Chapter 4

Zero divisor graphs with respect to different operations
4.1 Introduction

In this chapter, we study zero divisor graphs of lattices with respect to different operations such as direct product of lattices and 2-sum of lattices. We determine the diameter of the zero divisor graph of direct product of lattices with respect to different ideals. Further, we prove that the complement of the zero divisor graph of finite direct product of lattices is connected. Then we consider the zero divisor graph of 2-sum of lattices and characterize the diameter of zero divisor graph for lower dismantlable lattices. At the end of this chapter, we prove a representation theorem for lower dismantlable lattices.

4.2 Diameter of zero divisor graphs of finite direct product of lattices

In this section we study the zero divisor graphs of finite direct product of lattices.

Through out this section, we assume that all lattices have the smallest element 0.

We need the following definitions:

Definition 4.2.1. The product of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is showing by $G_1 \times G_2$ and is defined as following:
Consider any two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V = V_1 \times V_2$. Then $u$ and $v$ are adjacent in $G_1 \times G_2$ whenever $[u_1 = v_1$ and $u_2$ is adjacent to $v_2]$ or $[u_2 = v_2$ and $u_1$ is adjacent to $v_1]$. 
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The following Figure 4.2.1, illustrates the product of two graphs:

\[ G_1 \times G_2 \]

\( G_1 \times G_2 \)

Figure 4.2.1: The product of two graphs

**Definition 4.2.2.** Let \( L \) and \( K \) be lattices. Define \( \wedge \) and \( \vee \) in \( L \times K \) component-wise:

\[
< a_0, b_0 > \wedge < a_1, b_1 > = < a_0 \wedge a_1, b_0 \wedge b_1 > \\
< a_0, b_0 > \vee < a_1, b_1 > = < a_0 \vee a_1, b_0 \vee b_1 >
\]

This makes \( L \times K \) into a lattice, called the direct product of \( L \) and \( K \).

As an example see the following Figure 4.2.2

\[ N_5 \times C_2 \]

\( N_5 \times C_2 \)

Figure 4.2.2: The direct product \( N_5 \times C_2 \)

**Remark 4.2.3.** Let \( L_1 \) and \( L_2 \) be two lattices. Let \( G_{(0,0)}(L_1 \times L_2) \) be the zero divisor graph of product of lattices \( L = L_1 \times L_2 \) with respect to the
ideal \( I = (0,0) \). Now we give the set of vertices (edges) of \( G_{(0,0)}(L_1 \times L_2) \) in terms of vertex set (edge set) of \( G_{(0)}(L_1) \) and \( G_{(0)}(L_2) \) respectively. The set of vertices of \( G_{(0,0)}(L_1 \times L_2) \) is \( V(G_{(0,0)}(L_1 \times L_2)) = \{(a,b) \neq (0,0) \mid a \in V(G_{(0)}(L_1))\cup\{0\} \text{ or } b \in V(G_{(0)}(L_2))\cup\{0\}\} \) and two distinct vertices \((a,b)\) and \((x,y)\) are adjacent \((e = ((a,b),(x,y)) \in E(G_{(0,0)}(L)))\) if and only if one of the following conditions hold:

- either \( e \in G_{(0)}(L_1) \times G_{(0)}(L_2) \);
- or \( a = 0, x = 0 \) and \( (b,y) \in E(G_{(0)}(L_2)) \);
- or \( b = 0, y = 0 \) and \( (a,x) \in E(G_{(0)}(L_1)) \);
- or \( a = 0, x \neq 0, b \neq 0, y = 0 \);
- or \( a \neq 0, x = 0, b = 0, y \neq 0 \).

The following theorem is essentially due to Joshi [23].

**Theorem 4.2.4.** Let \( I \) be an ideal of a lattice \( L \). Then \( G_I(L) \) is a connected graph with \( \text{diam}(G_I(L)) \leq 3 \).

**Lemma 4.2.5.** Let \( L_1, L_2, \ldots, L_n \) be lattices with ideals \( I_1, I_2, \ldots, I_n \) respectively. Then \( I = I_1 \times I_2 \times \ldots \times I_n \) forms an ideal in \( L = L_1 \times L_2 \times \ldots \times L_n \).

**Proof.** Easy to prove. \( \square \)

**Remark 4.2.6.** Note that if one of \( I_j \)'s, \( j = \{1,2,\ldots,n\} \) is not prime, then \( I = I_1 \times I_2 \times \ldots \times I_n \) is not prime.

**Lemma 4.2.7.** Let \( L_1 \) and \( L_2 \) be two lattices with \( I_2 \), a non-prime ideal, then \( \text{diam}(G_{L_1 \times I_2}(L_1 \times L_2)) = \text{diam}(G_{I_2}(L_2)) \).
4.2 Diameter of zero divisor graphs of finite direct product of lattices

Proof. Suppose $diam(G_{L_1 \times L_2}(L_1 \times L_2)) = n > diam(G_{I_2}(L_2))$. Then $n = 2$ or $n = 3$. Let $(a_0, x_0), (a_1, x_1), ..., (a_n, x_n) \in Z_{L_1 \times L_2}(L_1 \times L_2)^*$ be such that $(a_0, x_0) - (a_1, x_1) - ... - (a_n, x_n)$ be a minimal path. This implies that $a_i \land a_{i+1} \in L_1$ and $x_i \land x_{i+1} \in I_1$ for $i = \{0, 1, ..., n-1\}$. Hence we have a path $x_0 - x_1 - ... - x_n$ in $G_{I_2(L_2)}$. Since $n > diam(G_{I_2}(L_2))$, $x_0 - x_1 - ... - x_n$ must not be a minimal path.

This can happen in two ways.

If there exist $i, j$ such that $0 \leq i < j \leq n$, $j \neq i + 1$, and $x_i - x_j$. Then $(a_i, x_i) - (a_j, x_j)$, a contradiction to $(a_0, x_0) - (a_1, x_1) - ... - (a_n, x_n)$ is a minimal path. So $diam(G_{L_1 \times L_2}(L_1 \times L_2)) = diam(G_{I_2}(L_2))$.

Suppose $diam(G_{L_1 \times L_2}(L_1 \times L_2)) = n < diam(G_{I_2}(L_2))$ such that $1 \leq n \leq 3$. Then there exist $x_0, x_1, ..., x_n \in Z_{I_2}(L_2)^*$ such that $x_0 - x_1 - ... - x_{n+1}$ is a minimal path. Since $L_1 = I_1$, $\forall a_0, a_1, ..., a_{n+1} \in L_1$, $(a_0, x_0) - (a_1, x_1) - ... - (a_n, x_n) - (a_{n+1}, x_{n+1})$ is a minimal path of length $n + 1$, a contradiction. Thus $diam(G_{L_1 \times L_2}(L_1 \times L_2)) = diam(G_{I_2}(L_2))$.

We recall the following definition from Chapter 2.

**Definition 4.2.8.** Let $I$ be an ideal of a lattice $L$. We define the set $Z_I(L)^* = \{ r \notin I \mid r \land a \in I \text{ for some } a \notin I \}$. Clearly, $Z_I(L) = Z_I(L)^* \cup I$

**Lemma 4.2.9.** Let $L_1, L_2, ..., L_{n-1}$ and $L_n$ be lattices with ideals $I_1, I_2, ..., I_n$ respectively such that $Z_{I_i}(L_i)^* \neq \emptyset$ for $\forall i$ and let $L = L_1 \times L_2 \times L_3 \times ... \times L_n$ ($n \geq 2$) and $I = I_1 \times I_2 \times I_3 \times ... \times I_n$ ($n \geq 2$). Then $diam(G_{I_1 \times I_2 \times ... \times I_n}(L_1 \times L_2 \times ... \times L_n)) > 1$.

