CHAPTER-II

ON (ε)-PARA α- SASAKIAN MANIFOLD*

The purpose of this chapter is to generalize some of the results of M.M. Tripathi and his colleague and some of those of ([9], [10], [18], [19], [20]). In this chapter we introduce the notion of (ε)-α-almost Para contact manifolds, and in particular, of (ε)-Para α- Sasakian manifolds and also studies the basic results in (ε)-Para α- Sasakian manifolds and its properties. Some typical identities for curvature tensor and Ricci tensor of (ε)-Para α- Sasakian manifolds are obtained.

2.1. Introduction

In 1976, an almost Para contact structure (ϕ,ξ,η) satisfying $\phi^2 = I - \eta \oplus \xi$ and $\eta(\xi) = 1$ on a differentiable manifold was introduced by Sato [16]. The structure is an analogues of the almost contact structure [2, 13] and is closely related to almost product structure.

In 1969, Takahashi [12] introduced almost contact manifolds equipped with associated pseudo-Riemannian metric. In 2010, Mukut Mani Tripathi, Erol Kilic, Seleen Yiiksel Perktas and sadik keles have studied the various properties of (ε)-Para sasakian manifolds and in 2008, Mukut Mani Tripathi, Mohit Kumar Dwivedi have studied the various properties of K-contact manifolds. Inspired by these papers and some other papers (See the exhaustive list in [1], [5], [6], [7], [9], [10],[15],[18],[19],[20]) we have

*The content of this chapter is published in the Cayley Journal of Mathematics Vol.1(1)(2012), 29-53.
introduced, in this chapter, the \((\varepsilon)\)-Para \(\alpha\)-sasakian manifolds with an indefinite metric and study the properties.

In section 2, we consider the \((2n+1)\) dimensional differentiable manifold \(M\) with \((\varepsilon)\) almost paracontact metric structure with indefinite metric \(g\). In this section, some background information for defining \((\varepsilon)\)-Para \(\alpha\)-sasakian manifold has been given. Further various basic results are studied. We find some typical identities for curvature tensor and Ricci tensor.

In section 3, we show that, for an \((\varepsilon)\)-Para \(\alpha\)-Sasakian manifold, the conditions of being symmetric, semi-symmetric, or of constant sectional curvature are all identical. Section 4 deals with the properties of Ricci-symmetric and Ricci-semi-symmetric metric \((\varepsilon)\)-Para \(\alpha\)-Sasakian manifolds.

Section 5 is devoted to the study of the generalized recurrent properties of \((\varepsilon)\)-Para \(\alpha\)-Sasakian manifolds.

In section 6, we prove that a conformally flat \((\varepsilon)\)-Para \(\alpha\)-Sasakian manifold \(M\) is \(\eta\)-Einstein manifold and the scalar curvature \(r\) is constant.

In section 7, we consider three cases of projective curvature tensor and give definitions of quasi projectively flat, \(\xi\)-projectively flat and \(\phi\)-projectively flat \((\varepsilon)\)-Para \(\alpha\)-Sasakian manifolds. We prove that a \((\varepsilon)\)-Para \(\alpha\)-Sasakian manifold is \(\xi\) projectively flat if and only if \(M\) is Einstein Sasakian. Also prove that a projectively flat Einstein \((\varepsilon)\)-Para \(\alpha\)-Sasakian manifold is locally isometric to sphere \(S^{(2n+1)}(c)\) Where \(c = -\varepsilon \alpha^2\) with
\( \varepsilon = -1 \) and hyperbolic to a sphere \( H^{(2n+1)}(c) \) where \( c = -\varepsilon \alpha^2 \) with \( \varepsilon = 1 \). We prove that a \( (\varepsilon) \)-Para \( \alpha \)-Sasakian manifold is quasi projectively flat. