Chapter 2

Zero divisor graphs of lattices and primal ideals

The paper titled “Zero divisor graphs of lattices and primal ideals” based on the text of this chapter has been published in the journal Asian-Eur. J. Math 5 (2012), 1250037-1250047.
2.1 Introduction

In this chapter, we introduce the concepts of primal and weakly primal ideals in lattices. Further, the diameter of the zero divisor graph of a lattice with respect to a non-primal ideal is determined in terms of number of minimal prime ideals.

2.2 Primal ideals and weakly primal ideals

We recall necessary concepts and terminology.

Definition 2.2.1. A subset $I$ of a lattice $L$ is called semi-ideal, if for $a \in I$, $b \in L$, $b \leq a$ implies $b \in I$. A semi-ideal $I$ of a lattice $L$ is called an ideal of $L$ if $a, b \in I$ implies $a \lor b \in I$. Dually, we have the concepts of semi-filter and filter. A proper semi-ideal (ideal) $I$ of a lattice $L$ is called prime, if $a, b \in L$ and $a \land b \in I$ imply $a \in I$ or $b \in I$. A prime ideal $P$ of a lattice $L$ is said to be minimal prime ideal belonging to an ideal $I$, if $I \subseteq P$ and there exists no prime ideal $Q$ such that $I \nsubseteq Q \subseteq P$. The set of all prime ideals in a lattice $L$ is denoted by $\text{Spec}(L)$. The set of minimal prime ideals belonging to $I$ is denoted by $\text{Min}(I)$.

For an ideal $I$ and a non-empty subset $A$ of a lattice $L$, define a subset $I : A$ of $L$ as follows: $I : A = \{ z \in L \mid z \land a \in I ; \forall a \in A \}$.

If $A = \{ x \}$ then we write $I : x$ instead of $I : \{ x \}$. Note that, if $x \leq y$ for $x, y \in L$, then $I : y \subseteq I : x$. Observe that $I \subseteq I : A$ and $I : A = \bigcap_{x \in A} I : x$. Note that $I : A$ need not be an ideal but it is a semi-ideal. Moreover, if $I = (0)$ then $I : x$ is nothing but
2.2 Primal ideals and weakly primal ideals

Ann(x) = \{y|y \land x = 0\}.

Now, we define the concept of a primal ideal analogues to primal ideals in rings introduced by Fuchs [13] (see also Atani [7]).

**Definition 2.2.2.** Let I be an ideal of a lattice L. A subset J of L is called prime to I if I : J = I. In particular, if J = \{a\}, then we say that a \in L is prime to I if I = I : a. Further, we denote by S(I), the set of elements of L that are not prime to I.

A proper ideal I of a lattice L is said to be primal if S(I) forms an ideal, more specifically I is called P-primal, where P = S(I).

**Lemma 2.2.3.** If I is a proper ideal of a lattice L, then S(I) is a prime semi-ideal containing I.

**Proof.** It is clear that S(I) is a semi-ideal of L containing I. Let a \land b \in S(I) and a, b \notin S(I). Suppose r \in I : a \land b, so r \land a \land b \in I, hence r \land a \in I : b = I, as b \notin S(I) which further yields r \in I : a = I. As r is arbitrary, I = I : (a \land b), a contradiction to a \land b \in S(I). □

![Diagram](image.png)

Figure 2.2.1: (c] is a non-primal ideal whereas (d] is a primal ideal.
Remark 2.2.4. Note that $S(I)$ is a semi-ideal, but need not be an ideal in general. Consider the lattice $L$ depicted in Figure 2.2.1 on Page 21. For $I = (d)$, $S(I) = (d)$ which is an ideal. But if we take $J = (c)$, then $S(J) = \{a, b, c, d, e, f, 0\}$ which is not an ideal. Note that 1 (whenever exists) is the only element of $L$ which is prime to any ideal $K$ and hence in particular, to an ideal $J$.

We now obtain a result regarding $S(I)$ to be union of minimal primes containing $I$. For this purpose we need the following result which is obtained by Joshi and Mundlik [27]. For the sake of completeness, we provide the proof of the same.

Theorem 2.2.5 (Joshi and Mundlik [27]). Let $P$ be a prime ideal and $I$ be a semiprime ideal of a lattice $L$. Then the following statements are equivalent.

a) $P$ is a minimal prime ideal belong to $I$.

b) For each $x \in P$, there exists $y \not\in P$ such that $x \land y \in I$.

c) Exactly one of $(x)$ or $I : x$ is contained in $P$.