**Proof.** Let $x_1 \in Z_{I_1}(L_1)^*$ and $y_1 \in Z_{I_2}(L_2)^*$. So there exist $x_2 \in L_1 \backslash I_1$...
and \( y_2 \in L_2 \setminus I_2 \) such that \( x_1 \land x_2 \in I_1 \) and \( y_1 \land y_2 \in I_2 \). Consider, 

\[(x_1, y_1, 0, ..., 0), (0, y_1, 0, ..., 0) \in L_1 \times L_2 \times ... \times L_n. \]

It is easy to see that 

\((x_1, y_1, 0, ..., 0), (0, y_1, 0, ..., 0) \in V(G_{I_1 \times I_2 \times ... \times I_n}(L_1 \times L_2 \times ... \times L_n)). \]

Since \((x_1, y_1, 0, ..., 0), (0, y_1, 0, ..., 0)\) are not adjacent, therefore \(diam(G_{I_1 \times I_2 \times ... \times I_n}(L_1 \times L_2 \times ... \times L_n)) > 1. \)

\[\square\]

**Theorem 4.2.10.** Let \( L_1, L_2, ..., L_{n-1} \) and \( L_n \) be lattices with ideals \( I_1, I_2, ..., I_n \) respectively, such that at least two of them are non-prime. Let \( L = L_1 \times L_2 \times L_3 \times ... \times L_n \) \((n \geq 2)\) and \( I = I_1 \times I_2 \times I_3 \times ... \times I_n \) \((n \geq 2)\). If \( \text{diam}(G_I(L)) = 2 \) then \( L_i - Z_{I_i}(L_i) = \emptyset \) for some \( i \in \{1, 2, ..., n\} \).

**Proof.** Since at least two of the ideals \( I_1, I_2, ..., I_n \) are non-prime, we have \( I \) is non-prime. This gives \( Z_I(L)^* \neq \emptyset \). Assume that \( \text{diam}(G_I(L)) = 2 \). We claim that \( L_i - Z_{I_i}(L_i) = \emptyset \) for some \( i \in \{1, 2, ..., n\} \). Suppose on the contrary that \( L_i - Z_{I_i}(L_i) \neq \emptyset \), \( \forall i \). Then there must exist \( x_i \in L_i - Z_{I_i}(L_i) \) for each \( i = \{1, 2, ..., n\} \). Without loss of generality, let \( I_1 \) and \( I_2 \) be two non-prime ideals. Then \( z_j \in Z_{I_j}(L_j)^* \) for \( j = \{1, 2\} \). So there is an element \( z_j' \) of \( Z_{I_j}(L_j)^* \) such that \( z_j \land z_j' \in I_j \) for \( j = \{1, 2\} \). If \( a = (z_1, x_2, x_3, ..., x_n) \) and \( b = (x_1, z_2, x_3, ..., x_n) \) then \( a \land a' \in I \) and \( b \land b' \in I \) where \( a' = (z_1', 0, ..., 0) \) and \( b' = (0, z_2', 0, ..., 0) \). So \( a, b \in Z_I(L)^* \). Clearly, \( a \land b \notin I \). Since \( \text{diam}(G_I(L)) = 2 \), there must be some \( c = (c_1, c_2, ..., c_n) \in Z_I(L)^* \) such that \( a \land c, b \land c \in I \). But \( a \land c = (z_1 \land c_1, x_2 \land c_2, ..., x_n \land c_n) \in I \), i.e, \( z_1 \land c_1 \in I_1 \) and \( x_i \land c_i \in I_i \) for \( i = \{2, 3, ..., n\} \) but \( x_i \in L_i - Z_{I_i}(L_i) \). Hence \( x_i \notin I_i \). This together with \( x_i \land c_i \in I_i \) gives \( c_i \in I_i \) for \( i = \{2, 3, ..., n\} \). .................... (1)

Similarly, \( b \land c \in I, \) but \( b \land c = (x_1 \land c_1, z_2 \land c_2, ..., x_n \land c_n) \in I, \) i.e, \( z_2 \land c_2 \in I_2 \) and \( x_i \land c_i \in I_i \) for \( i = \{1, 3, ..., n\} \) but \( x_i \in L_i - Z_{I_i}(L_i) \).
Therefore we must have \( c_i \in I_i \) for \( i = \{1, 3, ..., n\} \).  \( \cdots \) \( \cdots \) \( \cdots \) (2)

From (1) and (2) we get \( c = (c_1, c_2, ..., c_n) \in I \), a contradiction to the fact that \( c \not\in I \). Thus \( L_i = Z_{I_i}(L_i) \) for some \( i \in \{1, 2, ..., n\} \). \( \square \)

**Remark 4.2.11.** We provide an example of a lattice \( L \) such that \( L = Z_I(L) \) for an ideal \( I \) of \( L \). Consider the lattice of all proper subsets of \( \mathbb{N} \), the set of all natural numbers under inclusion. Then it is easy to observe that \( L = Z_{\{\emptyset\}}(L) \).

In view of Theorem 4.2.10, it is clear that \( \text{diam}(G_{I_{\{0\}}}(L_1 \times L_2 \times ... \times L_n)) = 3 \) whenever \( L_i \)'s are finite for every \( i \).

**Corollary 4.2.12.** If \( L_i - Z_{I_i}(L_i) \neq \emptyset \) for every \( i \in \{1, 2, ..., n\} \), then \( \text{diam}(G_I(L)) = 3 \). In particular \( \text{diam}(G_{\{0\}}(L)) = 3 \) for \( L = 2^n \), a Boolean lattice, for \( n \geq 3 \).

**Proof.** Follows from Theorem 4.2.10, Theorem 4.2.9, and Theorem 4.2.10. \( \square \)

**Theorem 4.2.13.** Let \( L_1, L_2, ..., L_{n-1} \) and \( L_n \) be lattices with ideals \( I_1, I_2, ..., I_n \) respectively, such that at least two of them are non-prime. Let \( L = L_1 \times L_2 \times L_3 \times ... \times L_n \) (\( n \geq 2 \)) and \( I = I_1 \times I_2 \times I_3 \times ... \times I_n \) (\( n \geq 2 \)). If \( \text{diam}(G_{I_1}(L_1)) = \text{diam}(G_{I_2}(L_2)) = ... = \text{diam}(G_{I_n}(L_n)) = 3 \) then \( \text{diam}(G_I(L)) = 3 \).

**Proof.** Since for each \( i \in \{1, 2, ..., n\} \), \( \text{diam}(G_{I_i}(L_i)) = 3 \), there exist non-adjacent vertices \( x_i, y_i \in Z_{I_i}(L_i)^* \) such that there is no \( z_i \in Z_{I_i}(L_i)^* \) with \( x_i \wedge z_i, y_i \wedge z_i \in I_i \). Consider \( x = (x_1, x_2, ..., x_n) \) and \( y = (y_1, y_2, ..., y_n) \).

For each \( i \in \{1, 2, ..., n\} \), there are elements \( x'_i, y'_i \in Z_{I_i}(L_i)^* \) such that \( x_i \wedge x'_i \in I_i \) and \( y_i \wedge y'_i \in I_i \), so \( x, y \in Z_I(L)^* \). As \( x \wedge y \notin I \) and
diam(G_I(L)) \neq 1$, therefore $diam(G_I(L)) = 2$ or $3$. If $diam(G_I(L)) = 2$, then there exist an element $a = (a_1, a_2, ..., a_n) \in Z_I(L)^*$ such that we have a path $x - a - y$ in $G_I(L)$. Therefore, we have $x_i \land a_i, y_i \land a_i \in I_i$. Hence $d(x_i, y_i) = 2$, which is a contradiction to the fact that $diam(G_{I_i}(L_i)) = 3$. So $diam(G_I(L)) = 3$.

