Chapter 4

The Closure Operator and the Cocircuits of the es-Splitting Matroid

In this chapter, we characterize the closure operator and the cocircuits of the es-splitting binary matroid $M^c_X$ in terms of the closure operator and the cocircuits of the original binary matroid $M$, respectively.

4.1 Introduction

We know that Shikare and Azanchiler [14] extended Slater’s [38] $n$-line splitting operation from graphs to binary matroids. We call the corresponding operation as es-splitting operation.

Azanchiler [15] explored a few properties concerning the es-splitting operation. He characterized the bases of the es-splitting matroid in terms of the bases of the original matroid $M$.

For convenience we use the following notations.
4.1 Introduction

1. $C_{OX}$ denotes the set of all circuits of a matroid $M$ each of which contains an odd number of elements of the set $X \subseteq E(M)$. The members of the set $C_{OX}$ are called $OX$-circuits.

2. $C_{EX}$ denotes the set of all circuits of a matroid $M$ each of which contains an even number of elements of the set $X \subseteq E(M)$. The members of the set $C_{EX}$ are called $EX$-circuits.

Let $cl$ and $cl'$ be the closure operators of the matroids $M$ and $M_X$, respectively. The following result characterizes the rank function of the matroid $M_X$ in terms of the rank function of the matroid $M$.

**Lemma 4.1.1.** Let $r$ and $r'$ be the rank functions of the matroids $M$ and $M_X$, respectively. Suppose that $A \subseteq E(M)$. Then

1. $r'(A) = r(A) + 1$, if $A$ contains an $OX$-circuit of the matroids $M$; $= r(A)$; otherwise.

2. $r'(A \cup a) = r(A) + 1$.

3. $r'(A \cup \gamma) = r(A)$ if not $A$ but $A \cup e$ contains an $OX$-circuit;
   $= r(A) + 2$; if $A$ contains an $OX$-circuit and $e \notin cl(A)$;
   $= r(A) + 1$; otherwise.

4. $r'(A \cup \{a, \gamma\}) = r(A) + 1$ if $e \in cl(A)$;
   $= r(A) + 2$ if $e \notin cl(A)$.

**Proof.** The proofs of the statements (1) and (2) are given in [15]. We prove (3) and (4).

3. If $A$ contains no $OX$-circuit of $M$, then by (1), $r'(A) = r(A)$. Further, if there is an $OX$-circuit $C$ of $M$ such that $e \in C \subseteq A \cup e$,
then $C' = (C \setminus e) \cup \gamma$ is a circuit of $M'_X$ contained in $A \cup \gamma$. Consequently, 
$r'(A \cup \gamma) = r'(A) = r(A)$.

Suppose, $A$ contains an $OX$-circuit of $M$ or $e \notin cl(A)$. Let $T$ and $T'$ be bases for $A$ in $M$ and $M'_X$, respectively. Then, $r(A) = |T|$ and $r'(A) = |T'| = |T| + 1 = r(A) + 1$. We show that $T' \cup \gamma$ is a basis for $A \cup \gamma$ in $M'_X$. On the contrary, suppose $T' \cup \gamma$ is dependent in $M'_X$ and $C'$ is a circuit of $M'_X$ contained in $T \cup \gamma$. Then, by Proposition 3.1.8, one of the following four cases occurs:

(i) $C' \in C_0$. Then $C'$ is a $EX$-circuit of $M$ and $\gamma \notin C'$. Now $C' \subseteq T' \cup \gamma$ implies that $C' \subseteq T'$; a contradiction.

(ii) $C' \in C_1$. So that $C' \subseteq T' \cup \gamma$ and hence $C' \subseteq T'$; a contradiction.

(iii) $C' \in C_2$. Then $C' = C \cup a \subseteq T' \cup \gamma$ implies that $a \in T'$; a contradiction.

(iv) Let $C' \in C_3$. Then one of the following three sub cases occurs.

(a) $C' = C \cup \{a, \gamma\}$. Then $C' \subseteq T' \cup \gamma$ this implies that $C \cup e \subseteq T' \subseteq A$; a contradiction.

(b) $C' = (C \setminus e) \cup \gamma$. This implies that $C' \subseteq T' \cup \gamma$ and hence $(C \setminus e) \subseteq T' \subseteq A$. Consequently, $e \in cl(A)$; a contradiction.

(c) $C' = (C \setminus e) \cup \{a, \gamma\}$. Then $C' \subseteq T' \cup \gamma$ and $(C \setminus e) \cup a \subseteq T' \subseteq A$. Consequently, $a \in A$; a contradiction.

Further, we prove that $T' \cup \gamma \subseteq E(M'_X)$ is a maximal independent set in $M'_X$ and is contained in $A \cup \gamma$. On the contrary, suppose $T' \cup \{\gamma, z\}$
is independent in $M^c_X$ and is contained in $A \cup \gamma$ where $z \in A - (T' \cup \gamma)$. Then $T' \cup \{\gamma, z\} \subseteq A \cup \gamma$ and $(T' \cup z) \subseteq A$. But this is a contradiction to the maximality of $T'$. Therefore, $T' \cup \gamma$ is a basis of $A \cup \gamma$. Thus, $r'(A \cup \gamma) = |T' \cup \gamma| = |T'| + 1 = r'(A) + 1 + 1 = r(A) + 2$.

If $A$ contains no $OX$-circuit of $M$ and $e \notin cl(A)$, then by (1), $r'(A) = r(A)$. Now by similar argument as given above, one can show that $r'(A \cup \gamma) = |T' \cup \gamma| = |T'| + 1 = r'(A) + 1 = r(A) + 1$.

If $A$ contains an $OX$-circuit $C_1$ of $M$ and $e \in cl(A)$, then one of the following two cases occurs.

(i) There is an $OX$-circuit $C$ of $M$ such that $e \in C \subseteq A \cup e$. Then $C' = (C \setminus e) \cup \gamma$ is a circuit of $M^c_X$ contained in $A \cup \gamma$. Consequently, $r'(A \cup \gamma) = r'(A) = r(A) + 1$.

(ii) There is no $OX$-circuit $C$ of $M$ such that $e \in C \subseteq A \cup e$. Then there is an $EX$-circuit $C$ of $M$ such that $e \in C \subseteq A \cup e$ which is also a circuit of $M^c_X$. Thus, $e \in cl'(A)$ and $r'(A \cup e) = r'(A)$. Then, by Proposition 3.1.8, $C_1 \cup \{e, \gamma\}$ is a circuit of $M^c_X$ contained in $A \cup \{e, \gamma\}$. Therefore, $r'(A \cup \{e, \gamma\}) = r'(A \cup \gamma) = r'(A \cup e) = r'(A) = r(A) + 1$.

(4) Let $A \subseteq E(M)$ and $e \in cl(A)$. Note that $\{a, e, \gamma\}$ forms a circuit of $M^c_X$, therefore, $r'(A \cup \{a, \gamma, e\}) = r'(A \cup \{a, \gamma\}) = r'(A \cup \{\gamma, e\}) = r'(A \cup \{a, e\})$. By statement (2), we have $r'(A \cup a) = r(A) + 1$.

If $e \in cl(A)$, then one of the following two cases occurs.

(i) There is an $OX$-circuit $C$ of $M$ such that $e \in C \subseteq A \cup e$. Then, $C' = (C \setminus e) \cup \gamma$ is a circuit of $M^c_X$ contained in $A \cup \gamma \subseteq A \cup \{a, \gamma\}$. 
Consequently, \( r'(A \cup \{a, \gamma\}) = r'(A \cup a) = r(A) + 1 \).

(ii) There is an \( EX\) -circuit \( C \) of \( M \) such that \( e \in C \subseteq A \cup e \). Then, by Proposition 3.1.8, \((C \setminus e) \cup \{a, \gamma\}\) is a circuit of \( M_X^e \) contained in \( A \cup \{a, \gamma\} \). Therefore, \( r'(A \cup \{a, \gamma\}) = r'(A \cup a) = r(A) + 1 \).

If \( e \notin \text{cl}(A) \), then by Proposition 3.1.8, \( A \cup \{a, \gamma\} \) contains no circuit \( C' \) of \( M_X^e \) such that \( \gamma \in C' \). Let \( T' \) be the basis of \( A \) in \( M_X^e \). Then, by argument similar as given above, one can show that \( T' \cup \{a, \gamma\} \) is a basis of \( A \cup \{a, \gamma\} \). Thus, \( r'(A \cup \{a, \gamma\}) = r'(A \cup a) + 1 = r(A) + 1 + 1 = r(A) + 2 \).

The relation between \( r(M) \) and \( r'(M_X^e) \) now follows from the Lemma 4.1.1.

**Corollary 4.1.2.** Let \( r \) and \( r' \) be the rank functions of the matroids \( M \) and \( M_X^e \), respectively. Then, \( r'(M_X^e) = r(M) + 1 \).

**Lemma 4.1.3.** Let \( M \) be a matroid on \( E \) with rank function \( r \), and let \( A \subseteq E \). Then \( \text{cl}(A \cup x) = \text{cl}(A) \) if and only if \( x \in \text{cl}(A) \).

**Proof.** Suppose \( x \in \text{cl}(A) \). Then \( r(A) = r(A \cup x) \). We prove that \( \text{cl}(A \cup x) \subseteq \text{cl}(A) \). Let \( y \in \text{cl}(A \cup x) \). If \( y \in (A \cup x) \), then we are through. If \( y \in \text{cl}(A \cup x) - (A \cup x) \), then \( r(A \cup x) = r(A \cup \{x, y\}) \). Therefore, \( r(A) = r(A \cup \{x, y\}) \) and \( y \in \text{cl}(A) \). Thus, \( \text{cl}(A \cup x) \subseteq \text{cl}(A) \).

We conclude that \( \text{cl}(A \cup x) = \text{cl}(A) \). \( \square \)

**Definition 4.1.4.** Let \( M \) be a matroid on a set \( E \) with rank function \( r \), \( X \subset E \) and \( A \subseteq E \). Then for a subset \( A \) of \( E \) define the sets \( \mathcal{T}(A) \) and \( \mathcal{F}(A) \) as follows.
1. $\mathcal{T}(A) = \{x \in (E - A) \mid x \neq e \text{ and there is an } OX\text{-circuit } C \text{ of } M \text{ such that } x \in C \text{ and } C \subseteq (A \cup e) \cup x\}$.