Proof. a) $\Rightarrow$ b): Let $P$ be a minimal prime ideal and let $I$ be a semiprime ideal of $L$. It is easy to prove that $L \setminus P$ is a filter maximal with respect to the property that $(L \setminus P) \cap I = \emptyset$. Let $x \in P$. Then $I \cap ((L \setminus P) \lor [x]) \neq \emptyset$. Thus there exists $y \not\in P$ such that $x \land y \in I$.

b) $\Rightarrow$ c): For any $x \in P$, let $I : x \subseteq P$. By the hypothesis, there exists $y \not\in P$ such that $x \land y \in I$. Thus $y \in I : x \subseteq P$, a contradiction.

c) $\Rightarrow$ a): Let $Q$ be a prime ideal such that $I \subseteq Q \not\subseteq P$. Then there exists $x \in P$ such that $x \not\in Q$. By the hypothesis, $I : x \not\subseteq P$. Hence
there is \( y \in I : x \) such that \( y \notin P \). But then \( x \land y \in I \subseteq Q \) and \( x \notin Q \) give \( y \in Q \), a contradiction to \( y \notin P \).

The following result is due to Rav [13].

**Theorem 2.2.6** (Rav [13]). Let \( L \) be a lattice and \( I \) be a semiprime ideal of \( L \). Then \( I \) can be represented as intersection of prime ideals belonging to \( I \). Essentially, \( I \) is the intersection of minimal prime ideals belonging to \( I \).

**Lemma 2.2.7.** Let \( I \) be a proper semiprime ideal of \( L \). Then \( S(I) = \bigcup_{Q \in \text{Min}(I)} Q \).

*Proof.* Let \( x \in S(I) \). If \( x \in I \) then we are through. Suppose \( x \in S(I) \) but \( x \notin I \). By Lemma 2.2.19, we have \( x \in Z_I(L)^* = S(I) \setminus I \). Therefore there exists \( y \notin I \) such that \( x \land y \in I \). We claim that \( x \in P \), for some minimal prime ideal \( P \) belonging to \( I \).

Suppose on the contrary that \( x \notin P \) for every minimal prime ideal \( P \) belonging to \( I \). Therefore by Theorem 2.2.5, \( I : x \subseteq P \), for every \( P \). Now \( y \in I : x \subseteq P \), for every \( P \). Therefore \( y \in P \), for every minimal prime ideal \( P \) belonging to \( I \). Therefore \( y \in \bigcap_{P \in \text{Min}(I)} P = I \), a contradiction to \( y \notin I \).

Conversely, assume that \( x \in P \), for some minimal prime ideal \( P \) belonging to \( I \). If \( x \in I \), then \( x \in S(I) \), as \( I \subseteq S(I) \). So we can assume that \( x \notin I \). Since \( P \) is a minimal prime ideal belonging to \( I \) and \( x \in P \), by Theorem 2.2.5, there exists \( y \notin P \) such that \( x \land y \in I \). Thus \( x \land y \in I \) for \( y \notin I \) (since \( I \subseteq P \)). Hence \( x \in S(I) \). \( \square \)
Lemma 2.2.8. Let $I$ be a proper ideal of a lattice $L$. Then $I$ is prime if and only if $S(I) = I$.

Proof. Let $I$ be a prime ideal. By Lemma 2.2.3, $I \subseteq S(I)$. Let $a \in S(I)$. Then $I \nsubseteq I : a$. Hence there exists $x \in I : a$ such that $x \notin I$. This together with primeness of $I$ gives $a \in I$. Thus $S(I) = I$.

Conversely, assume that $S(I) = I$ and $I$ is not prime. Hence there exist $a, b \notin I$ such that $a \land b \in I$.

But then $a, b \in S(I) = I$, a contradiction. \qed

The following example shows that $S(I)$ is an ideal though $I$ need not be prime.

Example: Consider $\mathbb{N}$, the set of natural numbers. Consider the set $L = \{ X \subseteq \mathbb{N} \mid |X| \text{ is finite} \} \cup \{ \mathbb{N} \}$. Then $L$ forms a lattice under set inclusion with the greatest element $\mathbb{N}$. Consider the non-prime ideal $I = (0]$ then $S(I)$, the set of elements of $L$ which are not prime to $I$, is $L \setminus \{ \mathbb{N} \}$ which is an ideal.

Now, we observe some properties of $P$-primal ideals.

Definition 2.2.9. An ideal $I$ of a lattice $L$ is said to be meet irreducible if for ideals $J$ and $K$ of $L$, $I = J \cap K$ implies that either $I = J$ or $I = K$.

Lemma 2.2.10. Let $L$ be a lattice, then every meet irreducible ideal of $L$ is $P$-primal.