**Theorem 4.2.14.** Let $L_1, L_2, ..., L_{n-1}$ and $L_n$ be lattices with ideals $I_1, I_2, ..., I_n$ respectively, such that at least two of them are non-prime. Let $L = L_1 \times L_2 \times L_3 \times ... \times L_n$ ($n \geq 2$) and $I = I_1 \times I_2 \times I_3 \times ... \times I_n$ ($n \geq 2$). Then $G_I(L)$ has a cycle of length $3$. Hence $gr(G_I(L)) = 3$

**Proof.** Take non-zero elements $a = (a_1, 0, ..., 0), b = (0, b_2, 0, ..., 0)$ and $c = (0, 0, c_3, 0, ..., 0)$ of a lattice $L$. Clearly, $a, b, c \in V(G_I(L))$ and $a \land b, a \land c, b \land c \in I$. Therefore, we get a cycle $a - b - c - a$, hence the girth is $3$. □

**Lemma 4.2.15.** Let $L_1, L_2, ..., L_{n-1}$ and $L_n$ be lattices with ideals $I_1, I_2, ..., I_n$ respectively, such that at least two of them are non-prime. Let $L = L_1 \times L_2 \times L_3 \times ... \times L_n$ ($n \geq 2$) and $I = I_1 \times I_2 \times I_3 \times ... \times I_n$ ($n \geq 2$). If $a$ is a cut vertex of $G_I(L)$, then there exists some $a_i \neq 0$; $(1 \leq i \leq n)$ such that $a = (0, 0, ..., a_i, ...0)$.

**Proof.** Let $a$ is a cut vertex of $G_I(L)$, with $a = (a_1, a_2, ..., a_i, ... a_n)$. Since $a$ is a cut vertex, for any two arbitrary elements $b, c \in V(G_I(L))$, the path between $b$ and $c$ goes through of $a$. Consider the element $d = (0, 0, ..., a_i, 0, ..., 0)$. Then we get a path $b - d - c$. Since $a$ is a cut vertex, we have $a = d$. Then $a = (0, 0, ..., a_i, ...0)$. □
4.3 Complement of zero divisor graphs of direct product of lattices

In this section, we study the connectivity of the complement of zero divisor graphs.

**Definition 4.3.1.** Let $G = (V, E)$ be a simple graph. The complement of $G$, denoted by $G^c$, is defined by setting $V(G^c) = V(G) = V$ and two distinct vertices $u, v \in V$ are joined by an edge in $G^c$ if and only if there exists no edge in $G$ joining $u$ and $v$.

We give examples of two lattices $L_1$ and $L_2$ such that $(G_{\{0\}}(L_i))^c$, the complement of the zero divisor graph of a lattice $L_i$ ($i = 1, 2$) is disconnected and connected respectively.

![Figure 4.3.1: Connected zero divisor graph whose complement is disconnected](image1)

![Figure 4.3.2: A zero divisor graph and its complement both connected](image2)
From Figure 4.3.1 on Page 66, it is clear that $G_0(L_1)$ is connected but not $(G_0(L_1))^c$ whereas in Figure 4.3.2 on Page 66, $G_0(L_2)$ and $(G_0(L_2))^c$ both are connected. Hence it is natural to ask the following question.

**Question:** When $(G_I(L))^c$ is connected?

We answer this question in the following theorem. To prove this theorem, we need the following results in sequel and the proof of Theorem 4.3.2 is mentioned at the end of Thesis.

We use the notation, $0 = (0, 0, ..., 0)$.

**Theorem 4.3.2.** Let $L_1, L_2, L_3, ..., L_n$ ($n \geq 3$) be co-atomic, 1-distributive semi-complemented lattices and $L = L_1 \times L_2 \times L_3 \times ... \times L_n$, then $(G_0(L))^c$ is connected.

**Lemma 4.3.3.** Let $L = L_1 \times L_2 \times ... \times L_n$. If $(G_0(L))^c$ is connected, then $diam(G_0(L))^c \geq 2$.

**Proof.** Let $a = (a_1, a_2, ..., a_n), b = (b_1, b_2, ..., b_n) \in Z_0(L)^*$ be two distinct elements. By Theorem 4.2.4, $G_0(L)$ is connected; hence there exists $c = (c_1, c_2, ..., c_n) \in Z_0(L)^*$ such that $c \wedge a = 0$. Hence, if $(G_0(L))^c$ is connected, then $d(a, c) \geq 2$ in $(G_0(L))^c$ and so $diam(G_0(L))^c \geq 2$. 

**Definition 4.3.4.** A lattice $L$ with 0 is said to be 0-distributive if $a \wedge b = 0$ and $a \wedge c = 0$ imply $a \wedge (b \vee c) = 0$ for $a, b, c \in L$.

A lattice $L$ with 1 is said to be 1-distributive if $a \vee b = 1$ and $a \vee c = 1$ imply $a \vee (b \wedge c) = 1$ for $a, b, c \in L$. 
A bounded lattice which is both 0-distributive and 1-distributive is called 0−1-distributive lattice.

**Theorem 4.3.5.** Let $L_1, L_2$ be 1-distributive lattices. Then direct product of $L_1$ and $L_2$ is also a 1-distributive lattice.

**Proof.** Let $L_1$ and $L_2$ are 1-distributive lattices. To show that $L = L_1 \times L_2$ is 1-distributive lattice, it is enough to show that if $(x_1, y_1) \lor (x_2, y_2) = (1, 1)$ and $(x_1, y_1) \lor (x_3, y_3) = (1, 1)$ then $(x_1, y_1) \lor ((x_2, y_2) \land (x_3, y_3)) = (1, 1)$ for any $x_i \in L_1$ and $y_i \in L_2$ where $i = \{1, 2, 3\}$. From the hypothesis we can conclude that $(x_1 \lor x_2, y_1 \lor y_2) = (1, 1) = (x_1 \lor x_3, y_1 \lor y_3)$, i.e, $x_1 \lor x_2 = x_1 \lor x_3 = 1$ and $y_1 \lor y_2 = y_1 \lor y_3 = 1$. Since $L_1$ and $L_2$ are 1-distributive lattices, therefore we have $x_1 \lor (x_2 \land x_3) = 1$ and $y_1 \lor (y_2 \land y_3) = 1$.

Therefore $(x_1, y_1) \lor ((x_2, y_2) \land (x_3, y_3)) = (x_1, y_1) \lor (x_2 \land x_3, y_2 \land y_3) = (x_1 \lor (x_2 \land x_3), y_1 \lor (y_2 \land y_3)) = (1, 1)$.

**Theorem 4.3.6.** Let $L_1, L_2, \ldots, L_n$ be 1-distributive lattice. Then $L = L_1 \times L_2 \times \ldots \times L_n$ is also a 1-distributive lattice.

**Proof.** Follows by using mathematical induction. \hfill \□

**Corollary 4.3.7.** Let $L_1, L_2, \ldots, L_n$ be 0-distributive lattice. Then $L = L_1 \times L_2 \times \ldots \times L_n$ is also 0-distributive lattice.

**Definition 4.3.8.** A bounded lattice $L$ is complemented if, for each element $x$, there exists at least one element $y$ such that $x \land y = 0$ and $x \lor y = 1$. In a lattice $L$ with 0, an element $y$ is called a semi-complement of $x$ if $x \land y = 0$; and $L$ is said to be semi-complemented(SC) if each
$x \in L$ (with $x \neq 1$, if 1 exists in $L$) admits at least one non-zero semi-complement.

**Definition 4.3.9.** A lattice $L$ is called atomic if $L$ has 0 and, for every $(\neq 0)a \in L$, there is an atom $p \leq a$.

**Definition 4.3.10.** A lattice $L$ is called co-atomic if $L$ has 1 and, for every $(\neq 1)a \in L$, there is a co-atom $q \geq a$.

**Lemma 4.3.11.** Let $L_1, L_2, ..., L_n$ be semi-complemented lattice. Then $L = L_1 \times L_2 \times ... \times L_n$ is also semi-complemented lattice.

**Proof.** By mathematical induction. \hfill \square

The following lemma is essentially due to Joshi and Mundlik [27].

**Lemma 4.3.12.** Let $L$ be a co-atomic lattice with the greatest element 1. Then the following are equivalent.

(a) $L$ is a 1-distributive lattice.

(b) $[q]$ is a prime ideal of $L$ for every co-atom $q \in L$.

**Lemma 4.3.13.** Let $L_1, L_2, L_3, ..., L_n$ ($n \geq 3$) be co-atomic, 1-distributive lattices. Then $L = L_1 \times L_2 \times L_3 ... \times L_n$ has at least three prime ideals.

**Proof.** By applying Theorem 4.3.10, the finite direct product of 1-distributive lattices is again a 1-distributive lattice.

We consider the elements $(q_1, 1, ..., 1), (1, q_2, 1, ..., 1), (1, 1, q_3, 1, ..., 1)$ in $L = L_1 \times L_2 \times L_3 ... \times L_n$, where $q_i$ are co-atoms of $L_i$. It is easy to see that $(q_1, 1, ..., 1), (1, q_2, 1, 1, ..., 1), (1, 1, q_3, 1, ..., 1)$ are co-atoms of $L$. By applying Lemma 4.3.12, we get at least three prime ideals in $L$. \hfill \square
Now, we close this section by proving Theorem 4.3.2.

Proof of Theorem 4.3.2. We claim that there exist \( x, y \in V((G_0(L))^c) \) such that \( x \wedge y = 0 \), where \( x = (x_1, x_2, \ldots, x_n) \), \( y = (y_1, y_2, \ldots, y_n) \). If \( x \wedge y \neq 0 \) for any \( x, y \in V((G_0(L))^c) \), then \( \text{diam}((G_0(L))^c) = 1 \), a contradiction to \( \text{diam}((G_0(L))^c) \geq 2 \), by Lemma 4.3.3. Thus \( x \) and \( y \) are not adjacent in \( (G_0(L))^c \). By Lemma 4.3.13, at least three prime ideals say \((q_1), (q_2), (q_3)\) do exist, where \( q_i \in L_i \) are co-atoms of \( L \) of the form \( q_1 = (d_1, 1, 1, \ldots, 1) \), \( q_2 = (1, d_2, 1, \ldots, 1) \) and \( q_3 = (1, 1, d_3, 1, \ldots, 1) \) where \( d_i \) are co-atoms of \( L_i \).

Let \( x \) and \( y \) be two non-adjacent vertices. We have the following cases:

(Case I) If \( x, y \in (q_1) \), then \( x \wedge q_1 = x \neq 0 \) and \( y \wedge q_1 = y \neq 0 \). Since \( L \) is semi-complemented and \( q_1 \in L \), therefore \( q_1 \in V((G_0(L))^c) \). Hence there is a path \( x - q_1 - y \) in \( (G_0(L))^c \).

(Case II) If \( x \in (q_1) \) and \( y \in (q_2) \). Since \( x \wedge y = 0 \in (q_3) \) and \( (q_3) \) is a prime ideal, therefore at least one of \( x \) or \( y \in (q_3) \). Without loss of generality, we assume that \( y \in (q_3) \). Therefore \( y \wedge q_2 = y \neq 0 \) and \( y \wedge q_3 = y \neq 0 \). We claim that \( (q_1) \cap (q_2) \neq \{0\} \) or \( (q_1) \cap (q_3) \neq \{0\} \). For otherwise \( (q_1) \cap (q_2) = (q_1 \wedge q_2) = \{0\} \) and \( (q_1) \cap (q_3) = (q_1 \wedge q_3) = \{0\} \), i.e. \( q_1 \wedge q_2 = 0 \) and \( q_1 \wedge q_3 = 0 \). But this gives \( q_1 \wedge q_2 \in (q_3) \). Since \( q_i \)'s are dual atoms, we have either \( q_1 = q_3 \) or \( q_2 = q_3 \), a contradiction to the fact that \( q_i \) are distinct. Hence without loss of generality, we assume that \( q_1 \wedge q_2 \neq 0 \). Then we get a path \( x - q_1 - q_2 - y \) in \( (G_0(L))^c \). \( \square \)
4.4 Zero divisor graphs of 2-sum of lattices

Through of this section, we assume that all lattices are finite. The concept of \textit{adjunct operation (2-sum) of lattices} firstly introduced by Thakare, Pawar and Waphare [18] to achieve a constructive characterization of dismantlable lattices. If $L_1$ and $L_2$ are two disjoint lattices and $(a, b)$ is a pair of elements in $L_1$ such that $a < b$ and $a \not< b$ (by $a < b$ we mean there is no $c$ such that $a < c < b$). Define the partial order $\leq$ on $L = L_1 \cup L_2$ with respect to the pair $(a, b)$: $x \leq y$ in $L$ if

- either $x, y \in L_1$ and $x \leq y$ in $L_1$;
- or $x, y \in L_2$ and $x \leq y$ in $L_2$;
- or $x \in L_1$, $y \in L_2$ and $x \leq a$ in $L_1$;
- or $x \in L_2$, $y \in L_1$ and $b \leq y$ in $L_1$.

It is easy to see that $L$ is a lattice containing $L_1$ and $L_2$ as sublattices. The procedure for obtaining $L$ in this way is called \textit{adjunct operation (or adjunct sum) of $L_1$ with $L_2$}. The pair $(a, b)$ is called an \textit{adjunct pair} and $L$ is called \textit{adjunct} of $L_1$ with $L_2$ with respect to the adjunct pair $(a, b)$ and write $L = L_1 \upharpoonright_a^b L_2$. A diagram of $L$ is obtained by placing a diagram of $L_1$ and a diagram of $L_2$ side by side in such a way that the largest element 1 of $L_2$ is at the lower position than $b$ and the least element 0 of $L_2$ is at the higher position than $a$ and then by adding the coverings $< 1, b >$ and $< a, 0 >$ as shown in Figure 4.4.1. This clearly gives $|E(L)| = |E(L_1)| + |E(L_2)| + 2$, where $E(L)$ is nothing but edge set of $L$. This also implies that the adjunct operation preserves all the covering relations of the individual lattices $L_1$ and $L_2$. 
4.4 Zero divisor graphs of 2-sum of lattices

Definition 4.4.1. Given two graphs $G_1$ and $G_2$, the union $G_1 \cup G_2$ is the graph with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

The join of $G_1$ and $G_2$, denoted by $G_1 + G_2$ is the graph with $V(G_1 + G_2) = V(G_1) \cup V(G_2)$, $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup J$ where $J = \{\{x_1, x_2\} \mid x_1 \in V(G_1), x_2 \in V(G_2)\}$.

The null graph on a set $S$ is the graph whose vertex set is $S$ and edge set is the empty set, we denote it by $N(S)$.

The following Theorem illustrates the zero divisor graph of 2-sum of lattices.

Theorem 4.4.2. Let $L_1$ and $L_2$ be two lattices. Define $L = L_1|_a^b L_2$. Then the following statements are true.

1. If $a \neq 0$ and $a \notin V(G_{\{0\}}(L_1))$ then $G_{\{0\}}(L) = G_{\{0\}}(L_1)$.

2. If $a \in V(G_{\{0\}}(L_1))$ then $G_{\{0\}}(L) = G_{\{0\}}(L_1) \cup (G_a + N(L_2))$, where $G_a = \{x \in L_1 \mid a, x$ are adjacent in $G_{\{0\}}(L_1)\}.$
3. If \( a = 0 \) then \( G_{[0]}(L) = G_{[0]}(L_1) \cup (N(L_1^\ast) + N(L_2)) \), where 
\[ |b| = \{ x \in L_1 \mid x \geq b \} \] 
is the principal dual ideal generated by \( b \) in 
\( L_1 \) and \( L_1^\ast = L_1 \{ 0 \} \).

Proof. (1) Consider \( a \neq 0 \) and \( a \notin V(G_{[0]}(L_1)) \). If \( x \in V(G_{[0]}(L)) \) is 
adjacent to some \( y \in L_2 \), i.e., \( x \wedge y = 0 \). As \( a \leq y \) in \( L \), we get \( a \wedge x = 0 \) 
gives \( a \in V(G_{[0]}(L_1)) \), a contradiction to the fact that \( a \notin V(G_{[0]}(L)) \). 
Hence no element of \( L_2 \) is adjacent to any vertex of \( G_{[0]}(L) \). Hence 
\( G_{[0]}(L) = G_{[0]}(L_1) \).

(2) Now, let \( a \in V(G_{[0]}(L_1)) \). If \( x \in V(G_{[0]}(L_1)) \), then there exists 
\( (\neq 0)y \in L \) such that \( x \wedge y = 0 \). This implies that \( x \) and \( y \) both can 
not be in \( L_2 \) otherwise \( a \leq x \wedge y = 0 \), a contradiction. If \( x, y \in L_1 \), 
then \( x \in V(G_{[0]}(L_1)) \). Without loss of generality, let \( x \in L_1 \) and 
\( y \in L_2 \) which gives \( x \wedge a = 0 \) as \( a \leq y \). Therefore \( x \in G_a \). Thus 
\( x \in V(G_a + N(L_2)) \). Hence \( V(G_{[0]}(L)) \subseteq V(G_{[0]}(L_1) \cup (G_a + N(L_2))) \). 
Let \( x \in V(G_{[0]}(L_1) \cup (G_a + N(L_2))) \). If \( x \in V(G_{[0]}(L_1)) \) then \( x \in 
V(G_{[0]}(L)) \). If \( x \in V((G_a + N(L_2))) \) then there exists \( (\neq 0)y \in L \) such 
that \( x \wedge y = 0 \). This gives \( x \) and \( y \) both can not be in \( L_2 \). If \( x, y \in L_1 \) 
then \( x \in G_{[0]}(L_1) \) and we are done. If \( x \in L_1 \), \( y \in L_2 \) and \( x, y \) are 
adjacent gives \( x \wedge a = 0 \). Hence \( x \in G_a \).

Therefore \( V(G_{[0]}(L_1) \cup (G_a + N(L_2))) \subseteq V(G_{[0]}(L)) \subseteq V(G_{[0]}(L_1) \cup 
(G_a + N(L_2))) \). Hence the equality holds. Let \( x \) and \( y \) are adjacent 
in \( G_{[0]}(L) \). Therefore \( x \wedge y = 0 \). Hence \( x \) and \( y \) both can not be in 
\( L_2 \). If \( x, y \in L_1 \) then \( x \) and \( y \) are adjacent in \( G_{[0]}(L_1) \). Without loss 
of generality, let \( x \in L_1 \) and \( y \in L_2 \). Therefore \( x \wedge a = 0 \). Hence 
\( x \in G_a \), i.e., \( x \) and \( y \) are adjacent in \( G_a + N(L_2) \). Now, let \( x \) and
y are adjacent in $G_{\{0\}}(L_1) \cup (G_a + N(L_2))$. If $x, y \in G_{\{0\}}(L_1)$ we are through. Let $x \in G_a$ and $y \in L_2$. Then $x \land a = 0$. We claim that $x \land y = 0$. Suppose $x \land y \neq 0$. Then we have two possibilities either $x \land y \in L_1$ or $x \land y \in L_2$. If $x \land y \in L_2$. Then by the definition of 2-sum, we have $a \leq x \land y$ which yields $a \leq x \land y \land a = 0$, a contradiction (as $y \geq a$). Thus $x \land y \notin L_2$. Hence $x \land y \in L_1$. Since $x \land y \leq y$ for $y \in L_2$, again by the definition of 2-sum, we have $x \land y \leq a$. This gives $x \land y \land x \leq a \land x = 0$, a contradiction to the fact that $x \land y \neq 0$. Thus we conclude that $x \land a = 0$ if and only if $x \land y = 0$ for any $y \in L_2$. Therefore $G_{\{0\}}(L) = G_{\{0\}}(L_1) \cup (G_a + N(L_2))$.

(3) Let $a = 0$. We claim that $G_{\{0\}}(L) = G_{\{0\}}(L_1) \cup (N(L_1^* \setminus \{b\}) + N(L_2))$. Let $x \in V(G_{\{0\}}(L))$. Then there exists a non-zero element $y$ such that $x \land y = 0$. Hence $x$ and $y$ both can not be in $L_2$. If $x, y \in L_1$, then $x \in G_{\{0\}}(L_1)$. Without loss of generality, let $x \in L_1$ and $y \in L_2$. If $x \geq b$ then by the definition of 2-sum, we have $x \geq y$. But $y = x \land y = 0$, a contradiction. Hence $x \neq b$, i.e, $x \in N(L_1^* \setminus \{b\})$. Therefore $x \in V(G_{\{0\}}(L_1) \cup (N(L_1^* \setminus \{b\}) + N(L_2)))$. Now, let $x \in V(G_{\{0\}}(L_1) \cup (N(L_1^* \setminus \{b\}) + N(L_2)))$. If $x \in G_{\{0\}}(L_1)$ then we are through. Let $x \in V(N(L_1^* \setminus \{b\}) + N(L_2))$. If $x \in N(L_1^* \setminus \{b\})$ then $x \neq b$. For any $y \in L_2$, we have $x \parallel y$ and by the definition of 2-sum and $a = 0$, we have $x \land y = 0$. Therefore $x \in V(G_{\{0\}}(L))$ as $y$ is non-zero element of $L$. If $x \in N(L_2)$ then for any atom $p$ of $L_1$, $p \land x = 0$. Therefore $x \in G_{\{0\}}(L)$. Hence we get $V(G_{\{0\}}(L)) = V(G_{\{0\}}(L_1) \cup (N(L_1^* \setminus \{b\}) + N(L_2)))$. Let $x$ and $y$ are adjacent in $G_{\{0\}}(L)$. Then $x$ and $y$ both can not be in $L_2$. If $x, y \in L_1$ then they are adjacent
in $G_{\{0\}}(L_1)$. Therefore they are adjacent in $G_{\{0\}}(L_1) \cup (N(L_1^* \setminus \{b\}) + N(L_2))$. Without loss of generality, let $x \in L_1$ and $y \in L_2$. As above, $x \nleq b$. Hence $x \in (N(L_1^* \setminus \{b\})$ and $y \in L_2$. Therefore they are adjacent in $N(L_1^* \setminus \{b\}) + N(L_2)$ and hence in $G_{\{0\}}(L_1) \cup (N(L_1^* \setminus \{b\}) + N(L_2))$.

Conversely, suppose that $x$ and $y$ are adjacent in $G_{\{0\}}(L_1) \cup (N(L_1^* \setminus \{b\}) + N(L_2))$. If both $x, y \in G_{\{0\}}(L_1)$, we are through. Also, $x$ and $y$ both can not be in $L_2$. Without loss of generality, let $x \in L_1$ and $y \in L_2$ then $x \nleq b$, i.e, $x \in N(L_1^* \setminus \{b\})$. Therefore $x \land y = 0$ in $L$. Hence $x$ and $y$ are adjacent in $G_{\{0\}}(L)$. Therefore we get $G_{\{0\}}(L) = G_{\{0\}}(L_1) \cup (N(L_1^* \setminus \{b\}) + N(L_2))$.

The following Figures, 4.4.2 and 4.4.3 and 4.4.4 illustrate Theorem 4.4.2.

![Figure 4.4.2: Theorem 4.4.2, Case (1)](image)
4.4 Zero divisor graphs of 2-sum of lattices

\[ G_{\{0\}}(L) = G_{\{0\}}(L_1) \cup (G_a + N(L_2)) \]

Figure 4.4.3: Theorem 4.4.2, Case (2)

\[ G_{\{0\}}(L_1) \cup (N(L_1 \setminus \{b\}) + N(L_2)) \]

Figure 4.4.4: Theorem 4.4.2, Case (3)
The following result is essentially due to Alizadeh et. al. \[1\]

**Theorem 4.4.3** (Alizadeh [1, Theorem 3.3]). Let \( L \) be a lattice with 0. Then \( G_{\{0\}}(L) \) is a connected graph and \( \text{diam}(G_{\{0\}}(L)) = 1 \) if and only if \( V(G_{\{0\}}(L)) = \text{At}(L) \), where \( \text{At}(L) \) denotes the set of atoms of \( L \).

**Proposition 4.4.4.** If \( \text{diam}(G_{\{0\}}(L)) = 1 \), where \( L = L_1 \uparrow aL_2 \) and \( a \neq 0 \), then \( V(G_{\{0\}}(L)) = \text{At}(L_1) \). Moreover, every atom in \( L_1 \) is strictly below \( a \).

