Chapter 3

The graph of equivalence classes of zero divisors

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3.1 Introduction

In this chapter, we introduce the graph $G_E(L)$ of equivalence classes of zero divisors of a meet semi-lattice $L$ with 0. Further, it is proved that for a semi-complemented meet semi-lattice $L$, $G_E(L) = G_{\{0\}}(L)$ if and only if $L$ is $SSC$. Also, we prove that if $L_1$ and $L_2$ be bounded $SSC$ meet semi-lattices, then $G_{\{0\}}(L_1) \cong G_{\{0\}}(L_2)$ if and only if $L_1 \cong L_2$. We verify the diameter and girth of the graph of equivalence classes of zero divisors of a meet semi-lattice $L$ and then we find out the relation between diameter and girth of $G_E(L)$ and $G_{\{0\}}(L)$. We prove some results regarding to cut vertices also. At last, we show that Beck’s Conjecture is true for $G_E(L)$.

3.2 Properties of the graph $G_E(L)$

We recall the necessary definitions and terminology.

A non-empty subset $I$ of a meet semi-lattice $L$ is said to be semi-ideal, if $y \leq x \in I$ implies that $y \in I$.

A proper semi-ideal(ideal) is said to be prime, if $a \land b \in I$ implies either $a \in I$ or $b \in I$.

A prime semi-ideal(ideal) $P$ of a meet semi-lattice $L$ is said to be minimal prime semi-ideal(ideal), if there is no prime ideal $Q$ such that $Q \nsubseteq P$. Let $L$ be a meet semi-lattice with 0. The set of all zero divisors of $L$ is denoted by $Z_{\{0\}}(L) = \left\{ x \in L \mid x \land y = 0 \text{ for some } y \in L \setminus \{0\} \right\}$ and the set of all non-zero zero divisors is denoted by
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$Z_{\{0\}}(L)^* = \left\{ x \in L \setminus \{0\} \mid x \land y = 0 \text{ for some } y \in L \setminus \{0\} \right\}$. Clearly, $Z_{\{0\}}(L)^* \cup \{0\} = Z_{\{0\}}(L)$.

Joshi [23] introduced the concept of the zero divisor graph of a poset $P$ having the smallest element 0 with respect to an ideal $I$ of $P$. We mention below this definition when the corresponding poset is a meet semi-lattice and $I = \{0\}$. Note that this definition coincides with the definition given by Lu and Wu [38] when $I = \{0\}$.

**Definition 3.2.1** (Joshi [23], Lu and Wu [38]). Let $L$ be a semi-lattice with 0. We associate a simple undirected graph, called the zero divisor graph of $L$ with respect to $\{0\}$, denoted by $G_{\{0\}}(L)$, with the set of vertices is $V(G_{\{0\}}(L)) = Z_{\{0\}}(L)^*$ and two distinct vertices $x, y$ are adjacent if and only if $x \land y = 0$.

For any $x, y \in L$, we say that $x \sim y$ if and only if $ann(x) = ann(y)$, where $ann(a) = \{b \mid a \land b = 0\}$. Note that $\sim$ is an equivalence relation on $L$. Furthermore, if $x_1 \sim x_2$ and $x_1 \land y = 0$ then $y \in ann(x_1) = ann(x_2)$ and hence $x_2 \land y = 0$.

We define $[x]$, the class of $x$ as follows: $[x] = \{z \in L \mid ann(z) = ann(x)\}$. Clearly, $a \in [a]$.

The graph of equivalence classes of zero divisors of commutative rings is well studied in Allen et. al. [8], Anderson and LaGrange [5], and Spiroff and Wickham [46].

Now, we introduce the graph $G_E(L)$ of equivalence classes of zero divisors of a meet semi-lattice $L$ with 0, denoted by $G_E(L)$. 
Definition 3.2.2. Let \( L \) be a meet semi-lattice with 0. We associated the simple undirected graph to \( L \) whose vertices are the equivalence classes of elements in \( \mathbb{Z}_{\{0\}}(L)^* \) and with each pair of distinct classes \([x]\) and \([y]\) joined by an edge if and only if \( [x] \land [y] = \{0\} = [0] \).

We illustrate this concept with an example in Figure 3.2.1.

Consider the following two infinite lattices \( L \) and \( L' \) in Figure 3.2.2.

We can see that the zero divisor graph of lattice \( L, G_{\{0\}}(L) \), is infinite but the graph of equivalence classes of zero divisors of the lattice \( L, \)
3.2 Properties of the graph \( G_E(L) \)

namely, \( G_E(L) \), is finite. Also for the lattice \( L' \), \( G_{\{0\}}(L') \) as well as \( G_E(L') \) both are infinite.

Now we have some properties of \( G_E(L) \).

**Lemma 3.2.3.** Let \( x \) and \( y \) be elements of \( L \). If \( 0 \neq \text{ann}(x) \subseteq \text{ann}(y) \) then \( \deg[x] \leq \deg[y] \).

**Proof.** Let \([u] \in V(G_E(L))\) such that \([u] - [x] \). Then \( u \wedge x = 0 \) which further yields \( u \in \text{ann}(x) \subseteq \text{ann}(y) \). Hence \([u] - [y] \). \( \square \)

**Lemma 3.2.4.** Let \( x \) and \( y \) be elements of \( L \). If \( 0 \neq \text{ann}(x) \nsubseteq \text{ann}(y) \) then \( \deg[x] < \deg[y] \).