Proof. Assume that $I$ is a meet irreducible ideal of $L$. Let $a, b \in S(I)$, i.e., $I \nsubseteq (I : a)$ and $I \nsubseteq (I : b)$. Since $I$ is an irreducible ideal, $I \nsubseteq (I : a) \cap (I : b) \subseteq (I : a \lor b)$ hence $a \lor b \in S(I)$. If $r \in L$ and
a ∈ S(I), then \( I \subseteq (I : a) \subseteq (I : r \land a) \). Therefore \( r \land a \in S(I) \). Thus \( S(I) \) is an ideal. By Lemma 2.2.8, \( S(I) \) is a prime ideal and hence \( I \) is \( P \)-primal.

**Definition 2.2.11.** For an ideal \( I \) and a prime ideal \( P \) of a lattice \( L \), we use the notation \( I(P) = \{ x \in L | x \land y \in I \text{ for some } y \in L \setminus P \} \).

**Lemma 2.2.12.** Let \( I \) be an ideal and \( P \) be a prime ideal of a lattice \( L \). Then we have \( I(P) = \bigcup_{z \notin P}(I : z) \).

**Proof.** Let \( x \in I(P) \). Then \( x \land y \in I \), for some \( y \notin P \). Therefore \( x \in I : y \subseteq \bigcup_{y \notin P} I : y \). Hence \( I(P) \subseteq \bigcup_{z \notin P}(I : z) \).

Conversely, let \( x \in \bigcup_{z \notin P}(I : z) \), i.e., \( x \in (I : z) \) for some \( I : z \) and \( z \notin P \), i.e., \( x \land z \in I \) for some \( z \notin P \), i.e., \( x \in I(P) \).

Therefore \( \bigcup_{z \notin P}(I : z) \subseteq I(P) \). Hence the equality holds.

**Lemma 2.2.13.** Let \( I \) be an ideal and \( P \) be a prime ideal of a lattice \( L \). Then \( x \in I(P) \) if and only if \( I : x \nsubseteq P \).

**Proof.** Let \( I : x \nsubseteq P \). Then there exists \( t \in I : x \) such that \( t \notin P \), i.e., \( t \land x \in I \) for \( t \notin P \), i.e., \( x \in I(P) \).

Conversely, let \( x \in I(P) \) and suppose \( I : x \subseteq P \). Then \( x \land y \in I \) for some \( y \notin P \). Therefore \( y \in I : x \subseteq P \). Hence \( y \in P \), a contradiction.

**Lemma 2.2.14.** Consider a lattice \( L \) with ideal \( I \) and prime ideal \( P \) then for any element \( x \in L \) we have \( (I : x)(P) = I(P) : x \).

**Proof.** Evidently, \( y \in (I : x)(P) \). Therefore there exists \( c \notin P \) such that \( c \land y \in (I : x) \), i.e., there exists \( c \notin P \) with \( c \land x \land y \in I \). Therefore \( x \land y \in I(P) \). Therefore \( y \in I(P) : x \).
Conversely, let $z \in I(P) : x$, then we get $z \land x \in I(P)$. Therefore $z \land x \land c \in I$ for some $c \notin P$, i.e., $z \land c \in (I : x)$ for $c \notin P$. Therefore $z \in (I : x)(P)$ for $c \notin P$. Hence equality holds. 

**Lemma 2.2.15.** Let $I$ be an ideal and $P$ be a prime ideal of a lattice $L$. Then the following statements hold.

(a) $I \subseteq I(P)$. 

(b) $I \subseteq J$ implies that $I(P) \subseteq J(P)$. 

(c) $(I \cap J)(P) = I(P) \cap J(P)$. 

(d) If $P'$ is a prime ideal containing $P$. Then $I(P') \subseteq I(P)$. 

**Proof.** (a) Obvious. 

(b) Let $x \in I(P)$, i.e., $x \land y \in I$ for some $y \notin P$. By the assumption we have $I \subseteq J$, therefore $x \land y \in I \subseteq J$, i.e., $x \land y \in J$ for $y \notin P$. Therefore $x \in J(P)$. 

(c) Since $I \cap J \subseteq I$, by applying property (b) we get $(I \cap J)(P) \subseteq I(P)$. Similarly we get $(I \cap J)(P) \subseteq J(P)$. Therefore $(I \cap J)(P) \subseteq I(P) \cap J(P)$. 

Conversely, let $x \in I(P) \cap J(P)$. Then $x \land y \in I$ for some $y \notin P$ and $x \land z \in J$ for some $z \notin P$. Since $I$ is an ideal and $P$ is prime, we have $x \land (y \land z) \in I \cap J$ for $y \land z \notin P$. Therefore $x \in (I \cap J)(P)$. 

(d) Let $x \in I(P')$, i.e., $x \land y \in I$ for some $y \notin P'$. But $P \subseteq P'$, therefore we have $x \land y \in I$ for $y \notin P$, i.e., $x \in I(P)$. Therefore $I(P') \subseteq I(P)$. 

**Lemma 2.2.16.** Let $I$ be a $P$-primal ideal of a lattice $L$. Then $I(P') = I$ for every prime ideal $P'$ containing $P$. In particular $I(P) = P$. 

**Proof.** We know that $I \subseteq I(P')$. Assume that $I \subsetneq I(P')$, i.e., there exists $x \in I(P')$ such that $x \notin I$. By Lemma 2.2.15, we have $x \in I(P)$. 

This gives $x \wedge y \in I$ for some $y \notin P$. But then $y \in S(I) = P$, a contradiction. \qed

**Lemma 2.2.17.** Let $L$ be a lattice with 1 and $I$ be a semiprime $P$-primal ideal. If $I(Q)$ is a $Q$-primal ideal for minimal prime ideal $Q$ containing $I$ then $Q \subseteq P$

**Proof.** First we prove that $S(I(Q)) = Q = \bigcup_{x \in L \setminus I(Q)} I(Q) : x$. Suppose $t \in S(I(Q))$, i.e., $t \wedge s \in I(Q)$ for some $s \notin I(Q)$. Therefore $t \in I(Q) : s$ for some $s \notin I(Q)$. Therefore $t \in \bigcup_{x \in L \setminus I(Q)} I(Q) : x$.

Conversely, suppose $t \in \bigcup_{x \in L \setminus I(Q)} I(Q) : x$, i.e., there is some $I(Q) : x$ such that $t \in I(Q) : x$ for $x \in L \setminus I(Q)$. Therefore $t \wedge x \in I(Q)$ for $x \notin I(Q)$. Therefore $t \in S(I(Q))$.

Since $I$ is semiprime ideal and $Q$ is minimal prime ideal containing $I$, then by Theorem 2.2.5, we can prove that $I(Q) = Q = S(I(Q))$. Therefore we get $S(I(Q)) = Q = \bigcup_{x \in L \setminus I(Q)} I(Q) : x$. If $Q \nsubseteq P$ then there exists $q \in Q \setminus P$ such that $q \wedge x \in I(Q)$ for some $x \notin I(Q)$. Thus there exists $c \notin Q$ such that $c \wedge q \wedge x \in I$. Now $q \notin P$, by applying Lemma 2.2.10, we have $I = I(P)$, hence $c \wedge x \in I(P) = I$. Since $c \notin Q$ and $c \wedge x \in I$. So $x \in I(Q)$, contrary to the assumption. Hence we conclude that $Q \subseteq P$. \qed

**Definition 2.2.18.** Let $I$ be an ideal of a lattice $L$. We define the set $Z_I(L)^* = \{r \notin I \mid r \wedge a \in I \text{ for some } a \notin I\}$.

**Lemma 2.2.19.** Let $I$ be a proper ideal of a lattice $L$. Then $Z_I(L)^* = S(I) \setminus I$. In particular, $Z_I(L)^* \cup I = S(I)$. 
Proof. Let \( x \in Z_I(L)^* \), then \( x \notin I \) and \( x \land y \in I \) for some \( y \notin I \). Hence \( x \in S(I) \setminus I \). Thus, \( Z_I(L)^* \subseteq S(I) \setminus I \). Let \( a \in S(I) \setminus I \). Then there exists \( y \notin I \) such that \( a \land y \in I \), which yields \( a \in Z_I(L)^* \).

\[ \]

Lemma 2.2.20. Let \( I \) and \( P \) be proper ideals of a lattice \( L \) and \( I \subseteq P \). Then \( I \) is a \( P \)-primal ideal of \( L \) if and only if \( Z_I(L)^* = P \setminus I \).

Proof. If \( I \) is a \( P \)-primal ideal of \( L \), then \( Z_I(L)^* = S(I) \setminus I = P \setminus I \), by Lemma 2.2.19. Conversely, assume that \( Z_I(L)^* = P \setminus I \). It suffices to show that \( P \) is exactly the set of elements of \( L \) that are not prime to \( I \). Since every element of \( I \) is not prime to \( I \), hence we can assume that \( x \in P \setminus I = Z_I(L)^* \). Then there exists \( y \notin I \) such that \( x \land y \in I \). So \( x \) is not prime to \( I \), that is, \( P \setminus I \subseteq S(I) \). Taking union with \( I \), we get \( P \subseteq S(I) \).

Next, suppose that \( s \in S(I) \). If \( s \in I \), then \( s \in P \), as \( I \subseteq P \). If \( s \notin I \) and as \( s \in S(I) \), there is an element \( t \notin I \) such that \( s \land t \in I \). So \( s \in Z_I(L)^* = P \setminus I \subseteq P \). Hence \( S(I) \subseteq P \). Thus \( S(I) = P \), that is, \( I \) is a \( P \)-primal ideal of \( L \).

\[ \]

Corollary 2.2.21. Let \( I \) be an ideal of a lattice \( L \). Then \( I \) is a \( P \)-primal ideal of \( L \) if and only if \( Z_I(L)^* \cup I \) is a (prime)ideal.

