\]

\[
+ (2k+1) (N(r, f) + N(r, g)) + (N(r, f) + N(r, g)) + 2N_2 \left( r, \frac{1}{f} \right) + N_2 \left( r, \frac{1}{f} \right) + N_2 \left( r, \frac{1}{f} \right) + N_2 \left( r, \frac{1}{f} \right)
\]

\[
+ 2N_2 \left( r, \frac{1}{f(n)} \right) + N_2 \left( r, \frac{1}{f(n)} \right) = 2 \left( N_2 \left( r, \frac{1}{f} \right) + N_2 \left( r, \frac{1}{g} \right) \right) + \left( N \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{g} \right) \right)
\]

\[
+ (2k+1) (N(r, f) + N(r, g)) + (N(r, f) + N(r, g)) + 2N_2 \left( r, \frac{1}{f} \right) + N_2 \left( r, \frac{1}{f} \right) + N_2 \left( r, \frac{1}{f} \right) + N_2 \left( r, \frac{1}{f} \right)
\]

\[
+ 2N_2 \left( r, \frac{1}{f(n)} \right) + N_2 \left( r, \frac{1}{f(n)} \right) = 2 \left( N_2 \left( r, \frac{1}{f} \right) + N_2 \left( r, \frac{1}{g} \right) \right) + \left( N \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{g} \right) \right)
\]

\[
+ (2k+1) (N(r, f) + N(r, g)) + (N(r, f) + N(r, g)) + 2N_2 \left( r, \frac{1}{f} \right) + N_2 \left( r, \frac{1}{f} \right) + N_2 \left( r, \frac{1}{f} \right) + N_2 \left( r, \frac{1}{f} \right)
\]

\[
+ 2N_2 \left( r, \frac{1}{f(n)} \right) + N_2 \left( r, \frac{1}{f(n)} \right) = 2 \left( N_2 \left( r, \frac{1}{f} \right) + N_2 \left( r, \frac{1}{g} \right) \right) + \left( N \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{g} \right) \right)
\]

\[
+ (2k+1) (N(r, f) + N(r, g)) + (N(r, f) + N(r, g)) + 2N_2 \left( r, \frac{1}{f} \right) + N_2 \left( r, \frac{1}{f} \right) + N_2 \left( r, \frac{1}{f} \right) + N_2 \left( r, \frac{1}{f} \right)
\]
Since $f, g \in A$, we have

\[ N(r, f) + N(r, \frac{1}{f}) = S(r, f) \quad \text{and} \quad N(r, g) + N(r, \frac{1}{g}) = S(r, g). \]

Therefore

\[ \sum_{j=1}^{3} N_2 \left( r, \frac{1}{f_j} \right) + \sum_{j=1}^{3} N(r, f_j) \leq \left( N \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{g} \right) \right) + (N(r, f) + N(r, g)) \]

\[ + 2N_L(r, g) + N_L(r, f) + S(r, f) + S(r, g) \]

(4.3.7)

Noting that

\[ 2N_L(r, g) + N_L(r, f) \leq 2N(r, f) = S(r, f) \]

or

\[ 2N_L(r, g) + N_L(r, f) \leq 2N(r, g) = S(r, g), \]

Thus (4.3.7) becomes

\[ \sum_{j=1}^{3} N_2 \left( r, \frac{1}{f_j} \right) + \sum_{j=1}^{3} N(r, f_j) \leq 2(T(r, f) + T(r, g)) + S(r, f) + S(r, g) \]

using Lemma 4.2.11 and 4.2.12, we get

\[ \sum_{j=1}^{3} N_2 \left( r, \frac{1}{f_j} \right) + \sum_{j=1}^{3} N(r, f_j) \leq 2T(r, f) + 2(n+1) \frac{n}{n-1} T(r, f) + S(r, f) \]

\[ = \frac{4n}{(n-1)} T(r, f) + S(r, f) \]

\[ \leq \frac{4n}{(n-1)(n+1)} T(r) + S(r, f) \]

\[ \leq \left( \frac{4n}{(n-1)(n+1)} + o(1) \right) T(r), \]

Since $n \geq 5$, \( \frac{4n}{(n-1)(n+1)} < 1 \), using Lemma 4.2.4, we get $F_2 = 1$ or $F_3 = 1$.

Next we consider two cases:
Case 1. $F_2 = 1$ i.e, $-Hg^n g^{(k)} = 1$

Using (4.3.3) we have

$$\frac{f^n f^{(k)} - 1}{g^n g^{(k)} - 1} g^n g^{(k)} = 1$$

by simple computing, we get

$$f^n f^{(k)} g^n g^{(k)} = 1.$$

By Lemma 4.2.13, we get the conclusion of Theorem 4.3.2.

Case 2. $F_3 = 1$ i.e, $H = 1$

Using (4.3.3), we have

$$\frac{f^n f^{(k)} - 1}{g^n g^{(k)} - 1} = 1$$

i.e, $f^n f^{(k)} = g^n g^{(k)}$ \hspace{1cm} (4.3.8)

By Lemma 4.2.3, we have

$$T(r, f^n f^{(k)}) = T(r, g^n g^{(k)})$$

$$(n + 1)T(r, f) = (n + 1)T(r, g)$$

$$T(r, f) = T(r, g)$$ \hspace{1cm} (4.3.9)

and also

$$S(r, f) = S(r, g)$$ \hspace{1cm} (4.3.10)

Let $h = \frac{g}{f}$. Then by (4.3.8), we have

$$h^n = \frac{f^{(k)}}{g^{(k)}} \hspace{1cm} h^{n+1} = \frac{g f^{(k)}}{g^{(k)}}$$
Suppose that \( h \) is not a constant. By (4.3.9), we have

\[
T(r, h) = T\left(r, \frac{g}{f}\right) \\
\leq T(r, g) + T(r, f) + O(1) \\
\leq 2T(r, f) + O(1)
\]

which implies that

\[
S(r, h) = S(r, f).
\]

Similarly

\[
S(r, h) = S(r, g).
\]

Thus, by (4.3.10)

\[
S(r, h) = S(r, f) = S(r, g).
\]

By the first fundamental theorem and Lemma 4.2.8, we have

\[
T(r, h^{n+1}) = T\left(r, \frac{gf^{(k)}}{fg^{(k)}}\right) \\
(n + 1)T(r, h) \leq T\left(r, \frac{f^{(k)}}{f}\right) + T\left(r, \frac{g}{g^{(k)}}\right) + O(1) \\
= T\left(r, \frac{f^{(k)}}{f}\right) + T\left(r, \frac{g^{(k)}}{g}\right) + O(1) \\
= S(r, f) + S(r, g) \\
= S(r, h)
\]

which is a contradiction since \( n \geq 5 \). Therefore \( h \) is a constant. Since \( f \) and \( g \) are transcendental meromorphic functions, we have \( h \neq 0 \).
Let $t = \frac{1}{h}$, which implies that $f = tg$. From (4.3.8), we obtain $t^{n+1} = 1$.

This completes the proof of the Theorem.

**Remark 4.3.1** Theorem 4.3.2 is an improvement of Theorem 4.1.1 in class $A$.

### 4.4 Open Problem

In this section, we pose the following open problem.

**Problem**: Does Theorem 4.3.2 hold without the condition $N(r,f) + N \left( r, \frac{1}{f} \right) = S(r,f)$ and $N(r,g) + N \left( r, \frac{1}{g} \right) = S(r,g)$?