2. $\mathcal{F}(A) = \{x \in (cl(A) - A) \mid \text{there is an } OX\text{-circuit } C \text{ of } M \text{ such that } x \in C \text{ and there is no } EX\text{-circuit of } M \text{ containing } x\}$.

We have the following proposition.

**Proposition 4.1.5.** If $e \in cl(A)$, then the set $\mathcal{T}(A) \subseteq cl(A)$.

**Proof.** Let $e \in cl(A)$ and $x \in \mathcal{T}(A)$. Then there is a circuit $C$ of the matroid $M$ such that $x \in C$ and $C \subseteq A \cup \{e, x\}$. Consequently, $x \in cl(A \cup e)$. Since $e \in cl(A)$, $cl(A \cup e) = cl(A)$. Thus $x \in cl(A)$. □

### 4.2 The Closure Operator of the es-Splitting Matroid

Throughout this section, we assume that $M$ is a binary matroid on a set $E$ and $M^e_X$ is the splitting matroid of $M$ with respect to a subset $X$ of $E$ where $e \in X$. Further, $A' \subseteq E \cup \{a, \gamma\}$ and $A = A' \setminus \{a, \gamma\}$.

In this section, we characterize the closure operator of the es-splitting matroid $M^e_X$ in terms of the closure operator of the original matroid $M$.

Firstly we prove a few Lemmas concerning closure operators of $M$ and $M^e_X$.

**Lemma 4.2.1.** Suppose that $A' \subset E \cup \{a, \gamma\}$, $a, \gamma \notin A'$ and $(A \cup e)$ contains no $OX$-circuit. Then $cl'(A') = cl(A) - \mathcal{F}(A)$.
Proof. Suppose, $a, \gamma \notin A'$ and $(A \cup e)$ contains no $OX$-circuit. Let $x \in cl'(A')$. Then $x \in A' = A$ or $x \in cl'(A') - A$. If $x \in A$, then we are through. Now if $x \in cl'(A') - A'$, then there is a circuit $C'$ of $M_eX$ such that $x \in C'$ and $C' \subseteq A' \cup x = A \cup x$.

If $x \in E$, then by Proposition 3.1.8, $C'$ is a circuit of $M$ or $C'$ is the union of two circuits $C_1$ and $C_2$, where $C_1, C_2 \in C_{OX}$. If $C'$ is a circuit of $M$, then $C'$ is an $EX$-circuit. Therefore, $x \in cl(A) - F(A)$. On the other hand, if $C' = C_1 \cup C_2$, then without loss of generality assume that $x \in C_1$. Then $C_1 \cup C_2 \subseteq A \cup x$ implies that $C_2 \subseteq A$, a contradiction.

If $x = a$, then $a \in C'$ and $C'$ is a circuit of $M_eX$ contained in $A \cup a$. This implies that $C' = C \cup a$ where $C \in C_{OX}$. Thus, $a \in C' \subseteq A \cup a$ and hence $C \subseteq A$. This is a contradiction to the fact that $A$ contains no member of $C_{OX}$. Therefore, $x \neq a$.

If $x = \gamma$, then one of the following cases occurs.

(i) $C' = C \cup \{e, \gamma\} \subseteq A \cup \gamma$ where $e \notin C$ and $C$ contains an odd number of elements of $X$ in $M$. Then $x \in C' = C \cup \{e, \gamma\} \subseteq A \cup \gamma$ and this implies that $C \subseteq A$ but this is a contradiction.

(ii) $C' = (C \setminus e) \cup \gamma \subseteq A \cup \gamma$ where $C$ contains an odd number of elements of $X$ and $e \in C$. It follows that $e \in C_{OX} \subseteq A \cup e$; a contradiction.

(iii) $C' = (C \setminus e) \cup \{a, \gamma\} \subseteq A \cup \gamma$ where $(C \setminus e)$ contains an odd number of elements of $X$ and $e \in C$. This implies that $a \in A$ which is also a contradiction. Therefore, $x \neq \gamma$ and $cl'(A') \subseteq cl(A) - F(A)$.

Conversely, let $x \in cl(A) - F(A)$. If $x \in A$, then there is nothing
to prove. If \( x \in (\text{cl}(A) - A) \) and \( x \notin \mathcal{F}(A) \), then there exists an \( \text{EX} \)-circuit \( C \in \mathcal{C}_{\text{EX}} \) of \( M \) such that \( x \in C \) and \( C \subseteq A \cup x \). Now \( C' = C \) is a circuit of \( M'_{\chi} \) and \( x \in C' \subseteq A \cup x \). This implies that \( x \in \text{cl}'(A') \). We conclude that \( \text{cl}(A) - \mathcal{F}(A) \subseteq \text{cl}'(A') \).

\[ \square \]

**Lemma 4.2.2.** Suppose that \( a, \gamma \notin A' \) and \( \text{cl}(A) \) contains no \( \text{OX} \)-circuit. Then \( \text{cl}'(A') = \text{cl}(A) \).

**Proof.** Suppose, \( a, \gamma \notin A' \) and \( \text{cl}(A) \) contains no \( \text{OX} \)-circuit. Then \( A' = A \). If \( x \in \text{cl}'(A') \), then \( x \in A' \) or \( x \in \text{cl}'(A') - A' \). If \( x \in A' \), then we are through. Now suppose \( x \in \text{cl}'(A') - A' \) and let \( C' \) be a circuit of \( M_{\chi} \) such that \( x \in C' \subseteq A' \cup x \).

If \( x \in E \), then by Proposition 3.1.8, \( C' \) is a circuit of \( M \) or \( C' \) is the union of two circuits \( C_1 \) and \( C_2 \), belonging to the set \( \mathcal{C}_{\text{OX}} \). If \( C' \) is a circuit of \( M \), then \( C' \) is an \( \text{EX} \)-circuit. Therefore, \( x \in \text{cl}(A) \). On the other hand, if \( C' = C_1 \cup C_2 \), then without loss of generality assume that \( x \in C_1 \). Then \( C_1 \cup C_2 \subseteq A \cup x \) implies that \( C_2 \subseteq A \), a contradiction.

If \( x = a \), then \( a \in C' \subseteq A \cup a \) in \( M'_{\chi} \) and this implies that \( C' = C \cup a \) where \( C \in \mathcal{C}_{\text{OX}} \) is a circuit of \( M \). But \( a \in C' \subseteq A \cup a \) implies that \( C \subseteq A \) and this is a contradiction to the fact that \( A \) contains no member of \( \mathcal{C}_{\text{OX}} \). Therefore, \( x \neq a \).

If \( x = \gamma \), then \( \gamma \in C' \) and \( C' \subseteq A \cup \gamma \). It follows that \( C' \) has one of the following three types of forms.

(i) \( C' = C \cup \{e, \gamma\} \subseteq A \cup \gamma \) where \( e \notin C \) and \( C \in \mathcal{C}_{\text{OX}} \) is a circuit of \( M \). Then \( C \subseteq A \) and we get a contradiction.

(ii) \( C' = (C \setminus e) \cup \gamma \subseteq A \cup \gamma \) where \( C \in \mathcal{C}_{\text{OX}} \) is a circuit of \( M \) and \( e \in C \).
Consequently, \( C \subseteq A \cup e \) and \( e \in cl(A) \). This is a contradiction to the fact that \( e \notin cl(A) \).

(iii) \( C' = (C \setminus e) \cup \{a, \gamma\} \subseteq A \cup \gamma \) where \( (C \setminus e) \in C_{OX} \) is a circuit of \( M \) and \( e \in C \). We conclude that \( C \subseteq A \cup e \) and hence \( e \in cl(A) \), a contradiction.

Conversely, let \( x \in cl(A) \). If \( x \in A \), then we are through. If \( x \in cl(A) - A \), then there is a circuit \( C \) of \( M \) such that \( x \in C \subseteq A \cup x \). As \( cl(A) \) contains no member of \( C_{OX} \), \( C \) contains an even number of elements of \( X \). Thus \( C \) is also a circuit of \( M^e_X \). Thus, \( x \in cl'(A') \). This completes the proof of the Lemma.

**Lemma 4.2.3.** Suppose \( e \notin cl(A) \) and one of the following conditions is true.

1. \( a, \gamma \notin A' \) and there is an OX-circuit in \( A \).
2. \( a \in A' \) and \( \gamma \notin A' \).

Then \( cl'(A') = cl(A) \cup a \).

**Proof.** Suppose \( e \notin cl(A) \), \( a, \gamma \notin A' \) and there is an OX-circuit in \( A \). Let \( x \in cl'(A') - A' \) and \( C' \) be a circuit of \( M^e_X \) such that \( x \in C' \subseteq A' \cup x \).

If \( x \in E \), then by Proposition 3.1.8, \( C' \) is a circuit of \( M \) or \( C' \) is the union of two circuits \( C_1 \) and \( C_2 \), from the set \( C_{OX} \). If \( C' \) is a circuit of \( M \), then \( C' \) is an EX-circuit. Therefore, \( x \in cl(A) \). On the other hand if \( C' = C_1 \cup C_2 \), then without loss of generality, assume that \( x \in C_1 \) and \( C_1 \subseteq A \cup x \). Consequently, \( x \in cl(A) \).

If \( x = a \), then \( x \in cl(A) \cup a \) and if \( x = \gamma \), then \( \gamma \in C' \subseteq A \cup \gamma \) and \( e \notin cl(A) \). This implies that \( C' \) has one of the following three forms.
(i) \( C' = C \cup \{e, \gamma\} \subseteq A \cup \gamma \) where \( e \notin C \) and \( C \in \mathcal{C}_{OX} \) is a circuit of \( M \). Then \( x \in C' = C \cup \{e, \gamma\} \subseteq A \cup \gamma \). This implies that \( e \in A \), a contradiction.

(ii) \( C' = (C \setminus e) \cup \gamma \subseteq A \cup \gamma \) where \( C \in \mathcal{C}_{OX} \) is a circuit of \( M \) and \( e \in C \). Then \( e \in cl(A) \) and this leads to a contradiction.

(iii) \( C' = (C \setminus e) \cup \{a, \gamma\} \subseteq A \cup \gamma \) where \( (C \setminus e) \in \mathcal{C}_{OX} \) is a circuit of \( M \) and \( e \in C \). We conclude that \( e \in cl(A) \), \( a \in A \) and this leads to a contradiction. Therefore, \( x \neq \gamma \) and \( cl'(A') \subseteq cl(A) \cup a \).