Theorem 2.2.22. Let \( I \) and \( J \) be two proper non-prime ideals of a lattice \( L \). Then \( Z_I(L)^* = Z_J(L)^* \) if and only if \( I = J \).

Proof. Let \( Z_I(L)^* = Z_J(L)^* \) and assume on the contrary that \( I \neq J \). Then without loss of generality, there exists \( x \in I \) such that \( x \notin J \). Since \( I \) and \( J \) are non-prime, \( Z_I(L)^* \) and \( Z_J(L)^* \) are non-empty. Let \( a \in Z_I(L)^* = Z_J(L)^* \). Hence, for \( a \in Z_J(L)^* \), there exists \( b \in Z_J(L)^* \).
such that $a \land b \in J$ and since $x \in I$, we have $x \land a$, $x \land b \in I$. We claim that $x \land a \notin J$ and $x \land b \notin J$.

If possible, $x \land a$ or $x \land b \in J$. Then, as $x \notin J$, $a \in Z_J(L)^*$ and without loss of generality assume that $x \land a \in J$, therefore $x \in Z_J(L)^* = Z_I(L)^*$, a contradiction to $x \in I$. Thus $x \land a, x \land b \notin J$.

Now, note that as $a \land b \in J$, we have that $(x \land a) \land (x \land b) \in J$, and $x \land a \notin J, x \land b \notin J$. Therefore $x \land a, x \land b \in Z_J(L)^* = Z_I(L)^*$. Hence $x \land a, x \land b \notin I$, a contradiction to the fact that $x \land a, x \land b \in I$. Thus $I = J$.

Conversely, if $I = J$ then obviously $Z_I(L)^* = Z_J(L)^*$.

**Remark 2.2.23.** Atani and Darani (Theorem 2.5, [8]) proved Theorem 2.2.22, under the assumption that $I$ and $J$ are $P$-primal ideals of a commutative ring $R$ with 1. From the proof of Theorem 2.2.22, it can be observed that an analogous proof can be given for commutative rings with unity.

**Corollary 2.2.24.** Let $I$ and $J$ be two non-prime ideals of a lattice $L$ such that $I$ is $P$-primal with $Z_I(L)^* = Z_J(L)^*$. Then $J$ is also a $P$-primal ideal of $L$.

We close this section by obtaining some properties of weakly primal ideals. First, we define the concept of weakly primal ideal of a lattice $L$ with 0.

**Definition 2.2.25.** A proper ideal $P$ of $L$ with 0 is said to be weakly prime if $0 \neq a \land b \in P$ implies that $a \in P$ or $b \in P$. We assume that $(0]$ is always weakly prime. An element $a \in L$ is called weakly prime
to an ideal \( I \) if \( 0 \neq r \land a \in I \) implies that \( r \in I \). Denote \( W_I(L)^* \), the set of all elements of \( L \) that are not weakly prime to \( I \). A proper ideal \( I \) of \( L \) is called \( P \)-weakly primal if the set \( P = W_I(L)^* \cup \{0\} \) forms an ideal. This ideal is always a weakly prime ideal. Let \( I \) be an ideal of \( L \).

We define the set \( Z_{I_0}(L)^* = \{ r \notin I \mid r \land a = 0 \text{ for some } a \notin I \} \).

**Lemma 2.2.26.** Let \( I \) be a non-prime \( P \)-weakly primal ideal of a lattice \( L \) with 0. Then \( Z_I(L)^* = (W_I(L)^* \setminus I) \cup Z_{I_0}(L)^* \).

**Proof.** Assume that \( I \) is a \( P \)-weakly primal ideal of \( L \). Let \( r \in Z_I(L)^* \). Then there is an element \( a \notin I \) with \( r \land a \in I \). If \( r \land a \neq 0 \), then \( r \in W_I(L)^* \). If \( r \land a = 0 \), then \( r \in Z_{I_0}(L)^* \). Hence \( Z_I(L)^* \subseteq (W_I(L)^* \setminus I) \cup Z_{I_0}(L)^* \). Let \( s \in (W_I(L)^* \setminus I) \cup Z_{I_0}(L)^* \). If \( s \in (W_I(L)^* \setminus I) \) then \( 0 \neq s \land b \in I \) for some \( b \notin I \), hence \( s \in Z_I(L)^* \). If \( s \in Z_{I_0}(L)^* \), then there is an element \( c \notin I \) such that \( s \land c = 0 \in I \) hence again \( s \in Z_I(L)^* \). \( \square \)

**Theorem 2.2.27.** Let \( I \) be a non-prime ideal of a lattice \( L \) with 0. Let \( P \) be an ideal of \( L \) with \( W_I(L)^* \subseteq P \) and \( (P \setminus I) \cap Z_{I_0}(L)^* = \phi \). Then \( Z_I(L)^* = (P \setminus I) \cup Z_{I_0}(L)^* \) if and only if \( I \) is a \( P \)-weakly primal ideal.