**Proof.** Let \( \text{diam}(G_{\{0\}}(L)) = 1 \). By applying Theorem 4.4.3, all vertices of \( G_{\{0\}}(L) \) are nothing but the atoms of \( L \). Since \( a \neq 0 \), \( \text{At}(L) = \text{At}(L_1) \). If \( p \) is an atom of \( L_1 \) such that \( a \land p = 0 \) then for any \( x \in L_2 \) we have \( x \land p = 0 \) if and only if \( x \land y = 0 \) for any \( y \in L_2 \). Thus \( d(a, x) = 2 \), a contradiction. Hence every atom of \( L_1 \) is below \( a \). We claim that \( a \) can not be an atom. If \( a \) is the only atom of \( L_1 \) then \( V(G_{\{0\}}(L)) = \emptyset \), as 0 is a prime ideal. If there is another atom of \( L_1 \) distinct from an atom \( a \). Then by applying above procedure we get a contradiction. Hence \( a \) can not be an atom. \( \square \)

The following example shown in Figure 4.4.5 on Page 78 illustrates this result.

Note that the assertion of Proposition 4.4.4, need not be true if \( a = 0 \). Consider two lattices \( L_1 \) and \( L_2 \) and its 2-sum shown in Figure 4.4.6 on Page 78. Then \( \text{diam}(G_{\{0\}}(L)) = 1 \), but not all vertices of \( G_{\{0\}}(L) \) are atoms of \( L_1 \).
4.4 Zero divisor graphs of 2-sum of lattices

Theorem 4.4.5. Let \( L_1 \) be a lattice such that \( \text{diam}(G_{\{0\}}(L_1)) = 3 \) and \( L_2 \) be any arbitrary lattice. If \( L = L_1 \upharpoonright a L_2 \) and \( a \neq 0 \), then \( \text{diam}(G_{\{0\}}(L)) = 3 \).

Proof. Since \( \text{diam}(G_{\{0\}}(L_1)) = 3 \), there exist \( x, y \in V(G_{\{0\}}(L_1)) \) such that \( d(x, y) = 3 \). Since adjacency in \( G_{\{0\}}(L_1) \) is preserved in \( G_{\{0\}}(L) \) where \( L = L_1 \upharpoonright a L_2 \), therefore we have the same path in \( G_{\{0\}}(L) \). Clearly, \( x \) and \( y \) can not be adjacent in \( G_{\{0\}}(L) \). Hence \( d(x, y) \neq 1 \) in \( G_{\{0\}}(L) \). If \( d(x, y) = 2 \) in \( G_{\{0\}}(L) \), then there is vertex \( e \) such that \( x - e - y \) is a path in \( G_{\{0\}}(L) \). Clearly, \( e \notin V(G_{\{0\}}(L_1)) \). Therefore \( e \in L_2 \).
Hence by applying the definition of 2-sum, we have $a \leq e$. Now, since $a \leq e$ and $x - e - y$ is a path in $G_{[0]}(L)$, we get $a \wedge x = 0 = a \wedge y$. Therefore $x - a - y$ is a path in $G_{[0]}(L_1)$, a contradiction to the fact that $d(x, y) = 3$ in $G_{[0]}(L_1)$. Therefore $d(x, y) = 3$ in $G_{[0]}(L)$. Hence by Theorem 3.2.17 of Chapter 3, we have $\text{diam}(G_{[0]}(L)) = 3$.

Lemma 4.4.6. Let $L$ be 2-sum of two chains $C_1$ and $C_2$ with adjunct pair $(a, 1)$, i.e, $L = C_1[a]C_2$. If $G_{[0]}(L) \neq \emptyset$ then $a = 0$ and $G_{[0]}(L)$ is a complete bipartite graph. Hence $\text{diam}(G_{[0]}(L)) \leq 2$. Moreover, $G_{[0]}(L) = K_{m,n}$, if $|C_1| = n + 2$ and $|C_2| = m$ where $m, n \in \mathbb{N}$.

Proof. Let $L = C_1[a]C_2$ and $G_{[0]}(L) \neq \emptyset$. If $a \neq 0$ then $a \notin V(G_{[0]}(C_1))$, as $V(G_{[0]}(C_1)) = \emptyset$. By applying Theorem 3.4.2, we get $G_{[0]}(L) = G_{[0]}(C_1) = \emptyset$, a contradiction. Therefore $a = 0$.

Now, every element of $C_1 \{0, 1\}$ is adjacent to each element of $C_2$. Hence $G_{[0]}(L)$ is a complete bipartite graph, in fact $G_{[0]}(L) = K_{m,n}$ whenever $|C_1| = n + 2$ and $|C_2| = m$ where $m, n \in \mathbb{N}$.

Remark 4.4.7. If $L$ is an adjunct of more than 2 chains, then $G_{[0]}(L)$ need not be bipartite. Consider the lattice depicted in the following Figure 4.4.7 on Page 79. Consider $L = C_1[1]C_2[1]C_3$ where $C_1 = \{0, x, 1\}$, $C_2 = \{y\}$ and $C_3 = \{z\}$. Then $G_{[0]}(L) \cong K_3$ which is not bipartite.

![Figure 4.4.7: A lattice with non-bipartite zero divisor graph]
The concept of dismantlable lattice is introduced by Rival [45].

**Definition 4.4.8** (Rival [45]). A finite lattice $L$ having $n$ elements is called dismantlable, if there exists a chain $L_1 \subset L_2 \subset \cdots \subset L_n (= L)$ of sublattices of $L$ such that $|L_i| = i$, for all $i$.

The following structure Theorem is due to Thakare, Pawar and Waphare [48].

**Theorem 4.4.9.** A finite lattice is dismantlable if and only if it is an adjunct of chains.

We call a dismantlable lattice $L$ to be a *lower dismantlable* if every adjunct pair $L$ is of the form $(0, b)$.

In the following examples, the lattice $L$ is lower dismantlable where as the lattice $L'$ is not lower dismantlable.

![Figure 4.4.8: Examples of lower dismantlable and non- lower dismantlable lattice](image)

**Definition 4.4.10.** Let $M_n = \{0, 1, a_1, a_2, \ldots, a_n\}$ be a lattice such that $0 < a_i < 1 \ \forall i$, where $i = 1, 2, \ldots, n$ with $a_i \land a_j = 0$ and $a_i \lor a_j = 1$ for every $i \neq j$. 
4.4 Zero divisor graphs of 2-sum of lattices

**Lemma 4.4.11.** If $L$ is a lower dismantlable lattice then, for non-zero elements $a, b \in L$, $a \parallel b$ if and only if $a \land b = 0$.

**Proof.** As $a$ and $b$ both are non-zero elements of $L$, this together with $a \land b = 0$ implies that $a \parallel b$. On the other hand if $a \parallel b$ and $a \land b \neq 0$ then it is clear that there is an adjunct pair $(a_1, b_1)$ in the adjunct representation of $L$ such that $a_1 = a \land b \neq 0$, a contradiction to the definition of lower dismantlability of $L$. 

**Theorem 4.4.12.** Let $L$ be a lower dismantlable lattice which is adjunct of $n$ chains, where $n \geq 2$. Then $V((G_{\{0\}}(L))) \neq \emptyset$ and $diam(G_{\{0\}}(L)) \leq 2$. Moreover, if we assume that $(0, 1)$ is an adjunct pair, then $diam(G_{\{0\}}(L)) = 1$ if and only if $L \cong M_n$.

**Proof.** Let $a, b \in V((G_{\{0\}}(L)))$. If $a \parallel b$, then by Lemma 4.4.11, $a \land b = 0$. Hence they are adjacent. If $a$ and $b$ are comparable, say $a \leq b$ then $a$ and $b$ are on the same chain. Since $L$ is an adjunct of more than two chains, then there is an element $(\neq 0)c$ such that $b \land c = 0$. Hence $a \land c = 0$. Thus we get a path $a-c-b$ of length 2. Therefore $d(a, b) \leq 2$. Hence in any case $diam(G_{\{0\}}(L)) \leq 2$.

Now, to prove moreover statement. Let $(0, 1)$ is an adjunct pair. If $L \cong M_n$, then $G_{\{0\}}(L) \cong K_n$. Hence $diam(G_{\{0\}}(L)) = 1$.

Conversely, suppose $diam(G_{\{0\}}(L)) = 1$. If $L = C_1 \upharpoonright \downharpoonleft C_2$, then $G_{\{0\}}(L) \neq \emptyset$ if and only if $|C_1| \geq 3$ and $|C_2| \geq 1$. If $|C_1| \geq 3$ and $|C_2| = 2$ then for $a, b \in C_2$ we have $d(a, b) = 2$. If $|C_1| \geq 4$ and $|C_2| = 1$ then for any $a, b \in C_1 \setminus \{0, 1\}$, $d(a, b) = 2$. Therefore $|C_1| = 3$ and $|C_2| = 1$ and the result follows by induction. 

□
Note that if we drop the condition of lower dismantlable, then the Theorem 4.4.12 need not be true. Consider $L = C_1 |_0C_2 |_bC_3 |_dC_4$ where $C_1 = \{0, a, e, 1\}$, $C_2 = \{b\}$, $C_3 = \{d\}$ and $C_4 = \{c\}$.

\[
\begin{array}{c}
\text{Figure 4.4.9: A dismantlable lattice } L \text{ with } \text{diam}(G_{\{0\}}(L)) = 3.
\end{array}
\]

Now, we characterize the $gr(G_{\{0\}}(L))$ for lower dismantlable lattices.

**Theorem 4.4.13.** Let $L$ be a lower dismantlable lattice. Then $gr(G_{\{0\}}(L)) \in \{3, 4, \infty\}$. In fact

- $gr(G_{\{0\}}(L)) = 3$ if and only if $L$ is 2-sum of at least 3-chains.
- $gr(G_{\{0\}}(L)) = 4$ if and only if $L = C_1 |_{\alpha_1}C_2$ with $|C_2| \geq 2$.
- $gr(G_{\{0\}}(L)) = \infty$ if and only if $L = C_1 |_{\alpha_1}C_2$ with $|C_2| = 1$.

**Proof.** If $L$ is 2-sum of more than 2-chains, then it contains at least 3 atoms. Hence $G_{\{0\}}(L)$ contains a triangle. Hence $gr(G_{\{0\}}(L)) = 3$. Conversely, let $gr(G_{\{0\}}(L)) = 3$. If $L = C_1 |_{\alpha_1}C_2$, then by Lemma 4.4.6, $G_{\{0\}}(L)$ is complete bipartite. Therefore it does not contain an odd cycle, a contradiction. Hence $L$ is 2-sum of at least 3 chains.

If $L = C_1 |_{\alpha_1}C_2$, then by Lemma 4.4.6, $G_{\{0\}}(L)$ is complete bipartite. Then $|C_2| = 1$ or $|V(G_{\{0\}}(L)) \cap C_1| = 1$ if and only if $gr(G_{\{0\}}(L)) = \infty$.

If $|C_2| \geq 2$ or $|V(G_{\{0\}}(L)) \cap C_1| \geq 2$, then $gr(G_{\{0\}}(L)) = 4$. \qed
Now, we recall some more definition from graph and lattice theory.

**Definition 4.4.14.** An element $x$ in a lattice $L$ is join-reducible (meet-reducible) in $L$ if there exist $y, z \in L$ both distinct from $x$, such that $y \lor z = x$ ($y \land z = x$); $x$ is join-irreducible (meet-irreducible) if it is not join-reducible (meet-reducible); $x$ is doubly irreducible if it is both join-irreducible and meet-irreducible. Therefore, an element $x$ is doubly reducible in a lattice $L$ if and only if $x$ has at most one lower cover or $x$ has at most one upper cover. The set of all meet irreducible (join-irreducible) elements in $L$ is denoted by $M(L)$ ($J(L)$). The set of all doubly irreducible in $L$ is denoted by $Irr(L)$ and its complement in $L$ is denoted by $Red(L)$. Thus, if $x \in Red(L)$ then $x$ is either join reducible or meet reducible. The cover graph of a lattice $L$, denoted by $CG(L)$, is the graph whose vertices are the elements of $L$ and whose edges are the pairs $(x, y)$ with $x, y \in L$ satisfying $x < y$ or $y < x$ (by $a < b$ we mean there is no $c$ such that $a < c < b$).

**Definition 4.4.15.** A graph is acyclic if it has no cycles. A tree is a connected acyclic graph. A tree is called a rooted tree if one vertex has been designated the root, in which case the edges have a natural orientation, towards or away from the root. A vertex $w$ is called an ancestor of $v$ if $w$ is on the unique path from $v$ to the root of the tree. Let $T$ be a rooted tree in which the root $R$ has at least two chains and $G(T)$ be the non-ancestor graph of $T$, i.e, $V(G(T)) = T \setminus \{R\}$ and two vertices are adjacent if and only if no one is an ancestor of the other. Denote the class of non-ancestor graph by $\mathcal{G}_T$.

The following result is essentially due to Kelly and Rival [31].
Theorem 4.4.16 (Kelly and Rival [31]). A finite lattice is dismantlable if and only if it contains no crown.

Theorem 4.4.17. The following statements are equivalent for a finite lattice $L$.

(a) $L$ is a lower dismantlable lattice.

(b) Every non-zero element of $L$ is a meet irreducible element.

(c) The cover graph $CG(L)$ of $L$ has the property that $CG(L)\backslash\{0\}$ is a tree.

Proof. (a) $\Rightarrow$ (b) Let $L$ be a lower dismantlable lattice. Then by Theorem 4.4.9, $L = C_1^{a_1}C_2^{a_2}...C_n^{a_n-1}C_n$, where each $C_i$ is a chain. Let $(\neq 0)a \in L$ be an element which is not meet irreducible. Then $a = b \land c$ for some $b, c \neq a$. But then $b\parallel c$. By Lemma 4.4.11, $a = b \land c = 0$, a contradiction to $a \neq 0$. Hence every non-zero element of $L$ is a meet irreducible element.

(b) $\Rightarrow$ (c) Let $CG(L)$ be the cover graph of $L$ and let $C : a_1 \prec a_2 \prec ... \prec a_n \prec a_1$ be a cycle in $CG(L)\backslash\{0\}$. For distinct $a_1, a_2, a_3$ we have the following 3 cases.

Case (1) $a_1 \prec a_2 \prec a_3$,

Case (2) $a_2 = a_1 \land a_3$,

Case (3) $a_2 = a_3 \lor a_1$

The second case is impossible, as $a_1 \land a_3 = a_2 \neq 0$, a contradiction to lower dismantlability of $L$.

Case (1): Let $a_1 \prec a_2 \prec a_3$ is a chain. Then for $a_4$, we have either $a_1 \prec a_2 \prec a_3 \prec a_4$ is a chain or $a_1 \prec a_2 \prec a_3$ and $a_4 \prec a_3$. Then,
$a_{i+1} \leq a_i$, for all $i \geq 4$ otherwise we get $a_4$ is a meet irreducible element. Hence $a_n \leq a_{n-1}$ and $a_n = a_{n-1} \land a_1 \neq 0$, as $C : a_1 - a_2 - \ldots - a_n - a_1$ is a cycle. This contradicts to the fact that $a \neq 0$. On the other hand if $a_1 < a_2 < a_3 < a_4$ is chain, then using the above arguments we have $a_1 < a_2 < a_3 < a_4 < a_5$ is a chain, continuing in this way we get $a_1 < a_2 < \ldots < a_n$ is a chain and $a_1 \leq a_n$. Hence $a_1$ and $a_n$ can not be adjacent, a contradiction to the fact that $C$ is a cycle in $CG(L) \setminus \{0\}$.

Now we consider Case (3). Here we can not have $a_3 \leq a_4$ otherwise $a_3$ is the meet of $a_2$ and $a_4$, a contradiction. Hence $a_4 \leq a_3$. In fact $a_{m+1} \leq a_m$, for $m \geq 3$. Using the above arguments, we get a contradiction in this case also. Hence $CG(L)$ is a connected acyclic graph. Therefore it is a tree.

$(c) \Rightarrow (a)$ If $L$ contains a crown then $CG(L)$ contains a cycle, a contradiction. Hence $L$ does not contain a crown. By applying Theorem 4.4.10, $L$ is a dismantlable lattice. Now, let $a$ and $b$ are incomparable elements of $L$. Suppose that $a \land b \neq 0$. Let $a \land b = a_1 \prec a_2 \prec \ldots \prec a_i = a \prec \ldots \prec a_n = a \lor b$, be a covering and also $a \land b = b_1 \prec b_2 \prec \ldots \prec b_j = b \prec \ldots \prec b_m = a \lor b$, be another covering, distinct from the first covering (such coverings exist, since $a \parallel b$). Then $a_1 - a_2 - \ldots - a_n = b_m - b_{m-1} - \ldots - b_1 = a_1$ is a cycle in $CG(L) \setminus \{0\}$, a contradiction to the fact that $CG(L) \setminus \{0\}$ is a tree. Thus $L$ is a lower dismantlable lattice.

Now, we prove a result regarding the complete bipartite graph of a lower dismantlable lattice.
Theorem 4.4.18. Let $L$ is a lower dismantlable lattice having $(0, 1)$ as an adjunct pair. Then the following statements are true.

(a) $|V(G_{[0]}(L))| = |L| - 2$.

(b) $G_{[0]}(L)$ is a complete bipartite, $K_{m,n}$ if and only if $L$ is adjunct of two chains only, i.e, $L = C_1[0]C_2$ with $|C_2| = n + 2$ and $|C_1| = m$.

(c) An equivalence class $[x]$ of $V(G_{[0]}(L))$ under the equivalence relation $a \sim b$ if and only if $N(a) = N(b)$, i.e, having same neighbors, contains a member of adjunct pair if and only if there is a pair of vertices $y, z \in V(G_{[0]}(L))$ such that $y$ is adjacent to $z$ and $x$ is not adjacent to any of $y$ and $z$.

Proof. (a) As $L$ is a lower dismantlable lattice having $(0, 1)$ as an adjunct pair, $L$ contains at least two chains in its adjunct representation. Also $a \wedge b = 0$ if and only if $a||b$. Hence any $a \in L\{0, 1\}$ is in $V(G_{[0]}(L))$. Hence $|V(G_{[0]}(L))| = |L| - 2$.

(b) Follows from Lemma 4.4.6 and the fact that the independent set of $K_{m,n}$ forms a chain in $L$. Note that $L$ can not be adjunct of more than two chains, otherwise we get three atoms and it will form a triangle.

(c) Let $[x] = \{y \in V(G_{[0]}(L)) \mid N(x) = N(y)\}$. Let $x_1 \in [x]$ be such that $(0, x_1)$ is an adjunct pair in $L$. Hence $x_1$ and $x$ are comparable. If $x \leq x_1$ then there exist $x_2 \in L$ such that $x_2 \leq x_1$ and $x \wedge x_2 = 0$. But then $x_1 \wedge x_2 \neq 0$, a contradiction to the fact that $x \sim x_1$. Therefore $x_1 \leq x$. Also since $(0, x_1)$ is an adjunct pair, there exist non-zero elements $y, z \in L$ such that $y, z \leq x_1$ and $y \wedge z = 0$. Hence $y$ and $z$ are
adjacent. Clearly, $x$ is not adjacent to any of $y$ and $z$.

Conversely, suppose there is a pair $y, z \in V(G_{\{0\}}(L_1))$ such that $y \land z = 0$, $x \land y \neq 0$ and $x \land z \neq 0$. Therefore $x$ and $y$ are comparable. So are $x$ and $z$. If $x \leq y$ or $x \leq z$, we get a contradiction to the fact that $x \land y \neq 0$ and $x \land z \neq 0$. Therefore $y \leq x$ and $z \leq x$. Hence $y \lor z \leq x$.

It is clear that $(0, y \lor z)$ is an adjunct pair. Also any element adjacent to $x$ is adjacent to $y \lor z$. Suppose $c$ be a vertex which is adjacent to $y \lor z$ but not to $x$, i.e., $c \land (y \lor z) = 0$ and $x \land c \neq 0$. Then $x$ and $c$ are comparable. Since $y \lor z \leq x$, the case $x \leq c$ is impossible. Hence $c \leq x$. If $y \lor z \in [x]$ then we are through, as $(0, y \lor z)$ is an adjacent pair. If not, i.e., $y \lor z \notin [x]$ then $N(x) \subsetneq N(y \lor z)$. Hence there exists $c_1 \in N(y \lor z)$ such that $c_1 \notin N(x)$. This gives $x$ and $c_1$ are comparable.

Using the above arguments, we get $c_1 \leq x$ and $N(x) \subseteq N(y \lor z \lor c_1)$. Continuing in this way we get an element say $c_n$ (as $L$ is finite) such that $N(x) = N(y \lor z \lor \bigvee_{i=1}^{n} c_i)$. Then $(0, y \lor z \lor \bigvee_{i=1}^{n} c_i)$ is an adjunct pair such that such that $y \lor z \lor \bigvee_{i=1}^{n} c_i \in [x]$. 

We conclude the Thesis by proving a realization problem of zero divisor graphs of lower dismantlable lattices, i.e., which graphs are the zero divisor graphs of lower dismantlable lattices.

**Theorem 4.4.19.** The following statements are equivalent for a simple undirected graph $G$.

(a) $G \in \mathcal{G}_T$.

(b) $G = G_{\{0\}}(L)$ for some lower dismantlable lattice $L$ having join reducible 1.
(c) $G$ is complement of the comparability graph of $(L\setminus\{0,1\},\leq)$ for some lower dismantlable lattice $L$ having join reducible 1.

Proof. $(a) \Rightarrow (b)$ Let $G \in \mathcal{G}_T$. Hence $G = V(T\setminus\{R\})$ for some rooted tree $T$ with the root $R$. Let $L = V(G) \cup \{R\} \cup \{0\}$. Define $\leq$ on $L$ by, $a \leq R$, $0 \leq a$ and $a \leq a$, for every $a \in L$. If $a \neq b$ then $a < b$ if and only if $b$ is an ancestor of $a$. Clearly, $(L, \leq)$ is poset. If $a \parallel b$ then no one is ancestor of the other, hence 0 is the only element below $a$ and $b$, i.e, $a \land b = 0$. Let $A = \{c \in L \mid c$ is a common ancestor of $a$ and $b\}$. Then $A \neq \emptyset$, as $R \in A$. We claim that the set $A$ forms a chain. Let $x, y \in A$ with $x \parallel y$. Hence $x$ and $y$ are ancestors of $a$ and $b$ respectively. But then $a - x - b - y - a$ is a cycle in the undirected graph of a rooted tree, a contradiction. Thus $A$ is a chain. Then the smallest element of $A$ (it exists due to finiteness of $L$) is nothing but $a \lor b$. Hence $L$ is a lattice. Since meet of any two incomparable elements is zero, $L$ does not contain crown, hence by Theorem 4.4.16, $L$ is a dismantlable lattice, say $L = C_1^{b_1a_1}C_2^{b_2a_2}...C_n^{b_na_n}$. Since meet of any two incomparable elements is zero, we get $a_i = 0$, $\forall i$. Therefore $L$ is lower dismantlable lattice $L$ having join reducible 1 (since the root of tree has at least two chains, hence 1 is join reducible).

$(b) \Rightarrow (c)$ Let $G = G_{\{0\}}(L)$ for some lower dismantlable lattice $L$ having join reducible element 1. Let $H = C(L\setminus\{0,1\})^c$, the complement of the comparability graph of $L\setminus\{0,1\}$. Clearly, $V(H) = V(G)$. Then $a$ and $b$ are adjacent in $G$ if and only if $a \parallel b$ if and only if $a$ and $b$ are adjacent in $H$. Hence $G = H$.

$(c) \Rightarrow (a)$ Let $G = C(L\setminus\{0,1\})^c$. Then the cover graph of $L\setminus\{0\}$ is a
rooted tree with root 1, say $T$. Let $H$ be an ancestor graph of $T$. Then clearly, $V(H) = V(G)$ and $a$ and $b$ are adjacent in $G$ if and only if $a\parallel b$ if and only if no one is ancestor of the other if and only if $a$ and $b$ are adjacent in $H$. Hence $G = H$. 