**Proof.** Obvious. \( \square \)

**Theorem 3.2.5.** Let \( x_1, x_2, ..., x_r \) be elements of a lattice \( L \) and suppose \( \text{ann}(x_1) \nsubseteq \text{ann}(x_2) \nsubseteq ... \nsubseteq \text{ann}(x_r) \) is a chain of associated primes of \( L \). If \( 3 \leq |V(G_E(L))| < \infty \) then \( \deg[x_1] < \deg[x_2] < ... < \deg[x_r] \).

**Proof.** Follows from Lemma 3.2.3 and Lemma 3.2.4. \( \square \)

**Theorem 3.2.6.** If \( \deg[y] > \deg[x] \) for every \([x] \in V(G_E(L))\), then \( \text{ann}(y) \) is maximal in \( \mathcal{L} \), where \( \mathcal{L} = \{\text{ann}(x)|0 \neq x \in L\} \) and hence is an associated prime.

**Proof.** Let \([y] \) be the unique vertex in \( G_E(L) \) with maximal degree. We claim that \( \text{ann}(y) \) is maximal in \( \mathcal{L} \). Let \( \text{ann}(y) \nsubseteq \text{ann}(z) \nsubseteq \text{ann}(0) = L \), such \( \text{ann}(z) \) with \( z \neq 0 \) exists due to \( \text{Clique}(G_{\{0\}}(L)) < \infty \). By Lemma 3.2.4, \( \deg[y] < \deg[z] \), a contradiction to the assumption that \( \deg[y] > \deg[x] \) for every \([x] \in V(G_E(L))\). Hence \( \text{ann}(y) = \text{ann}(z) \). Being \( \text{ann}(y) \) maximal in \( \mathcal{L} \), it is associated prime. \( \square \)
Theorem 3.2.7. Let $G_E(L)$ be a finite graph. Then any vertex of maximal degree is maximal in $L$ and hence is an associated prime.

Proof. Let $d$ denote the maximal degree of $G_E(L)$. If there is only one vertex of degree $d$, then Theorem 3.2.6 yields the desired result. Therefore we may assume that $|G_E(L)| > 3$ and that $G_E(L)$ has at least two vertices of degree $d$. Suppose that $[y_1]$ is a vertex of degree $d$. Then $\text{ann}(y_1) \subseteq \text{ann}(y_2)$ for some $y_2$ such that $\text{ann}(y_2)$ is maximal in $L$, and so $[y_2]$ is an associated prime. We claim that $[y_1] = [y_2]$. By Lemma 3.2.3, we have $d = \deg([y_1]) \leq \deg([y_2])$ and $d$ is a maximal degree, we have $\deg([y_2]) = d$. Therefore $\deg([y_1]) = \deg([y_2])$. The equality of degrees implies that $N([y_1]) = N([y_2])$. Denote this set by $N$. The connectivity of $G_E(L)$ implies that $d \geq 2$ and so $N \neq \emptyset$. To get the desired result, we must to show that $\text{ann}(y_1) = \text{ann}(y_2)$. Now, suppose that $\text{ann}(y_1) \subsetneq \text{ann}(y_2)$, i.e, there exists $t \in \text{ann}(y_2)$ such that $t \notin \text{ann}(y_1)$. Since $N([y_1]) = N([y_2])$ and $t \in N([y_2])$ gives $t \in \text{ann}(y_1)$, a contradiction. Hence $\text{ann}(y_1) = \text{ann}(y_2)$. \qed

Definition 3.2.8. A meet semi-lattice $L$ with 0 is said to be semi-complemented if, for $a \neq 1$, there exists $b \neq 0$ such that $a \wedge b = 0$ and $L$ is said to be section semi-complemented, (in brief SSC) if for $a, b \in L$ with $a < b$, there exists non-zero $c$ such that $0 < c \leq b$ and $a \wedge c = 0$. It is clear that every SSC lattice is a semi-complemented lattice. But not conversely. More details about SSC poset can be found in Thakare, et. al. [47, 48], Joshi [22].

An immediate consequence of the above definition is the following result.
Lemma 3.2.9. Let $L$ be a meet semi-lattice with $0$. Then $L$ is SSC if and only if for $a, b \in L$, $\text{ann}(a) = \text{ann}(b)$ implies $a = b$.

Proof. Suppose $L$ is SSC. Further, assume that there exist $a, b \in L$ such that $\text{ann}(a) = \text{ann}(b)$ and $a \neq b$. Since $L$ is SSC, there exists $c \in L$ such that $0 < c \leq a$ and $c \land b = 0$. This gives $c \in \text{ann}(b) = \text{ann}(a)$, a contradiction to the fact that $c \neq 0$. Hence $a = b$.

Conversely, assume that $\text{ann}(a) = \text{ann}(b) \Rightarrow a = b$ for $a, b \in L$. Let $x < y$. Then $\text{ann}(y) \not\subseteq \text{ann}(x)$. Hence there exists $t \in \text{ann}(x)$ such that $t \notin \text{ann}(y)$. This gives $0 < z = t \land y \leq y$ and $z \land x = 0$. Thus $L$ is SSC.

Remark 3.2.10. For a Boolean lattice, $[a] = \{a\}$ for every $a \in Z_{\{0\}}(L)$, hence in this case $G_{E}(L)$ and $G_{\{0\}}(L)$ are isomorphic. This fact is illustrated in the following example in Figure 3.2.3.

In view of Remark 3.2.10, we raise the following problem.

Problem: Find a class $\mathcal{L}$ of posets such that $G_{\{0\}}(L) \cong G_{E}(L)$ for $L \in \mathcal{L}$.
In the following theorem, we answer this problem.

**Theorem 3.2.11.** Let $L$ be a semi-complemented meet semi-lattice. Then $G_L = G_{\{0\}}(L)$ if and only if $L$ is SSC.

**Proof.** Let $L$ be a SSC meet semi-lattice. Then $G_L = G_{\{0\}}(L)$ follows from Lemma 3.2.9.

Conversely, assume that $G_L = G_{\{0\}}(L)$. This gives $[a] = \{a\}$, $\forall a \in Z_{\{0\}}(L)^*$. Let $a < b (a \neq 0)$.

If $b = 1$, then by semi-complementedness of $L$ we are through. Hence assume that $b \neq 1$. Since $L$ is semi-complemented, $a, b \in Z_{\{0\}}(L)^*$. Hence $[a] = \{a\}$ and $[b] = \{b\}$. This gives $\text{ann}(b) \subseteq \text{ann}(a)$. For otherwise, if $\text{ann}(a) = \text{ann}(b)$, then $b \in [a] = \{a\}$, a contradiction.

From the proof of Lemma 3.2.9, it is clear that $L$ is SSC. □

An immediate consequence of the Theorem 3.2.11 is the following corollary 3.2.14. As a preparation, we need the following Remark 3.2.12 and Lemma 3.2.13.

**Remark 3.2.12.** Let $L$ be a lattice and $L' = \{[x]| \ x \in L\}$, where $[x] = \{z \in L| \ \text{ann}(z) = \text{ann}(x)\}$. Then $L'$ is a meet semi-lattice under the partial order $[x] \leq [y]$ if and only if $\text{ann}(y) \subseteq \text{ann}(x)$.

For this, since $\text{ann}(x), \text{ann}(y) \subseteq \text{ann}(x \wedge y)$, we have $[x \wedge y] \leq [x], [y]$. We claim that $[x] \wedge [y] = [x \wedge y]$. Let $[t]$ be another lower bound of $[x]$ and $[y]$. Then $\text{ann}(x), \text{ann}(y) \subseteq \text{ann}(t)$. Now, we claim that $\text{ann}(x \wedge y) \subseteq \text{ann}(t)$. Let $z \in \text{ann}(x \wedge y)$. Then $z < z \wedge t \in \text{ann}(x) \subseteq \text{ann}(t)$. This gives $x \wedge z \wedge t = 0$, i.e., $z \wedge t \in \text{ann}(x) \subseteq \text{ann}(t)$. Hence $z \wedge t = 0$, that is, $z \in \text{ann}(t)$. This proves that $\text{ann}(x \wedge y) \subseteq \text{ann}(t)$. Thus $[t] \leq [x \wedge y]$.
and this yields \([x] \land [y] = [x \land y]\). Further \([1]\) is the largest element of \(L\). This proves that \(L\) is a bounded meet semi-lattice. It is easy to observe that if \(L\) is a finite lattice, then \(L\) is also a lattice.

In the following result, we prove that \(L\) is an SSC meet semi-lattice.

**Lemma 3.2.13.** \(L\) is an SSC meet semi-lattice.

**Proof.** Let \(0 < [x] \not\leq [y]\). Then \(\text{ann}(y) \not\subseteq \text{ann}(x)\). Hence there exists \(z \in \text{ann}(x)\) such that \(z \notin \text{ann}(y)\). Thus \(0 \neq z \land y = t\). But then \([0] < [t] \leq [y]\) and \([t] \land [x] = [0]\). Thus \(L\) is an SSC meet semi-lattice. 

**Corollary 3.2.14.** Let \(L\) be a semi-complemented meet semi-lattice. Then there exists an SSC meet semi-lattice \(L'\) (as constructed in Remark 3.2.12) such that \(G_E(L) = G_{\{0\}}(L')\).

**Proof.** Let \(L\) be a meet semi-lattice. By Theorem 3.2.11 and Remark 3.2.12, \(L'\) is an SSC meet semi-lattice. From the construction of \(L'\), it is clear that \(G_E(L) = G_{\{0\}}(L')\).

Hence from the Corollary 3.2.14, it is clear that the study of the graph of equivalence classes of zero divisors of \(L\) is nothing but the study of zero divisor graphs of SSC meet semi-lattices.

From Figure 3.2.4 on Page 46, it is clear that for the posets \(P_1, P_2, G_{\{0\}}(P_1) \cong G_{\{0\}}(P_2) \not\Rightarrow P_1 \cong P_2\). Hence it is worth to study the following problem.

**Problem:** Find the class of posets \(\mathcal{P}\) such that \(G_{\{0\}}(P_1) \cong G_{\{0\}}(P_2)\) if and only if \(P_1 \cong P_2\) for the posets \(P_1, P_2 \in \mathcal{P}\).
3.2 Properties of the graph $G_{E}(L)$

Figure 3.2.4: Non-isomorphic lattices with isomorphic zero-divisor graphs

In [25], Joshi and Khiste answered this problem in the case of Boolean posets.

**Theorem 3.2.15** (Joshi and Khiste [25, Theorem 2.11]). Let $P_1$ and $P_2$ be Boolean posets. Then $G_{\{0\}}(P_1) \cong G_{\{0\}}(P_2)$ if and only if $P_1 \cong P_2$.

As an immediate consequence of the above theorem is the following result which is due to LaGrange[33].

**Corollary 3.2.16** (LaGrange [33, Theorem 4.1]). Let $P_1$ and $P_2$ be Boolean algebras. Then $G_{\{0\}}(P_1) \cong G_{\{0\}}(P_2)$ if and only if $P_1 \cong P_2$.

It is well known that there is a 1−1 correspondence between Boolean algebras and Boolean rings, see [33]. Thus as a consequence of Corollary 3.2.16, we obtain the following result which is due to Mohammadian[40].

**Corollary 3.2.17** (Mohammadian [40, Theorem 7]). Let $P_1$ and $P_2$ be Boolean rings. Then $G_{\{0\}}(P_1) \cong G_{\{0\}}(P_2)$ if and only if $P_1 \cong P_2$.

Now, we are ready to extend this result to a class of SSC meet semi-lattices. Note that every Boolean lattice is SSC.

**Theorem 3.2.18.** Let $L_1$ and $L_2$ be bounded SSC meet semi-lattices. Then $G_{\{0\}}(L_1) \cong G_{\{0\}}(L_2)$ if and only if $L_1 \cong L_2$. 
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Proof. Since $L_1$ and $L_2$ are semi-complemented, every non-zero, non-unit element $a$ is in $V(G_{\{0\}}(L_i))$ for $i = 1, 2$. Let $\phi : V(G_{\{0\}}(L_1)) \rightarrow V(G_{\{0\}}(L_2))$ be a graph isomorphism. Define a map $f : L_1 \rightarrow L_2$ such that $f(a) = \phi(a)$, $\forall a \in V(G_{\{0\}}(L_1))$ along with $f(0_{L_1}) = 0_{L_2}$ and $f(1_{L_1}) = 1_{L_2}$, where $0_{L_i}(1_{L_i})$ is the smallest(largest) element of $L_i$. We claim that $f$ is a lattice isomorphism. Let $a \leq b$. First, we show that $f(a) \leq f(b)$. If $a = 0_{L_1}$ or $b = 1_{L_1}$, then we are through. Hence assume that $a \neq 0_{L_1}$ and $b \neq 1_{L_1}$. Suppose $f(a) \nsubseteq f(b)$. Since $L_2$ is SSC, there exists $f(c) \in L_2$ such that $0_{L_2} < f(c) \leq f(a)$ and $f(c) \land f(b) = 0_{L_2}$. Since $b \neq 0$, we have $f(b) \neq 0_{L_2}$. This gives $\phi(c)$ and $\phi(b)$ are adjacent in $G_{\{0\}}(L_2)$. This implies that $c \land b = 0_{L_1}$ in $L_1$ which further yields $c \land a = 0_{L_1}$ in $L_1$. Thus $a$ and $c$ are adjacent in $G_{\{0\}}(L_1)$ as $\phi(c) \leq \phi(a)$. Hence, we have $\phi(c) \land \phi(a) = 0_{L_2}$ in $L_2$. Hence $\phi(c) = 0_{L_2}$, a contradiction. Hence $f(a) \leq f(b)$ in $L_2$.

Conversely, assume that $f(a) \leq f(b)$ in $L_2$. We claim that $a \leq b$. Suppose on the contrary that $a \nsubseteq b$. Since $L_1$ is SSC, there exists $c \in L_1$ such that $0_{L_1} < c \leq a$ and $c \land b = 0_{L_1}$. Thus $c$ and $b$ are adjacent in $G_{\{0\}}(L_1)$. Then $\phi(c)$ and $\phi(b)$ are adjacent in $G_{\{0\}}(L_2)$. Then $\phi(c) \land \phi(b) = 0_{L_2}$ in $L_2$. Replacing the role of $a, b$ in the above proof by $c, a$ respectively, we have $\phi(c) \leq \phi(a)$. This implies that $f(c) \leq f(a)$. Hence $f(c) \leq f(a) \leq f(b)$. But this gives $\phi(c) = \phi(c) \land \phi(b) = 0_{L_2}$. Thus $\phi(c) = 0_{L_2}$, a contradiction to the fact that $\phi(c)$ is a vertex of $G_{\{0\}}(L_2)$. Hence $a \leq b$. Thus $L_1 \cong L_2$.

The converse is obvious. \qed
Lemma 3.2.19. Let $L_1$ be a bounded SSC meet semi-lattice and $L_2$ be a bounded semi-complemented meet semi-lattice such that $G_{\{0\}}(L_1) \cong G_{\{0\}}(L_2)$. Then $L_2$ is also SSC.

Proof. Since $L_1$ and $L_2$ are semi-complemented, every non-zero, non-unit element $a$ is in $V(G_{\{0\}}(L_i))$ for $i = 1, 2$. Let $\phi : V(G_{\{0\}}(L_1)) \rightarrow V(G_{\{0\}}(L_2))$ be a graph isomorphism. Define a map $f : L_1 \rightarrow L_2$ such that $f(a) = \phi(a)$, $\forall a \in V(G_{\{0\}}(L_1))$ along with $f(0_{L_1}) = 0_{L_2}$ and $f(1_{L_1}) = 1_{L_2}$, where $0_{L_i}(1_{L_i})$ is the smallest(largest) element of $L_i$. Since $\phi$ is a bijective map, $f$ is an onto map. Let $f(a) < f(b)$ in $L_2$. Since $L_2$ is semi-complemented, and if $f(a) = 0_{L_2}$ or $f(b) = 1_{L_2}$, then we are through. Hence assume that $f(a) \neq 0_{L_2}$ and $f(b) \neq 1_{L_2}$. Then clearly, $a \neq b$ and $b \neq 1_{L_1}$. Since $L_1$ is SSC, there exists $0_{L_1} < c \leq b$ and $c \wedge a = 0_{L_1}$. Since $c$ and $a$ are adjacent in $G_{\{0\}}(L_1)$, we have $\phi(c)$ and $\phi(a)$ are adjacent in $G_{\{0\}}(L_2)$, that is, $\phi(c) \wedge \phi(a) = 0_{L_2}$. Further $b$ and $c$ are non adjacent in $G_{\{0\}}(L_1)$. Then $\phi(b)$ and $\phi(c)$ are non adjacent in $G_{\{0\}}(L_2)$. Consider $0_{L_2} \neq \phi(b) \wedge \phi(c) = f(b) \wedge f(c)$. Clearly, $0_{L_2} < [f(b) \wedge f(c)] \leq f(c)$ and $[f(b) \wedge f(c)] \wedge f(a) = 0_{L_2}$ (as $\phi(c) \wedge \phi(a) = 0_{L_2}$ implies $f(c) \wedge f(a) = 0_{L_2}$). Thus $L_2$ is SSC. 

Theorem 3.2.20. Let $L_1$ and $L_2$ be bounded semi-complemented meet semi-lattices. If $G_E(L_1) \cong G_{\{0\}}(L_2)$. Then $L_2$ is SSC.

Proof. For the given lattice $L_1$, we can construct an SSC meet semi-lattice $L'_1$ (as given in Remark 3.2.12), such that $G_E(L_1) \cong G_{\{0\}}(L'_1)$. This gives $G_{\{0\}}(L'_1) = G_{\{0\}}(L_2)$. By Lemma 3.2.19, $L_2$ is SSC. 

The above result is analogous to the following result of Anderson and LaGrange [3].

**Theorem 3.2.21** (Anderson and LaGrange [3], Theorem 2.6). Let $R$ and $S$ be commutative reduced rings with $1 \neq 0$ and $Z(S) \neq \{0\}$. If $\Gamma_E(R) \cong \Gamma(S)$, then $S$ is a Boolean ring.

**Theorem 3.2.22** (Joshi [23], Theorem 2.4). The graph $G_{\{0\}}(L)$ is connected and $\text{diam}(G_{\{0\}}(L)) \leq 3$.

**Theorem 3.2.23.** The graph $G_E(L)$ is connected and $\text{diam}(G_E(L)) \leq \text{diam}(G_{\{0\}}(L)) \leq 3$.

**Proof.** Follows from corollary 3.2.14, Theorem 3.2.22 and the fact that adjacency in $G_E(L)$ implies adjacency in $G_{\{0\}}(L)$. Hence $\text{diam}(G_E(L)) \leq \text{diam}(G_{\{0\}}(L)) \leq 3$. 

**Theorem 3.2.24.** Let $L$ be an SSC lattice. Then $G_{\{0\}}(L)$ is either $K_2$ or contains a cycle of length 3.

**Proof.** If $V(G_{\{0\}}(L)) = 2$, then $G_{\{0\}}(L) \cong K_2$, by Theorem 3.2.22. Now assume that $V(G_{\{0\}}(L)) \geq 3$. Let $a, b, c \in V(G_{\{0\}}(L))$, such that $a - b - c$. Then $a \land c \neq 0$ and $a \neq c$. Since $L$ is SSC, there exists $x$ such that $0 < x \leq a$ and $c \land x = 0$. Clearly, $x \in V(G_{\{0\}}(L))$. Since $a \land b = 0$, we have $x \land b = 0$. Thus we have a cycle $b - x - c - b$ of length 3.

An immediate consequence of the above theorem is the following corollary.

**Corollary 3.2.25.** For an SSC meet semi-lattice $L$, $\text{gr}(G_{\{0\}}(L)) \neq 4$. 


3.3 Relation between diameter of $G_{\{0\}}(L)$ and $G_E(L)$

Corollary 3.2.26. Let $L$ be a meet semi-lattice, then $G_E(L)$ is either $K_2$ or contains a cycle of length 3.

Proof. Follows from corollary 3.2.14 and Theorem 3.2.24. □

Corollary 3.2.27. Let $L$ be a meet semi-lattice such that $G_E(L)$ contains a cycle, then $gr(G_{\{0\}}(L)) = gr(G_E(L)) = 3$.

Remark 3.2.28. In the above result, the condition of existence of a cycle in $G_E(L)$ is necessary. In Figure 3.2.1, on the Page 40, we can observe that $G_{\{0\}}(L)$ has a cycle but $G_E(L)$ has no cycle.

Corollary 3.2.29. $gr(G_E(L)) = \infty$ if and only if $G_E(L)$ is $K_2$.

Corollary 3.2.30. If $G_{\{0\}}(L)$ is a complete bipartite graph, then so is $G_E(L)$.

3.3 Relation between diameter of $G_{\{0\}}(L)$ and $G_E(L)$

The following result gives the relation between diameter of $G_{\{0\}}(L)$ and $G_E(L)$.

Theorem 3.3.1. The following statements are true.

(a) If $diam(G_{\{0\}}(L)) = 1$ then $diam(G_E(L)) = 1$.

(b) If $diam(G_{\{0\}}(L)) = 2$ then $diam(G_E(L)) = 1$ or 2.

(c) If $diam(G_{\{0\}}(L)) = 3$ if and only if $diam(G_E(L)) = 3$.

(d) If $diam(G_E(L)) = 1$ then $diam(G_{\{0\}}(L)) = 1$ or 2.

(e) If $diam(G_E(L)) = 2$ then $diam(G_{\{0\}}(L)) = 2$. 
3.3 Relation between diameter of $G_{\{0\}}(L)$ and $G_E(L)$

Proof. (a) It is clear that $\text{diam}(G_E(L)) \neq 0$. Hence the result follows from Theorem 3.2.23.

(b) Suppose $\text{diam}(G_{\{0\}}(L)) = 2$. So we have a path $a - b - c$ in $G_{\{0\}}(L)$. Since we have $a \land b = b \land c = 0$, we get $[a \land b] = [a] \land [b] = [0] = \{0\}$ and $[b \land c] = [b] \land [c] = [0] = \{0\}$. We have two possibility:

Case 1: If $[a], [b], [c]$ are distinct vertices of $G_E(L)$, we get the path $[a] - [b] - [c]$. Hence by Theorem 3.2.23, $\text{diam}(G_E(L)) = 2$.

Case 2: Clearly, $\text{ann}(a) \neq \text{ann}(b)$ and $\text{ann}(b) \neq \text{ann}(c)$. So $[a] \neq [b]$ and $[b] \neq [c]$. Hence the only possibility is that $\text{ann}(a) = \text{ann}(c)$, i.e., $[a] = [c]$. Hence we get a path $[a] - [b]$. Thus $\text{diam}(G_E(L)) = 1$.

(c) Let $\text{diam}(G_{\{0\}}(L)) = 3$. Then there exists a path $a - c - d - b$. Clearly, $[a], [b], [c]$ and $[d]$ are vertices of $G_E(L)$. Furthermore, $a \land b \neq 0$ implies that $[a]$ is not adjacent to $[b]$ in $G_E(L)$. Now we claim that $[a] \neq [b]$. If possible, $[a] = [b]$, then $\text{ann}(a) = \text{ann}(b)$. But then $c \in \text{ann}(a) = \text{ann}(b)$ gives $d(a, b) = 2$, a contradiction. Thus $[a] \neq [b]$. Now assume that $d([a], [b]) = 2$ in $G_E(L)$. Then there exists $[x] \in V(G_E(L))$ such that $[a] - [x] - [b]$. This yields that $d(a, b) = 2$, again a contradiction. Thus by Theorem 3.2.23, $\text{diam}(G_E(L)) = 3$.

The converse follows from Theorem 3.2.23.

(d) Suppose $\text{diam}(G_E(L)) = 1$. Then $\text{diam}(G_{\{0\}}(L)) \neq 3$, by (c). Hence the result.

(e) Suppose $\text{diam}(G_E(L)) = 2$ then there exists a path $[a] - [c] - [b]$ in $G_E(L)$. This yields a path $a - c - b$ in $G_{\{0\}}(L)$. Thus $\text{diam}(G_{\{0\}}(L)) \neq 1$. The result follows from (c) and the fact that $\text{diam}(G_{\{0\}}(L)) \neq 3$.  

Corollary 3.3.2. Let \( L \) be a meet semi-lattice with 0. Then the following statements are equivalent:

1. \( \text{diam}(G_E(L)) \leq 2 \).
2. \( \text{diam}(G_{\{0\}}(L)) \leq 2 \).

Remark 3.3.3. The assertions (a) and (d) of Theorem 3.3.1, are justified in Figures 3.2.1 on Page 40 and 3.3.1 on Page 52 whereas the assertions (b) and (e) are justified in Figures 3.3.1 on Page 52 and 3.3.2 on Page 52. Figure 3.2.3 on Page 43 verifies the assertion (c).

![Diagram](image)

Figure 3.3.1: \( \text{diam}(G_E(L)) = \text{diam}(G_{\{0\}}(L)) = 1 \)

![Diagram](image)

Figure 3.3.2: \( \text{diam}(G_E(L)) = \text{diam}(G_{\{0\}}(L)) = 2 \)
Now, we examine the properties of cut-vertices of $G_E(L)$.

**Definition 3.3.4.** A lattice $L$ with 0 is said to be 0-distributive, if \(a \land b = a \land c = 0\) implies that \(a \land (b \lor c) = 0\); Varlet [49].

**Theorem 3.3.5.** Let $L$ be a 0-distributive lattice. If $[a]$ is a cut-vertex of $G_E(L)$, then $[a] \cup \{0\}$ forms an ideal of $L$.

**Proof.** Let $[a]$ be a cut-vertex of $G_E(L)$. Let $a_1, a_2 \in [a] \cup \{0\}$. Clearly, if $a_1$ or $a_2$ is 0 then we have $a_1 \lor a_2 \in [a] \cup \{0\}$.

Now assume that $a_1$ and $a_2$ are non-zero elements of $[a]$. Since $a_1, a_2 \in [a]$, $ann(a_1) = ann(a) = ann(a_2)$. Now, it is enough to show that $ann(a_1 \lor a_2) = ann(a)$. Clearly, $ann(a_1 \lor a_2) \subseteq ann(a)$.

Let $t \in ann(a)$. This gives $t \land a_1 = t \land a_2 = t \land a = 0$. Since $L$ is 0-distributive, we have $t \land (a_1 \lor a_2) = 0$, i.e., $t \in ann(a_1 \lor a_2)$. Thus $ann(a) \subseteq ann(a_1 \lor a_2)$. This implies that $a_1 \lor a_2 \in [a]$.

Now suppose $x \in [a] \cup \{0\}$ and $y \leq x$. To show that $y \in [a] \cup \{0\}$. If $x = 0$ or $y = 0$, then $y \in [a] \cup \{0\}$. Let $y \neq 0$ and $y \leq x$. Then $ann(x) \subseteq ann(y)$. Since $[a]$ is a cut vertex of $G_E(L)$, therefore for any two arbitrary vertices $[b], [c] \in V(G_E(L))$ we have a path $[b] - [a] - [c]$. Therefore $b \in ann(a) = ann(x)$, i.e., $b \land x = 0$, which further yields $b \land y = 0$. Similarly, $c \land y = 0$. Therefore, we get a path $[b] - [y] - [c]$. If $[y] \neq [a]$ then we get contradiction. Therefore $[a] = [y]$. Thus $[a] \cup \{0\}$ is an ideal. $\square$

**Remark 3.3.6.** It is clear from the proof of the above theorem that in a general lattice, if $[a]$ is a cut vertex of $G_E(L)$ then $[a] \cup \{0\}$ is a semi-ideal.
3.3 Relation between diameter of $G_{0}(L)$ and $G_{E}(L)$

**Lemma 3.3.7.** If $[a]$ is a cut-vertex of $G_{E}(L)$, then $\text{ann}(a)$ is maximal in $\mathcal{L} = \{\text{ann}(x)|0 \neq x \in L\}$.

*Proof.* Let $[a]$ be a cut-vertex of $G_{E}(L)$, and let $G_{1}$ and $G_{2}$ be mutually separated sub graphs of $G_{E}(L)$ with $V(G_{1} \cup G_{2}) = V(G_{E}(L)) \setminus [a]$. Let $[b] \in G_{1}$ and $[c] \in G_{2}$, therefore we have the path $[b] - [a] - [c]$. Suppose $\text{ann}(a) \subseteq \text{ann}(x)$ for some $\text{ann}(x) \in \mathcal{L}$. Since $[b] - [a]$ is an edge, $b \in \text{ann}(a) \subseteq \text{ann}(x)$. Then $[b] \wedge [x] = \{0\}$. Similarly $[c] \wedge [x] = \{0\}$. Thus we have another path $[b] - [x] - [c]$ passing through of $[x]$. Since $[a]$ is a cut-vertex of $G_{E}(L)$, then $[x] = [a]$. Thus $\text{ann}(x) = \text{ann}(a)$. \hfill $\square$

**Corollary 3.3.8.** If $[a]$ is a cut-vertex of $G_{E}(L)$, then $\text{ann}(a)$ is an associated prime.

**Lemma 3.3.9.** If $[a]$ is a cut-vertex of $G_{E}(L)$, then all other associated primes of $G_{E}(L)$ are contained in only one component of $G_{E}(L) \setminus [a]$.

*Proof.* Suppose that $G_{1}$ and $G_{2}$ are two mutually separated connected components of $G_{E}(L) \setminus [a]$, and each contains an associated prime. By Lemma 3.3.7, these associated primes are adjacent and so $G_{1}$ and $G_{2}$ are connected, a contradiction. \hfill $\square$

**Theorem 3.3.10.** Let $|V(G_{E}(L))| \geq 4$ and $G_{E}(L)$ has at least 2 cut-vertices. Then $\text{diam}(G_{E}(L)) = 3$.

*Proof.* Let $[a]$ and $[b]$ be cut-vertices of $G_{E}(L)$. Since $[a]$ is a cut-vertex of $G_{E}(L)$, there is some $[a_{1}]$ such that any path connecting $[a_{1}]$ and $[b]$ must include $[a]$. Similarly, since $[b]$ is a cut-vertex, there is some $[b_{1}]$ such that any path connecting $[b_{1}]$ and $[a]$ must include $[b]$. Clearly,
[a_1] \neq [b_1] and any path from [a_1] to [b_1] must include [a] and [b] and so \( \text{diam}([a],[b]) \geq 3 \). By Theorem 3.2.23, \( \text{diam}(G_E(L)) = 3 \). □

It is easy to see that Figure 3.2.3 on Page 43 fulfills the conditions of Theorem 3.3.10.

### 3.4 Beck’s Conjecture for \( G_E(L) \)

In this section, we prove that the Beck’s Conjecture is true for graph \( G_E(L) \) of equivalence classes of zero divisors of meet semi-lattices. As a preparation, we need the following easy lemma.

**Lemma 3.4.1.** Let \( P \) be a minimal prime semi-ideal of a meet semi-lattice \( L \) with 0. Then for any \( x \in P \), there exists \( y \notin P \) such that \( x \land y = 0 \).

**Proof.** Follows from the fact that \( L \setminus P \) is a maximal filter of \( L \) and \( x \notin L \setminus P \). □

**Lemma 3.4.2.** Let \( L \) be a meet semi-lattice with 0 and let \( L' \) be a meet semi-lattice with 0 (as constructed in Remark 3.2.12). If \( P \) is a minimal prime semi-ideal of \( L \) then \( P' = \{[x] \mid x \in P\} \) is a minimal prime semi-ideal of \( L' \).

**Proof.** First, we prove that \( P' \) is a semi-ideal. Let \([y] < [x] \in P'\). On the contrary, assume that \([y] \notin P'\). Hence \( \text{ann}(y) \subseteq P \). Further, \([y] < [x]\) gives \( \text{ann}(x) \not\subseteq \text{ann}(y) \) which yields \( \text{ann}(x) \subseteq P \). Since \( x \in P \), by Lemma 3.4.1, there exists \( y \notin P \) such that \( x \land y = 0 \), that
is, $y \in \text{ann}(x)$. But then $y \in \text{ann}(x) \subseteq P$, a contradiction. Thus $P'$ is a semi-ideal.

Let $[x] \land [y] \in P'$. Then $x \land y \in P$. By primeness of $P$, either $x \in P$ or $y \in P$. This gives either $[x] \in P'$ or $[y] \in P'$. Thus $P'$ is prime. Let $Q'$ be a prime semi-ideal of $L'$ such that $Q' \not\subseteq P'$. Hence there exists $[x] \in P'$ such that $[x] \notin Q'$. Again by Lemma 3.4.1, there exists $y \notin P$ such that $x \land y = 0$. This gives $[x] \land [y] = 0 \in Q'$. But then $[y] \in Q' \subseteq P'$, a contradiction. Thus $P'$ is a minimal prime semi-ideal of $L'$.

On similar lines, we can prove the following result.

**Lemma 3.4.3.** Let $L$ be a meet semi-lattice with 0 and let $L'$ be a meet semi-lattice with 0 (as constructed in Remark 3.2.12). If $P'$ is a minimal prime semi-ideal of $L'$ then $P = \{x \mid [x] \in P'\}$ is a minimal prime semi-ideal of $L$.

Let us denote the set of all minimal prime semi-ideals of $L$ by $\text{Min}_s(L)$.

**Theorem 3.4.4.** Let $L$ be a meet semi-lattice with 0 and let $L'$ be a meet semi-lattice with 0 (as constructed in Remark 3.2.12). Let $\phi : \text{Min}_s(L) \to \text{Min}_s(L')$ be a map. Then $\phi$ is bijective.

**Proof.** Let $\phi : \text{Min}_s(L) \to \text{Min}_s(L')$ be a map defined by $\phi(P) = P' = \{[x] \mid x \in P\}$. By Lemma 3.4.2, $P'$ is a minimal prime semi-lattice. Clearly, $\phi$ is well defined. Let $\phi(P) = \phi(Q)$. Then for $x \in P$, we have $[x] \in P' = \phi(P) = \phi(Q) = Q'$. This gives $x \in Q$. Hence $P \subseteq Q$. Similarly, $Q \subseteq P$. Thus $\phi$ is one-to-one. Let $Q'$ be any minimal prime semi-ideal of $L'$. By primeness of $P'$, either $[x] \in P'$ or $[y] \in P'$. Thus $\phi(P)$ is a minimal prime semi-ideal of $L'$.

On similar lines, we can prove the following result.

**Lemma 3.4.3.** Let $L$ be a meet semi-lattice with 0 and let $L'$ be a meet semi-lattice with 0 (as constructed in Remark 3.2.12). If $P'$ is a minimal prime semi-ideal of $L'$ then $P = \{x \mid [x] \in P'\}$ is a minimal prime semi-ideal of $L$.

Let us denote the set of all minimal prime semi-ideals of $L$ by $\text{Min}_s(L)$.

**Theorem 3.4.4.** Let $L$ be a meet semi-lattice with 0 and let $L'$ be a meet semi-lattice with 0 (as constructed in Remark 3.2.12). Let $\phi : \text{Min}_s(L) \to \text{Min}_s(L')$ be a map. Then $\phi$ is bijective.

**Proof.** Let $\phi : \text{Min}_s(L) \to \text{Min}_s(L')$ be a map defined by $\phi(P) = P' = \{[x] \mid x \in P\}$. By Lemma 3.4.2, $P'$ is a minimal prime semi-lattice. Clearly, $\phi$ is well defined. Let $\phi(P) = \phi(Q)$. Then for $x \in P$, we have $[x] \in P' = \phi(P) = \phi(Q) = Q'$. This gives $x \in Q$. Hence $P \subseteq Q$. Similarly, $Q \subseteq P$. Thus $\phi$ is one-to-one. Let $Q'$ be any minimal prime semi-ideal of $L'$.
3.4 Beck’s Conjecture for $G_E(L)$

semi-ideal in $L'$. Then $Q = \{ x \mid [x] \in Q' \}$ is a minimal prime semi-ideal of $L$, by Lemma 3.4.3. It is clear that $\phi(Q) = Q'$. Thus $\phi$ is onto. □

Joshi [23] (See also, Halaš and Jukl [19], Nimbhakar, et. al, [41] Lu and Wu [38]) proved the Beck’s Conjecture for the zero divisor graphs of posets with respect to an ideal. We quote this result when the poset is a meet semi-lattice and ideal is a zero ideal.

**Theorem 3.4.5** (Joshi [23], Theorem 2.9]). Let $L$ be a meet semi-lattice with 0. If $\text{Clique}(G_{\{0\}}(L)) < \infty$ then $\text{Clique}(G_{\{0\}}(L)) = \chi(G_{\{0\}}(L)) = n$, where $n$ is the number of minimal prime semi-ideals of $L$.

We close this chapter by proving Beck’s Conjecture for $G_E(L)$.

**Theorem 3.4.6.** Let $L$ be a meet semi-lattice with 0. If $\text{Clique}(G_E(L)) < \infty$ then $\text{Clique}(G_E(L)) = \chi(G_E(L)) = n$, where $n$ is the number of minimal prime semi-ideals of $L$.

**Proof.** Let $\text{Clique}(G_E(L)) < \infty$. Then $\text{Clique}(G_{\{0\}}(L')) < \infty$, by Corollary 3.2.12, where $L'$ is the semi-lattice constructed as in Remark 3.2.12. By Theorem 3.4.5, $\text{Clique}(G_{\{0\}}(L')) = \chi(G_{\{0\}}(L')) = n$, where $n$ is the number of minimal prime semi-ideals of $L'$. Now, by Theorem 3.4.4, number of minimal prime semi-ideals of $L$ is also $n$. Thus we have $\text{Clique}(G_E(L)) = \chi(G_E(L)) = n$. □

**Corollary 3.4.7.** Let $L$ be a meet semi-lattice with 0. If $\text{Clique}(G_E(L)) < \infty$ then $\chi(G_E(L)) = \chi(G_{\{0\}}(L)) = \text{Clique}(G_E(L)) = \text{Clique}(G_{\{0\}}(L)) = n$, where $n$ is the number of all minimal prime semi-ideals of $L$. 