**Proof.** In view of Lemma 2.2.26, it suffices to show that if \( Z_I(L)^* = (P \setminus I) \cup Z_{I_0}(L)^* \) then \( I \) is a \( P \)-weakly primal ideal of \( L \). We show that \( P \setminus \{0\} \) consists of elements of \( L \) that are not weakly prime to \( I \). Let \( s \in P \setminus \{0\} \). Since every non-zero element of \( I \) is not weakly prime to \( I \), hence we can assume that \( s \notin I \). Therefore \( s \in P \setminus I \subseteq Z_I(L)^* \). This implies that \( s \land b \in I \) for some \( b \notin I \). Since \( (P \setminus I) \cap Z_{I_0}(L)^* = \phi \), we must have \( s \land b \neq 0 \). Hence \( s \) is not weakly prime to \( I \). Hence \( P \setminus \{0\} \subseteq W_I(L)^* \). This together with \( W_I(L)^* \subseteq P \setminus \{0\} \), we have \( I \) is a \( P \)-weakly primal ideal of \( L \). \( \square \)
2.3 Zero divisor graphs with respect to non-primal ideals

The concept of a zero divisor graph of a poset $P$ with respect to an ideal $I$ is due to Joshi [23]. We consider this definition when $P$ is a lattice.

**Definition 2.3.1.** Let $I$ be an ideal of a lattice $L$ with $0$. We associate an undirected graph, called the zero divisor graph of $P$ with respect to the ideal $I$, denoted by $G_I(L)$, in which the set of vertices is $V(G_I(L)) = \{x \in I \mid x \land y \in I \text{ for some } y \notin I\} = Z_I(L)^*$ and two distinct vertices $x, y$ are adjacent if and only if $x \land y \in I$. When $I = \{0\}$ then the corresponding zero divisor graph is denoted by $G_{\{0\}}(L)$.

The following theorem is essentially due to Joshi [23].

**Theorem 2.3.2.** Let $L$ be a lattice with $0$ and $I$ be an ideal of $L$. Then $G_I(L)$ is connected and $\text{diam}(G_I(L)) \leq 3$.

**Lemma 2.3.3.** Let $I$ be a proper non-primal semiprime ideal of a lattice $L$ with $0$. Then there exist vertices $a, b \in V(G_I(L))$ such that $a \lor b$ is prime to $I$.

**Proof.** Suppose $I$ is not a primal ideal, that is, $S(I)$ is not an ideal. Therefore there exist $a, b \in S(I)$ with $a \lor b \notin S(I)$. Since $a \lor b \notin S(I)$, $I : (a \lor b) = I$ and $a \lor b \notin I$. We claim that $a, b \notin I$. Without loss of generality assume that $a \in I$ and $b \notin I$. Since $b \in S(I) \setminus I$, by Lemma 2.2.13, $b \in Z_I(L)^*$. Hence there exists $c \notin I$ such that $b \land c \in I$. Since $I$ is semiprime and $a \land c , b \land c \in I$, we have $(a \lor b) \land c \in I$, that is,
2.3 Zero divisor graphs with respect to non-primal ideals

\( c \in I : (a \lor b) = I \), a contradiction to \( c \notin I \). Hence \( a, b \notin I \). Thus 
\( a, b \in S(I) \backslash I = V(G_I(L)) \), by Lemma 2.2.19 and moreover, \( a \lor b \) is prime to \( I \).

The following result is an analogue of Theorem 2.1 of Lucas [37].

**Lemma 2.3.4.** Let \( L \) be a lattice with \( 0 \) and \( I \) be a proper non-primal semiprime ideal of \( L \). If \( L \) has more than two minimal prime ideals belonging to (containing) \( I \), then \( \text{diam}(G_I(L)) = 3 \).

**Proof.** Since \( I \) is not a primal ideal, by Lemma 2.3.3, there exist \( a, b \in V(G_I(L)) \) such that \( I : (a \lor b) = I \). We claim that \( d(a, b) = 3 \).

**Case (I):** Let \( d(a, b) = 2 \). Then there is an element \( c \in V(G_I(L)) \) such that \( a \land c \in I \) and \( c \land b \in I \). Since \( I \) is semiprime, \( c \land (a \lor b) \in I \). So \( c \in I : (a \lor b) = I \), a contradiction to the fact that \( c \notin I \).

**Case (II):** Let \( d(a, b) = 1 \), i.e., \( a \land b \in I \). Since \( a, b \notin I \), by Theorem 2.2.6, there exist minimal prime ideals \( Q \) and \( R \) belonging to \( I \) such that \( a \notin R \) and \( b \notin Q \). Since \( a \land b \in I \) and \( L \) has more than two minimal prime ideals, there exists a third minimal prime ideal \( P \) (distinct from \( Q \) and \( R \)) such that \( a \land b \in I \subseteq P \). Therefore either \( a \in P \) or \( b \in P \). We claim that each minimal prime ideal contains only one of \( a \) and \( b \) but not both of them. If possible, assume that \( a, b \in P \) and \( P \) is minimal prime ideal belonging to a semiprime ideal \( I \). By Theorem 2.2.5, \( I : a \not\in P \) and \( I : b \not\in P \), which further yields \( I : a \cap I : b \not\in P \).

Since \( I \) is semiprime, \( I : a \cap I : b = I : (a \lor b) = I \). Thus, we have \( I = I : (a \lor b) \not\in P \), a contradiction to \( I \subseteq P \).

Hence without loss of generality assume that \( a \in (P \cap Q) \backslash R \) and \( b \in R \backslash (P \cup Q) \). Take any \( q_r \in (Q \cap R) \backslash P \). Clearly, \( b \land q_r \notin I \).
otherwise \(b \land q_r \in P\) and \(P\) is prime, therefore either \(b \in P\) or \(q_r \in P\), a contradiction. Hence \(b \land q_r \notin I\). Clearly, \(x = a \lor (b \land q_r) \notin I\). Moreover \(x \lor b = a \lor (b \land q_r) \lor b = a \lor b\). Hence \(I : a \lor b = I = I : x \lor b\). We claim that \(x \land b \notin I\). If \(x \land b \in I\) then \(x \in I : b\), i.e., \(a \lor (b \land q_r) \in I : b\) and hence \(b \land q_r \in I : b\) which yields \(b \land q_r \in I\), a contradiction. Hence \(x \land b \notin I\).

Note that \(x \in Q\) and \(Q\) is a minimal prime ideal belonging to \(I\), by Theorem 2.2.5 there exists \(y \notin Q\) such that \(x \land y \in I\). Clearly, \(y \notin I\). Hence \(x \in V(G_1(L))\). Thus, we have \(x, b \in V(G_1(L))\) such that \(x\) and \(b\) are non adjacent, as \(x \land b \notin I\) and \(x \lor b\) is prime to \(I\). Now, by Case(I), \(\text{diam}(G_1(L)) = 3\). \(\square\)

**Theorem 2.3.5.** Let \(L\) be a lattice with \(0\) and \(I\) be a non-primal semiprime ideal of \(L\). Then \(\text{diam}(G_1(L)) \leq 2\) if and only if \(L\) has exactly two minimal prime ideals belonging to \(I\).

**Proof.** Since \(I\) is not a primal ideal, i.e., \(S(I)\) is not an ideal, by Lemma 2.3.3, there exist \(a, b \in V(G_1(L))\) such that \(I : a \lor b = I\). Since \(a, b \in V(G_1(L))\), there exist elements \(c, d \notin I\) such that \(a \land c \in I\) and \(b \land d \in I\). Clearly \(c \neq d\). For otherwise, we get that \((a \lor b) \land c \in I\) (as \(I\) is semiprime). Hence, \(c \in I : a \lor b = I\), a contradiction to \(c \notin I\). Therefore \(c \neq d\). Now, assume that \(\text{diam}(G_1(L)) \leq 2\). By Lemma 2.3.4, there exist at most two minimal ideals \(P\) and \(Q\) belonging to \(I\). Further, by Theorem 2.2.7, \(S(I)\) is the union of at least two minimal prime ideals belonging to \(I\), otherwise \(S(I) = P\), a contradiction to the fact that \(S(I)\) is not an ideal. Thus \(L\) has exactly two minimal prime ideals belonging to \(I\).
Conversely, assume that $L$ has exactly two distinct minimal prime ideals, say $P$ and $Q$, belonging to $I$. Therefore, there exist elements $a$ and $b$ such that $a \in P \setminus Q$ and $b \in Q \setminus P$. But then $a \wedge b \in P \cap Q$. Since $I$ is semiprime, by Theorem 2.2.6, $I = P \cap Q$. Hence $a \wedge b \in I$.

Let $x, y \in V(G_I(L))$ and $x \neq y$. Then clearly, $x$ or $y$ can not be in both $P$ and $Q$. Without loss of generality, we may assume that $x \in P \setminus Q$. If $x \wedge y \in I$ then they are adjacent and so $d(x, y) = 1$. Now, suppose $x \wedge y \notin I$. We claim that $y \notin Q$. For otherwise, if $y \in Q$ then $x \wedge y \in P \cap Q = I$, a contradiction. Moreover, by Lemma 2.2.7, $S(I) = P \cup Q$. Since $y \notin Q$, we have by Lemma 2.2.19, $y \in V(G_I(L)) = S(I) \setminus I$. Therefore $y \in P$ which means $x \vee y \in P$. Therefore $(x \vee y) \wedge b \in P$. Moreover, as $b \in Q \setminus P$ we have $(x \vee y) \wedge b \in Q$. Thus $(x \vee y) \wedge b \in P \cap Q = I$. Since $x, y \in P$ and $b \in Q$, we have $x \wedge b, y \wedge b \in P \cap Q = I$. Therefore $d(x, y) = 2$. Hence $\text{diam}(G_I(L)) \leq 2$.

**Theorem 2.3.6.** Let $L$ be a lattice with 0 and $I$ be a non-primal semiprime ideal of $L$. Then $L$ has more than two minimal prime ideals belonging to $I$ if and only if $\text{diam}(G_I(L)) = 3$.

**Proof.** Follows from Lemma 2.3.4 and Theorem 2.2.6. □

**Definition 2.3.7.** The set of associated primes belonging to an ideal $I$ of a lattice $L$ is denoted by $\text{Ass}_1(I)$ and it is the set of prime ideals $P$ of $L$ such that there exists $x \in L$ with $P = I : x$.

The following result is due to Joshi [23] and the notions of a prime semi-ideal(ideal) and a semiprime ideal in a poset $P$ coincides with the corresponding notions in lattices when $P$ happens to be a lattice.
Theorem 2.3.8. Let $I$ be an ideal of a poset $P$ with 0. Then the following hold:

(a) If $P_1$ and $P_2$ are prime semi-ideals and $I = P_1 \cap P_2$. Then $G_I(P)$ is a complete bipartite graph.

(b) If $I$ is a semiprime ideal then $G_I(P)$ is a complete bipartite graph if and only if there exist prime ideals $P_1$ and $P_2$ such that $I = P_1 \cap P_2$.

Theorem 2.3.9. Let $L$ be a lattice with 0 and $I$ be a non-primal semiprime ideal. Let $V(G_I(L)) > 2$. Then $diam(G_I(L)) = 2$ if and only if $I = P_1 \cap P_2$ for some prime ideals $P_1$ and $P_2$ belonging to $I$.

Proof. Let $I = P_1 \cap P_2$. Then by Theorem 2.3.8, $G_I(L)$ is a complete bipartite graph. Hence $diam(G_I(L)) = 2$.

Conversely, suppose $diam(G_I(L)) = 2$ then by Theorem 2.3.5, $L$ has exactly two minimal prime ideals belonging to $I$ say, $P_1$ and $P_2$. Since $I$ is semiprime, by Theorem 2.2.9, we have $P_1 \cap P_2 = I$, and this completes the proof. \qed

We close this chapter by proving result about the girth of a zero divisor graph.

Theorem 2.3.10. Let $L$ be a lattice with 0 and $Ass_I(L) = \{P_1, P_2\}$. Then the following statements are true.

(i) If $P_1 \cap P_2 \neq I$, then $gr(G_I(L)) = 3$.

(ii) If $P_1 \cap P_2 = I$ and $|P_i \setminus I| \geq 3$, $i = 1, 2$, then $gr(G_I(L)) = 4$.

Proof. (i) Let $P_1 = I : x_1$ and $P_2 = I : x_2$ and $|P_1 \cap P_2| \geq 3$. Since $P_1$ and $P_2$ are distinct prime ideals belonging to $I$ of $Ass_I(L)$ then by primeness of $I : x_i$, we have $x_1 \land x_2 \in I$. Now take $a \in P_1 \cap P_2$ such that
2.3 Zero divisor graphs with respect to non-primal ideals

$a \notin I$. Therefore $a \in P_1 = I : x_1$ and $a \in P_2 = I : x_2$, i.e., $a \wedge x_1 \in I$ and $a \wedge x_2 \in I$. Hence we get $a - x_1 - x_2 - a$. Therefore $gr(G_I(L)) = 3.$

$(ii)$ Let $P_i = I : x_i$, $i = 1, 2$. By primeness of $I : x_i$, it is clear that $I$ is a semiprime ideal. Further, properness of $I : x_i$ gives $x_i \notin I$. By primeness of $I : x_i$, we have $x_1 \wedge x_2 \in I$. Let $a \in P_1 \setminus I \cup \{x_2\}$ and $b \in P_2 \setminus I \cup \{x_1\}$. Since $a \wedge b \in P_1 \cap P_2 = I$ we have $a - x_1 - x_2 - b - a$. Thus $G_I(L)$ has a cycle, moreover by Theorem 2.3.8, $G_I(L)$ is a complete bipartite graph and hence $gr(G_I(L)) = 4.$ $\square$