Conversely, let \( x \in cl(A) \cup a \). If \( x = a \), then \( a \in cl'(A') \) as \( A \) contains an element \( C \) of \( \mathcal{C}_{OX} \). Moreover, \( a \in C' = C \cup a \subseteq A \cup a \) in \( M_X^e \). If \( x \in A \), then we are through. Suppose \( x \in cl(A) - A \) and let \( C \) be a circuit of \( M \) contained in \( X \) such that \( x \in C \subseteq A \cup x \). If \( C \in \mathcal{C}_{EX} \) is a circuit of \( M \), then \( C' = C \); otherwise \( C' = C \cup a \) is a circuit of \( M_X^e \). Further, \( x \in C \) implies that \( x \in cl'(A \cup a) \) whereas \( a \in cl'(A') \) implies \( cl'(A \cup a) = cl'(A') \). Thus \( x \in cl'(A') \) and \( cl(A) \cup a \subseteq cl'(A') \), as desired. Second statement follows by argument similar to one as given for statement 1.

\[ \square \]

**Lemma 4.2.4.** Let \( a, \gamma \notin A' \) and let \( A \cup e \) contains an \( OX \)-circuit but \( A \) contains no \( OX \)-circuit. Then \( cl'(A') = (cl(A) - \mathcal{F}(A)) \cup \gamma \).

**Proof.** Suppose \( a, \gamma \notin A' \) and \( A \cup e \) contains an \( OX \)-circuit but \( A \) contains no \( OX \)-circuit. Let \( x \in cl'(A') \). If \( x \in A' \) or \( x = \gamma \), then we are through. Suppose that \( x \in cl'(A') - A' \). Then there is a circuit \( C' \) of \( M_X^e \) such that \( x \in C' \subseteq A' \cup x \).
If $x \in E$, then by Proposition 3.1.8, $C'$ is a circuit of $M$ or $C'$ is the union of two circuits $C_1$ and $C_2$, from the set $C_{OX}$. If $C'$ is a circuit of $M$, then $C'$ is an EX-circuit. Therefore, $x \in cl(A) - \mathcal{F}(A)$. On the other hand, if $C' = C_1 \cup C_2$, then without loss of generality assume that $x \in C_1$. Then $C_1 \cup C_2 \subseteq A \cup x$ implies that $C_2 \subseteq A$, a contradiction.

If $x = a$, then there is a circuit $C'$ of $M_X^e$ such that $a \in C' \subseteq A \cup a$. This implies that $C' = C \cup a$ for some $C \in C_{OX}$. Consequently, $C \subseteq A$. This is a contradiction to the fact that $A$ contains no member of $C_{OX}$. Therefore, $x \neq a$.

Conversely, let $x \in cl(A) - (\mathcal{F}(A) \cup \gamma)$. If $x \in (cl(A) - A) - \mathcal{F}(A)$, then there is a circuit $C \in C_{EX}$ such that $x \in C \subseteq A \cup x = A' \cup x$. This implies $x \in cl'(A')$. If $x = \gamma$ and there is a member $C$ of $C_{OX}$ such that $e \in C \subseteq A \cup e$, then $C' = (C \setminus e) \cup \gamma$ is a circuit of $M_X^e$ contained in $A \cup \gamma = A' \cup \gamma$. Therefore, $\gamma \in cl'(A')$. Thus, we conclude that $(cl(A) - \mathcal{F}(A)) \cup \gamma \subseteq cl'(A')$. This completes the proof.

*Lemma 4.2.5.* Let $a \notin A'$, $\gamma \in A'$, $e \notin cl(A)$ and $cl(A)$ contains an OX-circuit but $A$ contains no OX-circuit. Then $cl'(A') = (cl(A) - \mathcal{F}(A)) \cup \gamma \cup \mathcal{T}(A)$.

*Proof.* Let $x \in cl'(A') - (A')$. Then there is a circuit say $C'$ of $M_X^e$ such that $x \in C' \subseteq A' \cup x = A \cup \{\gamma, x\}$.

If $\gamma \in C'$, then $C' \subseteq A \cup \{\gamma, x\}$ and one of the following four cases occurs.

(i) $C' = C \cup \{e, \gamma\}$ where $C$ is a member of $C_{OX}$ and $e \notin C$. Then $x \in C' = C \cup \{e, \gamma\} \subseteq A \cup \{\gamma, x\}$ and this implies that $x \in
(C \cup e) \subseteq A \cup x$. Consequently, $x = e \in cl(A)$ and $C \subseteq A$ but this is a contradiction.

(ii) $C' = (C \setminus e) \cup \{a, \gamma\}$ where $C$ is a circuit of $M$ containing $e$ and $C \setminus e$ is a member of $C_{OX}$. Then $x \in C' = (C \setminus e) \cup \{a, \gamma\}$ and $C' \subseteq A \cup \{\gamma, x\}$ therefore, $(C \setminus e) \cup a \subseteq A \cup x$. Consequently, $x = a$ and $(C \setminus e) \subseteq A$. As $C \subseteq A \cup e$, it follows that $C$ is a member of $C_{EX}$ contained in $cl(A)$. This is a contradiction to the fact that $e \notin cl(A)$.

(iii) $C' = (C \setminus e) \cup \gamma$ where $C$ is a circuit of $M$ containing $e$ and $C \in C_{OX}$. Then $x \in C' = (C \setminus e) \cup \gamma \subseteq A \cup \{\gamma, x\}$ and hence $x \in C \setminus e \subseteq A \cup x$. Thus, $x \in C \subseteq A \cup \{e, x\}$ and $x \in T(A)$ as desired.

(iv) $C' = \{a, e, \gamma\}$. Then $x \in C' = \{a, e, \gamma\} \subseteq A \cup \{\gamma, x\}$ and $x = a$ or $x = e$. In either case, we get a contradiction.

If $\gamma \notin C'$, then $C' \subseteq A \cup x$ and one of the following two cases occurs.

(i) $C'$ is a circuit of $M^e_X$ containing an even number of elements of $X$. Then $C' = C$ or $C' = C_1 \cup C_2$ where $C, C_1, C_2 \in C_{OX}$. Then by argument similar to one as in the proof of Lemma 4.2.1, we conclude that $x \in cl(A) - F(A)$.

(ii) $C'$ is a circuit of $M^e_X$ containing an odd number of elements of $X$. Then $C' = C \cup a \subseteq A \cup x$ where $C \in C_{OX}$. This implies that $x = a$ and $C \subseteq A$, a contradiction. Therefore, $C'$ contains an even number of elements of $X$. 
Consequently, $cl'(A) \subseteq (cl(A) - \mathcal{F}(A)) \cup \gamma \cup \mathcal{T}(A)$.

Conversely, let $x \in (cl(A) - \mathcal{F}(A)) \cup \gamma \cup \mathcal{T}(A)$. If $x \in (A \cup \gamma)$, then we are through. In the case $x \in \mathcal{T}(A)$, there is a circuit $C'$ of $M$ such that $x \in C' \subseteq (A \cup \{e, x\})$. Then $C' = (C \setminus e) \cup \gamma$ is a circuit of $M^e_X$ and $x \in C' \subseteq A \cup \{\gamma, x\}$. We conclude that $x \in cl'(A \cup \gamma)$. If $x \in [(cl(A) - \mathcal{F}(A)) \cup \gamma - (A \cup \gamma)]$, then there is a circuit $C \in \mathcal{C}_{OX}$ of $M$ such that $x \in C \subseteq A \cup x$. Thus, $C' = C$ is a circuit of $M^e_X$ and $x \in C'$ of $M^e_X$ contained in $A \cup \{\gamma, x\}$. This implies $x \in cl'(A')$ and we conclude that $(cl(A) - \mathcal{F}(A)) \cup \gamma \subseteq cl'(A')$. \hfill \Box

**Lemma 4.2.6.** Let $cl(A)$ contains no $OX$-circuit, $a \notin A'$, $\gamma \in A'$ and $e \notin cl(A)$. Then $cl'(A') = cl(A) \cup \gamma \cup \mathcal{T}(A)$.

**Proof.** Suppose $cl(A)$ contains no $OX$-circuit, $a \notin A'$, $\gamma \in A'$ and $e \notin cl(A)$. If $x \in A' = A \cup \gamma$, then $x \in (cl(A) \cup \gamma)$. If $x \in cl'(A') - A'$, then there is a circuit $C'$ of $M^e_X$ such that $x \in C'$ and $C' \subseteq A' \cup x$. If $\gamma \notin C'$, then $x \in C' \subseteq A \cup x$. This implies that $x \in cl(A)$ in $M$. If $\gamma \in C'$, then $C'$ has one of the following forms.

(i) $C' = C \cup \{e, \gamma\}$ where $C \in \mathcal{C}_{OX}$ and $e \notin C$. Then $x \in C' = C' \cup \{e, \gamma\} \subseteq A \cup \{\gamma, x\}$. This implies that $C \subseteq A$ and $x = e$ which is not possible as $A$ contains no member of $\mathcal{C}_{OX}$.

(ii) $C' = (C \setminus e) \cup \{a, \gamma\}$ where $C$ is a circuit of $M$ such that $e \in C$ and $C \setminus e$ contains an odd number of elements of $X$. Then $x \in C' = (C \setminus e) \cup \{a, \gamma\} \subseteq A \cup \{\gamma, x\}$ and $(C \setminus e) \cup a \subseteq A \cup x$. Therefore, $x = a$ and $(C \setminus e) \subseteq A$. Further, $C \subseteq A \cup e$ implies that $e \in cl(A)$ and this is a contradiction to the fact that $e \notin cl(A)$. 

(iii) \( C' = (C \setminus e) \cup \gamma \) where \( e \in C \) and \( C \in \mathcal{C}_{OX} \) is a circuit of \( M \).

Then \( x \in C' = (C \setminus e) \cup \gamma \subseteq A \cup \{\gamma, x\} \) and this implies that
\( x \in C \setminus e \subseteq A \cup x \). That is \( x \in C \subseteq A \cup \{e, x\} \). We conclude that
\( x \in \mathcal{T}(A) \).

(iv) \( C' = \{a, e, \gamma\} \). Then \( x \in C' = \{a, e, \gamma\} \subseteq A \cup \{\gamma, x\} \) and we con-
clude that \( x = a \) or \( x = e \). In either case, we get a contradiction.

We conclude that, \( cl'(A') \subseteq cl(A) \cup \gamma \cup \mathcal{T}(A) \).

Conversely, let \( x \in cl(A) \cup \gamma \cup \mathcal{T}(A) \). If \( x = \gamma \) or \( x \in A \), then
\( x \in cl'(A \cup \gamma) \). If \( x \in cl(A) \setminus A \), then there is a circuit say \( C \) of \( M \) such that
\( x \in C \subseteq A \cup x \). As \( cl(A) \) contains no member of \( \mathcal{C}_{OX} \), \( C \in \mathcal{C}_{EX} \). Thus, \( C \) is also a circuit of \( M_X^\xi \) and \( x \in cl'(A') \). In the case \( x \in \mathcal{T}(A) \), there is a circuit \( C \) of \( M \) such that \( x \in C \subseteq A \cup \{e, x\} \). Then
\( C' = (C \setminus e) \cup \gamma \) is a circuit of \( M_X^\xi \) and \( x \in C' \subseteq A \cup \{\gamma, x\} \). We
conclude that \( x \in cl'(A \cup \gamma) = cl'(A') \).

\( \square \)

**Lemma 4.2.7.** Suppose one of the following conditions is true

1. \( a, \gamma \in A' \).
2. \( a \in A', \gamma \notin A' \) and \( e \in cl(A) \).
3. \( a \notin A', \gamma \in A' \) and \( A \) contains an \( OX \)-circuit.
4. \( a \notin A', \gamma \in A' \) and \( e \in cl(A) \).
5. \( a, \gamma \notin A' \), \( A \) contains an \( OX \)-circuit and \( e \in cl(A) \).

Then \( cl'(A') = cl(A) \cup \{a, \gamma\} \).
4.2 The Closure Operator of the es-Splitting Matroid

Proof. Assume that \( x \in \text{cl}'(A') - A' \). Then there is a circuit \( C' \) of \( M_X^e \) such that \( x \in C' \subseteq A' \cup x \). If \( x \in E \), then \( C' \) is a circuit of \( M \) or \( C' = C_1 \cup C_2 \), where \( C_1, C_2 \in \mathcal{C}_{OX} \). Without loss of generality, assume that \( x \in C_1 \subseteq A \cup x \). Then \( x \in \text{cl}(A) \). If \( x \in \{a, \gamma\} \), then we are through. Therefore, we conclude that \( \text{cl}'(A') \subseteq \text{cl}(A) \cup \{a, \gamma\} \).

Conversely, let \( x \in \text{cl}(A) \cup \{a, \gamma\} \). If \( x = a \), then in Case 1) and 2) \( a \in A' \).

Case 3) If \( C \) is a member of \( \mathcal{C}_{OX} \) contained in \( A \), then \( C' = C \cup a \subseteq A \cup a \) forms a circuit of \( M_X^e \). Thus, \( a \in \text{cl}'(A') \).

Case 4) If \( \gamma \in A' \) and \( e \in \text{cl}(A) \), then there is a \( EX \) or \( OX \)-circuit \( C \) containing \( e \). Then \( C' = (C \setminus e) \cup \{a, \gamma\} \) or \( C' = (C \setminus e) \cup \gamma \) are circuits of \( M_X^e \) contained in \( A' \cup a = A \cup \{a, \gamma\} \).

Case 5) If \( A \) contains an \( OX \)-circuit, then \( a \in \text{cl}'(A') \).

Now assume that \( x \in \text{cl}(A) - A \) and let \( C \) be a circuit of \( M \) such that \( x \in C \subseteq A \cup x \). If \( C \in \mathcal{C}_{EX} \), then \( C' = C \); otherwise \( C' = C \cup a \) is a circuit of \( M_X^e \). Thus, \( x \in C \) implies that \( x \in \text{cl}'(A \cup a) \). As \( a \in \text{cl}'(A') \), it follows that \( x \in \text{cl}'(A \cup a) = \text{cl}'(A') \).

If \( x = \gamma \), then in Cases 1, 3 and 4, \( \gamma \in A' \) implies \( \gamma \in \text{cl}'(A') \). In Cases 2 and 5, \( e \in \text{cl}(A) \) then one of the following two cases occurs.

(i) \( e \in C \subseteq A \cup e \) where \( C \in \mathcal{C}_{OX} \) and \( C \subseteq \text{cl}(A) \). Then \( C' = (C \setminus e) \cup \gamma \subseteq A \cup \gamma \) and we conclude that \( \gamma \in \text{cl}'(A') \).

(ii) \( e \in C \subseteq A \cup e \) where \( C \in \mathcal{C}_{EX} \) and \( C \subseteq \text{cl}(A) \). Then \( C' = ((C \setminus e) \cup \{a, \gamma\}) \subseteq A \cup \{a, \gamma\} \) and since \( a \in \text{cl}'(A') \), \( \gamma \in \text{cl}'(A \cup a) = \text{cl}'(A') \). This implies that \( \text{cl}(A) \cup \{a, \gamma\} \subseteq \text{cl}'(A') \).

\[ \square \]
Thus, we proved the following theorem.

**Theorem 4.2.8.** Let $M$ be a binary matroid on a set $E$ and $M_X^e$ be the splitting matroid of $M$ with respect to a subset $X$ of $E$ where $e \in X$. Suppose $A' \subseteq E \cup \{a, \gamma\}$ and $A = A' \setminus \{a, \gamma\}$. Then $cl'(A')$ has exactly one of the following forms: $cl(A) - F(A)$, $cl(A) \cup a$, $(cl(A) - F(A)) \cup \gamma$ and $(cl(A) - F(A)) \cup \gamma \cup T(A)$, $(cl(A) \cup \gamma) \cup T(A)$, $cl(A) \cup \{a, \gamma\}$.

The proof of Theorem 4.2.8 follows from Lemmas 4.2.1, 4.2.2, 4.2.3, 4.2.4, 4.2.5, 4.2.6 and 4.2.7.

### 4.3 Cocircuits of the es-Splitting Matroid

Allan Mills [20] described the cocircuits of the splitting matroid $M_{x,y}$ in terms of the cocircuits of the original matroid $M$. In Chapter 2, we have seen that if $M$ is an $n$-connected binary matroid and $|X| < n$, then $X$ will be a cocircuit of $M_X$ of size less than $n$ (see [20]). Therefore, $M_X$ can not be $n$-connected. Thus, before we study connectivity of es-splitting matroids it is necessary to have enough information about cocircuits of it. In this section, we characterize the cocircuits of the es-splitting binary matroid $M_X^e$ in terms of the cocircuits of the original binary matroid $M$.

We assume that $M$ is a binary matroid of rank $m$ on a set $E = \{1, 2, 3, \cdots, n\}$ and $A$ is a standard matrix representation of $M$ over $GF(2)$. Let $M_X^e = M[A_X^e]$ be the es-splitting matroid of $M$ with respect to a set $X \subseteq E$ and $e \in X$. Then the set $E' = \{1, 2, 3, \cdots n, a, \gamma\}$ is the
ground set of $M_X^e$. $\mathcal{R}(A_X^e)$ denotes the space spanned by the rows of $A_X^e$.

Given a field $F = GF(2)$ and a natural number $n$, the $n$-dimensional vector space over $F$ is denoted by $V(n, F)$. The support of a vector $v = (v_1, v_2, v_3, \ldots, v_n)$ is $\{i \mid v_i \neq 0\}$ and is denoted by $\text{supp}(v)$. The row space of an $(m \times n)$ matrix $A$ over the field $F$ is denoted by $\mathcal{R}(A)$ where $\mathcal{R}(A)$ is the subspace of $V(n, F)$, spanned by the rows of $A$. The next lemma due to Tutte [41] relates the cocircuits of the vector matroid $M[A]$ of a matrix $A$ to the minimal supports of vectors in $\mathcal{R}(A)$.

**Lemma 4.3.1.** Let $A$ be an $m \times n$ matrix over a field $F$ and $M = M[A]$. Then the set of cocircuits of $M$ coincides with the set of minimal non-empty supports of vectors from the row space of $A$.

Recall that the matrix $A_X^e$ is obtained from the matrix $A$ by adjoining an extra row say $\delta_X$ to $A$ and then adjoining two columns labelled $a$ and $\gamma$ to the resulting matrix as specified in Definition 3.1.7. Thus, we have a row vector $\delta_X' = [a_1, a_2, a_3, \ldots, a_n, a_{n+1}, a_{n+2}]_{1 \times (n+2)}$ of $A_X^e$ with the property that $a_i = 1$ if $i \in X$ or $i = n + 1$ and is zero otherwise. For convenience, we assume that this is the last row (i.e. $(m + 1)^{th}$ row) of the matrix $A_X^e$. Thus, the $\text{supp}(\delta_X') = X \cup a$.

Let $A'$ be the matrix obtained from $A_X^e$ by deleting the $(m + 1)^{th}$ row and $\mathcal{R}(A')$ be the space spanned by row vectors of the matrix $A'$. Further, let $I_n' = [I_n | 0]_{n \times (n+2)}$, $\delta_a = [0, 0, 0, ... 1, 0]_{1 \times (n+2)}$, $\delta_\gamma = [0, 0, 0, ... 0, 1]_{1 \times (n+2)}$ and $'*'$ be the usual matrix multiplication.

**Proposition 4.3.2.** Let $u' \in \mathcal{R}(A')$. Then there is a vector $u \in \mathcal{R}(A)$
such that

\[ u' = u \ast I'_n \quad \text{if } e \notin \text{supp}(u); \text{and} \]
\[ u' = u \ast I'_n + \delta \quad \text{if } e \in \text{supp}(u). \]

(4.3.1)  

(4.3.2)

In fact, \( u \) is a vector of size \( 1 \times n \) whose co-ordinates coincide with the first \( n \) co-ordinates of \( u' \).

The next Lemma is a basic linear algebra result and its proof is straightforward.

**Lemma 4.3.3.** Let \( B \) be the matrix obtained from \( A \) by adjoining the row vector \( z \). Then \( R(B) = R(A) \cup \{ y + z \mid y \in R(A) \} \).

Applying Lemma 4.3.3 to the matrices \( A' \) and \( A'_X \), we obtain the following corollary.

**Corollary 4.3.4.** \( R(A'_X) = R(A') \cup \{ u' + \delta'_X \mid u' \in R(A') \} \).

**Remark 4.3.5.** If \( v' \in R(A'_X) \), then one of the following two cases occurs.

**Case (I)** \( v' \in R(A') \). Then in the light of Proposition 4.3.2, \( v' = v \ast I'_n \) or \( v' = v \ast I'_n + \delta \gamma \) where \( v \in R(A) \). Further,

\[ \text{supp}(v') = \text{supp}(v) \cup \gamma \text{ if } e \in \text{supp}(v); \text{and} \]
\[ \text{supp}(v') = \text{supp}(v) \text{ if } e \notin \text{supp}(v). \]

(4.3.3)  

(4.3.4)

**Case (II)** \( v' \notin R(A') \). Here \( v' = u' + \delta'_X \) for some \( u' \in R(A') \). Therefore, \( \text{supp}(v') = \text{supp}(u') \Delta \text{supp}(\delta'_X) \). But \( \text{supp}(\delta'_X) = X \cup a \) and by equations (4.3.3) and (4.3.4), \( \text{supp}(u') = \text{supp}(u) \) or \( \text{supp}(u) \cup \gamma \).
Thus,

\[ \text{supp}(v') = (\text{supp}(u) \cup \gamma) \Delta (X \cup a) \text{ if } e \in \text{supp}(u); \text{and} \]

\[ \text{supp}(v') = \text{supp}(u) \Delta (X \cup a) \text{ if } e \notin \text{supp}(u) \] (4.3.5)

(4.3.6)

From equations (4.3.6) and (4.3.5), we observe that, if \( v' = u' + \delta_X \) for some \( u' \in \mathcal{R}(A') \) then \( a \in \text{supp}(v') \).

We illustrate the above discussion with the help of an example.

**Example 4.3.6.** Consider the Fano matroid mentioned in Example 1.2.25.

The Fano matroid \( F_7 \) is a connected matroid. Consider the following representation of \( F_7 \) over \( GF(2) \).

\[
A = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1
\end{pmatrix}
\]

The matrix \( A \) has three row vectors, \( u_1 = [1, 0, 0, 1, 0, 1, 1] \), \( u_2 = [0, 1, 0, 1, 1, 0, 1] \), and \( u_3 = [0, 0, 1, 0, 1, 1, 1] \). \( \mathcal{R}(A) = \{0, u_1, u_2, u_3, u_1 + u_2, u_1 + u_3, u_2 + u_3, u_1 + u_2 + u_3\} \). We denote by \( \mathcal{S}(A) \) the collection of non empty supports of vectors in \( \mathcal{R}(A) \). One can check that \( \mathcal{S}(A) = \{\{1, 4, 6, 7\}, \{2, 4, 5, 7\}, \{3, 5, 6, 7\}, \{1, 2, 5, 6\}, \{1, 3, 4, 5\}, \{2, 3, 4, 6\}, \{1, 2, 3, 7\}\} \). Further, each of these sets is minimal in the collection of non empty supports of vectors in \( \mathcal{R}(A) \). Thus, in view of Lemma 4.3.1, the set of all cocircuits of \( F_7 \) is given by
Let $X = \{1, 2, 3\}$ and $e = 1$. The representation of es-splitting matroid $(F_7)^c_X$ over $GF(2)$ is given by the matrix $A_X^c$ where

$$A_X^c = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & a & \gamma \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0
\end{pmatrix}.$$

The set $E' = \{1, 2, 3, 4, 5, 6, 7, a, \gamma\}$ is a ground set of the es-splitting matroid $(F_7)^c_X$. Note that $\delta_X' = [1, 1, 1, 0, 0, 0, 1, 0]$ is the row vector corresponding to the 4th row of $A_X^c$ and the $supp(\delta_X') = \{1, 2, 3, a\}$.

Let $A'$ be the matrix obtained from $A_X^c$ by deleting the 4th row and $\mathcal{R}(A')$ be the space spanned by row vectors of $A'$. Here $I_7' = [I_7 | 0]_{7 \times 9}$, $\delta_a' = [0, 0, 0, 0, 0, 0, 0, 1, 0]$ and $'*'$ is a usual matrix multiplication. Then for every $u' \in \mathcal{R}(A')$ there is a vector $u \in \mathcal{R}(A)$ such that $u' = u \cdot I_7'$ if 1 $\not\in supp(u)$ and $u' = u \cdot I_n' + \delta$ if 1 $\in supp(u)$. The vectors $u_1' = [1, 0, 0, 1, 0, 1, 1, 0, 1]$, $u_2' = [0, 1, 0, 1, 0, 1, 0, 1, 0]$ and $u_3' = [0, 0, 1, 0, 1, 1, 1, 0, 0]$ are the row vectors of $A'$. Thus, $\mathcal{R}(A') = \{0, u_1', u_2', u_3', u_1' + u_2', u_1' + u_3', u_2' + u_3', u_1' + u_2' + u_3'\}$ and $\mathcal{S}(A') = \{\{1, 4, 6, 7, \gamma\}, \{2, 4, 5, 7\}, \{3, 5, 6, 7\}, \{1, 2, 5, 6, \gamma\}, \{1, 3, 4, 5, \gamma\}, \{2, 3, 4, 6\}, \{1, 2, 3, 7, \gamma\}\}$. Applying Lemma 4.3.3 to $A$ and $A_X^c$, we have
4.3 Cocircuits of the es-Splitting Matroid

\(\mathcal{R}(A^e_X) = \mathcal{R}(A') \cup \{u' + \delta'_{X} \mid u' \in \mathcal{R}(A')\}\).

Thus, \(\mathcal{S}(A^e_X) = \{\{1, 4, 6, 7, \gamma\}, \{2, 4, 5, 7\}, \{3, 5, 6, 7\}, \{1, 2, 5, 6, \gamma\}, \{1, 3, 4, 5, \gamma\}, \{2, 3, 4, 6\}, \{1, 2, 3, 7, \gamma\}\} \cup \{\{1, 2, 3, a\}, \{2, 3, 4, 6, 7, a, \gamma\}, \{1, 3, 4, 5, 7, a\}, \{1, 2, 5, 6, 7, a\}, \{3, 5, 6, a, \gamma\}, \{2, 4, 5, a, \gamma\}, \{1, 4, 6, a\}, \{7, a, \gamma\}\}\).

Consequently, the set of all cocircuits of \((F_7)^e_X\) is
\(\mathcal{C}^*(F_7)^e_X = \{\{1, 4, 6, 7, \gamma\}, \{2, 4, 5, 7\}, \{3, 5, 6, 7\}, \{1, 2, 5, 6, \gamma\}, \{1, 3, 4, 5, \gamma\}, \{2, 3, 4, 6\}, \{1, 2, 3, 7, \gamma\}\} \cup \{\{1, 2, 3, a\}, \{1, 3, 4, 5, 7, a\}, \{1, 2, 5, 6, 7, a\}, \{3, 5, 6, a, \gamma\}, \{2, 4, 5, a, \gamma\}, \{1, 4, 6, a\}, \{7, a, \gamma\}\}\).

In order to describe the cocircuits of the es-splitting binary matroid \(M^e_X\) in terms of the cocircuits of the original binary matroid \(M\), first, we prove a few lemmas which will be used in the proof of the main theorem.

In the following discussion, \(\text{supp}(u)\) is minimal with respect to \(\mathcal{R}(A)\) means \(\text{supp}(u)\) is minimal in the collection of non empty supports of vectors in \(\mathcal{R}(A)\).

In the next lemma, we prove that, if \(\text{supp}(u')\) is minimal with respect to \(\mathcal{R}(A')\), then the support of the corresponding vector \(u\) is minimal with respect to \(\mathcal{R}(A)\).

**Lemma 4.3.7.** Let \(u' \in \mathcal{R}(A')\) and \(u' = u \ast I'_n + \delta_\gamma\) where \(u \in \mathcal{R}(A)\). Then \(\text{supp}(u')\) is minimal with respect to \(\mathcal{R}(A')\) if and only if \(\text{supp}(u)\) is minimal with respect to \(\mathcal{R}(A)\).

**Proof.** Suppose \(\text{supp}(u')\) is minimal with respect to \(\mathcal{R}(A')\) and \(\text{supp}(u)\) is not minimal with respect to \(\mathcal{R}(A)\). Then there is a vector \(v \in \mathcal{R}(A)\) such that \(\text{supp}(v) \subset \text{supp}(u)\). Let \(v' = v \ast I'_n + \delta_\gamma\) if \(e \in \text{supp}(v)\);
otherwise $v' = v$. Then $v' \in \mathcal{R}(A')$ and $supp(v') \subset supp(u')$, a contradiction. Converse of the statement in the Lemma follows by similar argument.

For convenience, we use the following notation.

**Notation:** $T_0 = \{supp(u) | u \in \mathcal{R}(A) \wedge supp(u) \subset X\}$.

**Lemma 4.3.8.** Let $v' = u' + \delta_X'$ be the element of $\mathcal{R}(A_X^c)$ where $u' \in \mathcal{R}(A')$. Suppose that $u' = u * I'_n + \delta_\gamma$ where $u \in \mathcal{R}(A)$, $supp(u) \subset X$ and $e \in supp(u)$. If $supp(v')$ is minimal with respect to $\mathcal{R}(A_X^c)$, then $supp(u)$ is maximal in $T_0$.

**Proof.** Let $v' = u' + \delta_X' \in \mathcal{R}(A_X^c)$ where $u' \in \mathcal{R}(A')$ and $supp(v')$ is minimal with respect to $\mathcal{R}(A_X^c)$. Since $e \in supp(u)$, we have $u' = u * I'_n + \delta_\gamma \in \mathcal{R}(A')$. So, by equation (4.3.5), $supp(v') = (supp(u) \Delta X) \cup \{a, \gamma\} = (X - supp(u)) \cup \{a, \gamma\}$. We show that $supp(u)$ is maximal in $T_0$. On the contrary, suppose that there is a vector $w \in \mathcal{R}(A)$ such that $supp(u) \subset supp(w) \subset X$. As $e \in supp(u) \subset supp(w)$, $w' = w * I'_n + \delta_\gamma'$ is an element of $\mathcal{R}(A')$. Further, the vector $p' = w' + \delta_X' \in \mathcal{R}(A_X^c)$ and $supp(p') = (supp(w) \Delta X) \cup \{a, \gamma\} \subset supp(v')$. This is a contradiction to the minimality of $supp(v')$ with respect to $\mathcal{R}(A_X^c)$. We conclude that, $supp(u)$ is maximal in $T_0$.

The following lemma establishes a natural relation between the cocircuits of $M$ and the cocircuit of $M_X^c$.

**Lemma 4.3.9.** Let $C^*$ be a cocircuit of a matroid $M$. Then $C^*$ or $C^* \cup \gamma$ is a cocircuit of $M_X^c$. 
Proof. Let $C^*$ be a cocircuit of $M$. By definition $C^*$ is minimal with respect to $R(A)$. Let $C^* = \text{supp}(u)$ for some vector $u \in R(A)$. Then $u' = u*I_n' + \delta'_\gamma$ if $e \in \text{supp}(u)$ and $u' = u*I_n'$ if $e \notin \text{supp}(u)$ is a vector in $R(A')$. Further, by equations (4.3.3) and (4.3.4), $\text{supp}(u') = \text{supp}(u) \cup \gamma$ if $e \in \text{supp}(u)$; otherwise $\text{supp}(u') = \text{supp}(u)$. Thus, $\text{supp}(u') = C^*$ or $C^* \cup \gamma$ and, by Lemma 4.3.7, $\text{supp}(u')$ is minimal with respect to $R(A_X')$. Now it is enough to show that $\text{supp}(u')$ is minimal with respect to $R(A_X^e)$.

On the contrary, suppose $\text{supp}(u')$ is not minimal with respect to $R(A_X^e)$. Then there is a vector $v' \in R(A_X^e)$ such that $v' \notin R(A')$ and $\text{supp}(v') \subset \text{supp}(u')$. Further, $\text{supp}(v')$ is minimal with respect to $R(A_X^e)$. Now, $v' = w' + \delta'_X$ for some $w' \in R(A')$ and $a \in \text{supp}(v') \subset \text{supp}(u')$; a contradiction. Thus, $C^*$ or $C^* \cup \gamma$ is a cocircuit of $M_X^e$. $\square$

In fact, one can observe that if $C^*$ is a cocircuit of $M$ and $e \notin C^*$, then $C^*$ is a cocircuit of $M_X^e$; otherwise $C^* \cup \gamma$ is a cocircuit of $M_X^e$. For convenience, we use the following notations.

$$C_1^* = \{C^* \mid C^* \in C^*(M) \text{ and } e \notin C^*\}; \text{ and}$$

$$C_2^* = \{C^* \cup \gamma \mid C^* \in C^*(M) \text{ and } e \in C^*\}.$$ 

Thus, $C_1^*$ and $C_2^*$ are collections of cocircuits of $M_X^e$. In Example 4.3.6, we observe that if $X = \{1, 2, 3\}$ and $e = 1$, then the members of the following collections are cocircuits of $(F^e)_X$.

$$C_1^* = \{\{2, 4, 5, 7\}, \{3, 5, 6, 7\}, \{2, 3, 4, 6\}\}; \text{ and}$$

$$C_2^* = \{\{1, 4, 6, 7, \gamma\}, \{1, 2, 5, 6, \gamma\}, \{1, 3, 4, 5, \gamma\}, \{1, 2, 3, 7, \gamma\}\}.$$
In the following result, we give a sufficient condition so that the set \( X \cup a \) is a cocircuit of \( M^e_X \).

**Lemma 4.3.10.** If \( X \) is a cocircuit of \( M \) or contains no member of \( C^*_1 \), then \( X \cup a \) is a cocircuit of \( M^e_X \).

**Proof.** Assume that \( X \) is a cocircuit of \( M \) or contains no member of \( C^*_1 \). Suppose \( \text{supp}(\delta'_X) = X \cup a \) is not a cocircuit of \( M^e_X \). Then there is a vector \( v' \in R(A^e_X) \) such that \( \text{supp}(v') \subset \text{supp}(\delta'_X) \) and \( \text{supp}(v') \) is minimal with respect to \( R(A^e_X) \). Now one of the following two cases occurs.

**Case (I)** \( v' = u' \) where \( u' \in R(A') \). By equations (4.3.1) and (4.3.2), \( u' = u*I'_n + \delta \gamma \) or \( u' = u*I'_n \) for some \( u \in R(A) \). Therefore, by equations (4.3.3) and (4.3.4), \( \text{supp}(v') = \text{supp}(u') = \text{supp}(u) \) or \( \text{supp}(u) \cup \gamma \). Now \( \gamma \notin (X \cup a) \) and \( \text{supp}(v') \subset \text{supp}(\delta'_X) \) implies that \( \text{supp}(u') = \text{supp}(u) \subset X \). Thus, \( \text{supp}(v') = \text{supp}(u) \) is minimal with respect to \( R(A^e_X), R(A') \) and \( R(A) \). Consequently, \( \text{supp}(u) \) is a cocircuit of \( M \) and \( e \notin \text{supp}(u) \); a contradiction to the assumption that \( X \) contains no member of \( C^*_1 \).

**Case (II)** \( v' = u' + \delta'_X \) where \( u' \in R(A') \). Then, by equations (4.3.6) and (4.3.5), \( \text{supp}(v') = (\text{supp}(u) \Delta X) \cup a \) or \( (\text{supp}(u) \Delta X) \cup \{a, \gamma\} \).

Now \( \text{supp}(v') \subset \text{supp}(\delta'_X) = X \cup a \) implies that \( \text{supp}(u) \subset X \) and \( e \notin \text{supp}(u) \). If \( \text{supp}(u) \) is minimal with respect to \( R(A) \), then we get a contradiction. If \( \text{supp}(u) \) is not minimal with respect to \( R(A) \), then there is a vector \( w \in R(A) \) such that \( \text{supp}(w) \subset \text{supp}(u) \) and \( \text{supp}(w) \) is minimal with respect to \( R(A) \); a contradiction. Therefore, \( \text{supp}(\delta'_X) \) is minimal with respect to \( R(A^e_X) \). Thus, \( X \cup a \) is a cocircuit of \( M^e_X \). \( \square \)
We use the following notations in the next results. For $u \in \mathcal{R}(A)$,
\[ F_1(\text{supp}(u)) = \{ w \mid w \in \mathcal{R}(A), e \in \text{supp}(w), (\text{supp}(w) - X) \subset (\text{supp}(u) - X) \text{ and } (\text{supp}(u) \cap X) \subset (\text{supp}(w) \cap X) \}; \]
\[ F_2(\text{supp}(u)) = \{ w \mid w \in \mathcal{R}(A), e \notin \text{supp}(w), (\text{supp}(w) - X) \subset (\text{supp}(u) - X) \text{ and } (\text{supp}(u) \cap X) \subset (\text{supp}(w) \cap X) \}; \]
\[ D^* = \{ D^* = C_1^* \cup C_2^* \cup \ldots \cup C_k^* \mid C_i^* \in \mathcal{C}^*(M), C_i^* \cap X \neq \emptyset (i = 1, 2, \ldots k) \text{ and } C_i^* \cap C_j^* = \emptyset \text{ for } i, j \in \{1, 2, \ldots k\} \}. \]

The following result describes the cocircuits of $M_X^c$ which arise from the cocircuits of $M$.

**Lemma 4.3.11.** Let $M$ be a binary matroid and let $X \subset E(M)$. Then the members of the following collections are cocircuits of $M_X^c$.

\[ C_3^* = \{ (C^* \Delta X) \cup \{a, \gamma\} \mid C^* \in \mathcal{C}^*(M), e \in C^*, C^* \cap X \neq \emptyset, \mathcal{F}_1(C^*) = \emptyset \text{ and } (C^* \Delta X) \cup \{a, \gamma\} \text{ contains no member of } C_1^* \text{ or } C_2^* \}; \]
\[ C_4^* = \{ (C^* \Delta X) \cup a \mid C^* \in \mathcal{C}^*(M), e \notin C^*, C^* \cap X \neq \emptyset, \mathcal{F}_2(C^*) = \emptyset \text{ and } (C^* \Delta X) \cup a \text{ contains no member of } C_1^* \}; \]

**Proof.** Let $C^* \in \mathcal{C}^*(M), e \in C^*, X_1 = C^* \cap X, X_2 = X - X_1, \mathcal{F}_1(C^*) = \emptyset \text{ and } (C^* \Delta X) \cup \{a, \gamma\} \text{ contains no member of } C_1^* \text{ and } C_2^*$. We prove that $(C^* \Delta X) \cup \{a, \gamma\}$ is a cocircuit of $M_X^c$. Let $C^* = \text{supp}(u)$ for some vector $u \in \mathcal{R}(A)$ and $\text{supp}(u)$ is minimal with respect to $\mathcal{R}(A)$. Then, by equation (4.3.2), $u' = u \ast I_n' + \delta_\gamma$ is a vector in $\mathcal{R}(A')$ and $\text{supp}(u') = C^* \cup \gamma$. Let $v' = u' + \delta_X'$. Then $\text{supp}(v') = (C^* \Delta X) \cup \{a, \gamma\}$. Suppose $\text{supp}(v')$ is not minimal with respect to $\mathcal{R}(A_X^c)$. Then there
exists a vector \( w' \in \mathcal{R}(A_X^e) \) such that \( \text{supp}(w') \subseteq \text{supp}(v') \) and \( \text{supp}(w') \) is minimal with respect to \( \mathcal{R}(A_X^e) \). Now there are two possibilities: \( w' \in \mathcal{R}(A') \) or \( w' = p' + \delta_X \) where \( p' \in \mathcal{R}(A') \).

If \( w' \in \mathcal{R}(A') \), then \( w' = w * I_n' + \delta \gamma \) or \( w' = w * I_n' \) for some \( w \in \mathcal{R}(A) \) and \( \text{supp}(w') \) is minimal with respect to \( \mathcal{R}(A') \). Therefore, \( \text{supp}(w') = \text{supp}(w) \cup \gamma \) or \( \text{supp}(w) \) and, by Lemma 4.3.7, \( \text{supp}(w) \) is minimal with respect to \( \mathcal{R}(A) \). Consequently, \( \text{supp}(w') \subset \text{supp}(v') = (C^* \Delta X) \cup \{a, \gamma\} \) implies that \( \text{supp}(w) \subset (C^* \Delta X) \); a contradiction.

Now if \( w' = p' + \delta_X \) where \( p' \in \mathcal{R}(A') \), then \( \text{supp}(w') = (Y - X_3) \cup X_4 \cup \{a, \gamma\} \) where \( Y = \text{supp}(p) \), \( X_3 = Y \cap X \), \( X_4 = X - X_3 \). We have, \( \text{supp}(w') = (C^* - X) \cup X_2 \cup \{a, \gamma\} \). So that \( \text{supp}(w') \subset \text{supp}(v') \) implies that \( \text{supp}(p) - X = (Y - X_3) \subset (C^* - X) \) and \( X_4 \subset X_2 \). This implies that \( X_1 \subset X_3 \) and \( p \in \mathcal{F}_1(C^*) \); a contradiction. We conclude that \( \text{supp}(v') \) is minimal with respect to \( \mathcal{R}(A_X^e) \). Consequently, every member of the collection \( \mathcal{C}_X^* \) is a cocircuit of \( M_X^e \).

Let \( C^* \in C^*(M) \), \( e \notin C^* \), \( X_1 = C^* \cap X \), \( X_2 = X - X_1 \), \( \mathcal{F}_2(C^*) = \phi \) and \( (C^* \Delta X) \cup a \) contains no member of \( \mathcal{C}_1^* \) or \( \mathcal{C}_2^* \). We show that \( (C^* \Delta X) \cup a \) is a cocircuit of \( M_X^e \). Let \( C^* = \text{supp}(u) \) for some vector \( u \in \mathcal{R}(A) \) and \( \text{supp}(u) \) is minimal with respect to \( \mathcal{R}(A) \). Then, by equation (4.3.1), \( u' = u * I_n' \) is a vector in \( \mathcal{R}(A') \) and \( \text{supp}(u') = C^* \). Let \( v' = u' + \delta_X \). Then \( \text{supp}(v') = (C^* \Delta X) \cup a \). Now, by the argument similar to one as given above, we can show that \( \text{supp}(v') \) is minimal with respect to \( \mathcal{R}(A_X^e) \). This proves that every member of the collection \( \mathcal{C}_4^* \) is a cocircuit of \( M_X^e \).

The following example illustrates the above Lemma.
Example 4.3.12. Consider the Fano matroid $F_7$ mentioned in Example 4.3.6. If $X = \{1, 2, 3\}$ and $e = 1$, then $X \cup a = \{1, 2, 3, a\}$ and the members of the following collections are cocircuits of $(F_7)^e_X$.

\[ C_1^* = \{2, 4, 5, 7\}, \{3, 5, 6, 7\}, \{2, 3, 4, 6\} \];
\[ C_2^* = \{\{1, 4, 6, 7, \gamma\}, \{1, 2, 5, 6, \gamma\}, \{1, 3, 4, 5, \gamma\}, \{1, 2, 3, 7, \gamma\}\}; \]
\[ C_3^* = \{\{7, a, \gamma\}, \{3, 5, 6, a, \gamma\}, \{2, 4, 5, a, \gamma\}\}; \]
\[ C_4^* = \{\{1, 3, 4, 5, 7, a\}, \{1, 2, 5, 6, 7, a\}, \{1, 2, 4, 6, a\}\}. \]

In the following result, we describe the cocircuits of $M_X^e$ which arise from the disjoint unions of cocircuits of $M$.

Lemma 4.3.13. Let $D^*(M)$ be the set of all disjoint unions of cocircuits of the matroid $M$ and let $X \subset E(M)$. Then the members of the following collections are cocircuits of $M_X^e$.

\[ C_5^* = \{(D^* \Delta X) \cup \{a, \gamma\} \mid e \in D^*, F_1(D^*) = \phi \text{ and } (D^* \Delta X) \cup \{a, \gamma\} \text{ contains no member of } C_1^* \text{ or } C_2^*\}; \]
\[ C_6^* = \{(D^* \Delta X) \cup a \mid e \notin D^*, F_1(D^*) = \phi \text{ and } (D^* \Delta X) \cup a \text{ contains no member of } C_1^* \text{ or } C_2^*\}. \]

Proof. Suppose that $D^* \in D^*$, $e \in D^*$, $F_1(D^*) = \phi$ and $(D^* \Delta X) \cup \{a, \gamma\}$ contains no member of $C_1^*$ or $C_2^*$. Then for each $C_i^* \in D^*$, $C_i^* = \text{supp}(u_i)$ where $u_i \in \mathcal{R}(A)$ is minimal with respect to $\mathcal{R}(A)$ for $i = 1, 2, ..., k$. Let $u = u_1 + u_2 + u_3 + ... + u_k$. Then $e \in \text{supp}(u)$ and $u' = u * I_n' + \delta$ is a vector in $\mathcal{R}(A')$. Further, $\text{supp}(u') = D^* \cup \gamma$ or $D^*$. Let $v' = u' + \delta_X'$. Therefore, $\text{supp}(v') = (D^* \Delta X) \cup \{a, \gamma\}$.
We show that \( \text{supp}(v') \) is minimal with respect to \( \mathcal{R}(A_{X}^{c}) \). On the contrary suppose there is a vector \( w' \in \mathcal{R}(A_{X}^{c}) \) such that \( \text{supp}(w') \subset \text{supp}(v') \) and \( \text{supp}(w') \) is minimal with respect to \( \mathcal{R}(A_{X}^{c}) \). There are two possibilities: \( w' \in \mathcal{R}(A') \) or \( w' = p' + \delta_{X} \) where \( p' \in \mathcal{R}(A') \).

If \( w' \in \mathcal{R}(A') \), then \( w' = w * I_{n}' + \delta_{Y} \) for some \( w \in \mathcal{R}(A) \) and \( \text{supp}(w') = \text{supp}(w) \cup \gamma \) is minimal with respect to \( \mathcal{R}(A') \). Now, by Lemma 4.3.7, \( \text{supp}(w) \) is minimal with respect to \( \mathcal{R}(A) \). Further, \( \text{supp}(w') \subset \text{supp}(v') \) implies that \( \text{supp}(w) \subset (D^{*} \Delta X) \), a contradiction.

Now if \( w' = p' + \delta_{X} \) where \( p' \in \mathcal{R}(A') \), then \( \text{supp}(w') = (Y - X_{3}) \cup X_{4} \cup \{a, \gamma\} \) where \( Y = \text{supp}(p), X_{3} \subset X, X_{4} = X - X_{3} \). Note that \( \text{supp}(v') = (D^{*} - X) \cup X_{2} \cup \{a, \gamma\} \) where \( X_{1} = D^{*} \cap X \) and \( X_{2} = X - X_{1} \). Now, \( \text{supp}(w') \subset \text{supp}(v') \) implies that \( \text{supp}(p) - X = (Y - X_{3}) \subset (D^{*} - X) \) and \( X_{4} \subset X_{2} \). Consequently, \( X_{1} \subset X_{3} \) and \( p \in \mathcal{F}_{1}(D^{*}) \). This is a contradiction. Therefore, \( \text{supp}(v') \) is minimal with respect to \( \mathcal{R}(A_{X}^{c}) \). This proves that every member of the collection \( C_{5}^{*} \) is a cocircuit of \( M_{X}^{c} \).

Let \( D^{*} \in D^{*}, e \notin D^{*} \), \( \mathcal{F}_{2}(C^{*}) = \phi \) and the set \( (D^{*} \Delta X) \cup a \) contains no member of \( C_{1}^{*} \) and \( C_{2}^{*} \). Then for each \( C_{i}^{*} \in D^{*}, C_{i}^{*} = \text{supp}(u_{i}) \) where \( u_{i} \in \mathcal{R}(A) \) is minimal with respect to \( \mathcal{R}(A) \) for \( i = 1, 2, \cdots, k \). Let \( u = u_{1} + u_{2} + u_{3} + \cdots + u_{k} \) then \( e \notin \text{supp}(u) \) and \( u' = u * I_{n}' \) is a vector in \( \mathcal{R}(A') \). Further, \( \text{supp}(u') = D^{*} \). Let \( v' = u' + \delta_{X}' \). Then \( \text{supp}(v') = (D^{*} \Delta X) \cup a \). Now, by arguments similar to one as above, we can show that \( \text{supp}(v') \) is minimal with respect to \( \mathcal{R}(A_{X}^{c}) \). Thus, every member of the collection \( C_{6}^{*} \) is a cocircuit of \( M_{X}^{c} \). \( \Box \)

Lemmas 4.3.9, 4.3.10, 4.3.11 and 4.3.13 provide collections of the
cocircuits of the matroid $M^e_X$ for different types of $X$. Conversely, one can prove that if $C^{*'}$ is a cocircuit of $M^e_X$, then it is a member of one of these collections. Consequently, the following Theorem characterizes the cocircuits of the matroid $M^e_X$.

**Theorem 4.3.14.** A subset $C^{*'}$ of $E(M^e_X)$ is a cocircuit of $M^e_X$ if and only if it is either $X \cup a$ or a member of one of the collections $C^*_1, C^*_2, C^*_3, C^*_4, C^*_5$ and $C^*_6$.

**Proof.** If $C^{*'}$ is a cocircuit of $M^e_X$, then there is a vector $v' \in R(A^e_X)$ such that $\text{supp}(v') = C^{*'}$. Now, by Proposition 4.3.4, $v' \in R(A')$ or $u' + \delta'_X$, where $u' \in R(A')$ and $\text{supp}(v')$ is minimal with respect to $R(A^e_X)$.

**Case (I)** $v' \in R(A')$. By equations (4.3.3) and (4.3.4), we have $\text{supp}(v') = \text{supp}(v)$ or $\text{supp}(v) \cup \gamma$ for some $v \in R(A)$. By Lemma 4.3.7, $\text{supp}(v)$ is minimal with respect to $R(A)$. Thus, $\text{supp}(v) = C^*$ for some $C^* \in C^*(M)$ and $\text{supp}(v') = C^{*'} = C^*$ or $C^* \cup \gamma$. Thus, $C^{*'} \in C^*_1$ or $C^*_2$.

**Case (II)** $v' = u' + \delta'_X$. Then $\text{supp}(v') = \text{supp}(u') \Delta \text{supp}(\delta'_X)$. One of the following two sub-cases occurs.

**Subcase (i) $e \notin \text{supp}(u')$.** By equation (4.3.1), $\text{supp}(u') = \text{supp}(u)$ for some $u \in R(A)$ and $\text{supp}(v') = (\text{supp}(u) \Delta X) \cup a$. If $\text{supp}(u) = \phi$, then $C^{*'} = \text{supp}(v') = X \cup a$.

If $\text{supp}(u)$ is minimal with respect to $R(A)$, then $\text{supp}(u) = C^*$ for some $C^* \in C^*(M)$. Consequently, if $C^* \subset X$, then $C^{*'} = \text{supp}(v') = (X - C^*) \cup a$; otherwise $C^{*'} = \text{supp}(v') = (C^* \Delta X) \cup a$. In either case, $C^{*'}$ is a member of $C^*_4$. 
If \( \text{supp}(u) \) is not minimal with respect to \( \mathcal{R}(A) \), then it is a disjoint union of cocircuits of \( M \). So, let \( \text{supp}(u) = D^* \). If \( D^* \subset X \), then \( C^* = \text{supp}(v^*) = (X - D^*) \cup \{a\} \); otherwise \( C^* = \text{supp}(v^*) = (D^* \Delta X) \cup \{a\} \). In both cases \( C^* \) is a member of \( C_6^* \).

**Subcase (ii) \( e \in \text{supp}(u^*) \).** Then, by equation (4.3.2), \( \text{supp}(u^*) = \text{supp}(u) \cup \gamma \) for some \( u \in \mathcal{R}(A) \) and \( \text{supp}(v^*) = (\text{supp}(u) \Delta X) \cup \{a, \gamma\} \). If \( \text{supp}(u) = C^* \) for some \( C^* \in C^*(M) \), then \( C^* = \text{supp}(v^*) = (C^* \Delta X) \cup \{a, \gamma\} \) is a member of \( C_3^* \). In particular, if \( C^* = X \), then \( C^* = \text{supp}(v^*) = \{a, \gamma\} \).

If \( \text{supp}(u) \) is not minimal with respect to \( \mathcal{R}(A) \), then it is a disjoint union of cocircuits of \( M \). So, let \( \text{supp}(u) = D^* \). If \( D^* \subset X \), then \( C^* = \text{supp}(v^*) = (X - D^*) \cup \{a, \gamma\} \); otherwise \( C^* = \text{supp}(v^*) = (D^* \Delta X) \cup \{a, \gamma\} \). Thus, in either case \( C^* \) is a member of \( C_5^* \). This completes the proof of the theorem.

\[ \square \]

The type of collection of cocircuits of \( M_X^* \) depends on the nature of the set \( X \). We consider the following three cases concerning the set \( X \).

1. \( X \) contains no cocircuit of \( M \);
2. \( X \) is a disjoint union of cocircuits of \( M \); and
3. \( X \) is not a disjoint union of cocircuits of \( M \) but contains a disjoint union of cocircuits of \( M \).

At a time only one of the three cases arises, so the set of cocircuits of \( M_X^* \) can be completely described with the help of Lemmas 4.3.9, 4.3.10, 4.3.11, 4.3.13 and Theorem 4.3.14.
In fact, the following corollary describes the set of cocircuits of $M_X^c$ when $X$ contains no cocircuit of $M$.

**Corollary 4.3.15.** If $X$ contains no cocircuit of $M$, then the set of cocircuits of $M_X^c$ consists of $X \cup a$ and the members of the collections $C_1^c, C_2^c, C_3^c, C_4^c, C_5^c$ and $C_6^c$.

The set of all cocircuits of $M_X^c$ when $X$ is a disjoint union of cocircuits of $M$ is described in the next result.

**Corollary 4.3.16.** Let $X$ be a disjoint union of cocircuits of $M$. Then $\{a, \gamma\}$ and the members of the collections $C_1^c, C_2^c, C_4^c$ and $C_6^c$ are cocircuits of $M_X^c$. Further, if $X$ is a cocircuit of $M$, then $X \cup a$ is a cocircuit of $M_X^c$.

*Proof.* Suppose $X$ is disjoint union of cocircuits $C_1^c, C_2^c, \ldots, C_k^c$ of $M$. Then, by Lemma 4.3.9, $C_i^c$ or $C_i^c \cup \gamma$ is a cocircuit of $M_X^c$ and then there are vectors $u_i' \in \mathcal{R}(A')$, $(i = 1, 2, \ldots, k)$ such that $\text{supp}(u_i') = C_i^c$ or $C_i^c \cup \gamma$. Now, $u = u_1 + u_2 + u_3 + \cdots + u_k$, $e \notin \text{supp}(u)$ and $u' = u * I_n'$ is a vector in $\mathcal{R}(A')$. Further, $\text{supp}(u') = X \cup \gamma$. Let $v' = u' + \delta_X'$. Then $\text{supp}(v') = (X \cup \gamma) \Delta (X \cup a) = \{a, \gamma\}$. One can verify that the set $\{a, \gamma\}$ is minimal with respect to $\mathcal{R}(A_X^c)$. We conclude that $\{a, \gamma\}$ is a cocircuit of $M_X^c$. Therefore, $C_3^c \cup C_5^c = \{a, \gamma\}$ and, by Lemmas 4.3.9, 4.3.11 and 4.3.13, the members of the collections $C_1^c, C_2^c, C_4^c$ and $C_6^c$ are cocircuits of $M_X^c$.

Further, if $X$ is a cocircuit of $M$, then, by Lemma 4.3.10, the set $X \cup a$ is a cocircuit of $M_X^c$. This completes the proof. $\square$

If $X$ is not a disjoint union of cocircuits of $M$ but contains disjoint
union of cocircuits of \( M \), then the cocircuits of \( M_X^c \) are described by the following result.

**Corollary 4.3.17.** Suppose \( X \) contains a disjoint union of cocircuits of \( M \). Then the members of the collections \( C_1^*, \ C_2^*, \ C_3^*, \ C_4^*, \ C_5^* \) and \( C_6^* \) are cocircuits of \( M_X^c \). Further, if \( X \) contains no member of \( C_1^* \), then in addition to the above collection, the set \( X \cup a \) is a cocircuit of \( M_X^c \).

With the help of the following example, we describe the cocircuits of \( M_X^c \) when \( X \) is a disjoint union of cocircuits of \( M \).

**Example 4.3.18.** Let \( G = K_{2,3} \) be the complete bipartite graph and \( M = M(G) \) be the cycle matroid of \( G \). Thus, \( E(M) = \{1, 2, 3, 4, 5, 6\} \) and the matrix \( A \) represents \( M \) over \( GF(2) \).

![Figure 1](image)

\[
A = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}.
\]

Here, \( C^*(M) = \{\{2,5\}, \{3,6\}, \{4,5,6\}, \{2,4,6\}, \{3,4,5\}, \{2,3,4\}, \{1,5,6\}, \{1,2,6\}, \{1,3,5\}, \{1,4\}, \{1,2,3\}\} \).
Let $X = \{1, 2, 4, 5\} \subset E(M)$. Then

\[
A^e_X = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & a & \gamma \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0
\end{pmatrix}.
\]

The set $E' = \{1, 2, 3, 4, 5, 6, a, \gamma\}$ is the ground set of the es-splitting matroid $M^e_X$. Note that $\delta'_X = [1, 1, 0, 1, 1, 0, 1, 0]$ is a row vector corresponding to the 5th row of $A^e_X$ and $supp(\delta'_X) = \{1, 2, 4, 5, a\}$.

The set $\{a, \gamma\}$ and the members of the following collections are cocircuits of $M^e_X$.

- $C^*_1 = \{\{2, 5\}, \{3, 6\}, \{4, 5, 6\}, \{2, 4, 6\}, \{3, 4, 5\}, \{2, 3, 4\}\};$
- $C^*_2 = \{\{1, 5, 6, \gamma\}, \{1, 2, 6, \gamma\}, \{1, 3, 5, \gamma\}, \{1, 4, \gamma\}, \{1, 2, 3, \gamma\}\};$ and
- $C^*_4 = \{\{1, 2, 6, a\}, \{1, 5, 6, a\}, \{1, 2, 3, a\}, \{1, 3, 5, a\}, \{1, 4, a\}\}.$

The following example illustrates the Corollary 4.3.17.

**Example 4.3.19.** Let $G = K_{2,3}$ be the complete bipartite graph, $M = M(G)$ be the cycle matroid of $G$ and $E(M) = \{1, 2, 3, 4, 5, 6\}$.

\[C^*(M) = \{\{2, 5\}, \{3, 6\}, \{4, 5, 6\}, \{2, 4, 6\}, \{3, 4, 5\}, \{2, 3, 4\}, \{1, 5, 6, \gamma\}, \{1, 2, 6, \gamma\}, \{1, 3, 5, \gamma\}, \{1, 4, \gamma\}, \{1, 2, 3, \gamma\}\}.
\]

Let $X = \{1, 2, 3, 4, 5\} \subset E(M)$. Then the representation of the es-splitting matroid $M^e_X$ over $GF(2)$ is given by the matrix
4.3 Cocircuits of the es-Splitting Matroid

\[ A^e_X = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & a & \gamma \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \end{pmatrix} \]

Note that \( \delta'_X = [1, 1, 1, 1, 0, 1, 0] \) is a row vector corresponding to the 5\textsuperscript{th} row of \( A^e_X \) and \( supp(\delta'_X) = \{1, 2, 3, 4, 5, a\} \).

Thus, the set of cocircuits of \( M^e_X \) consists of the members of the following collections.

\[ C^*_1 = \{ \{2, 5\}, \{3, 6\}, \{4, 5, 6\}, \{2, 4, 6\}, \{3, 4, 5\}, \{2, 3, 4\} \}; \]
\[ C^*_2 = \{ \{1, 5, 6, \gamma\}, \{1, 2, 6, \gamma\}, \{1, 3, 5, \gamma\}, \{1, 4, \gamma\}, \{1, 2, 3, \gamma\} \}; \]
\[ C^*_3 = \{ \{2, 4, a, \gamma\}, \{4, 5, a, \gamma\} \}; \]
\[ C^*_4 = \{ \{1, 2, a\}, \{1, 5, a\}, \{1, 3, 4, a\} \}; \]
\[ C^*_5 = \{ \{3, a, \gamma\}, \{6, a, \gamma\} \}; \] and
\[ C^*_6 = \{ \{1, 4, 6, a\} \} . \]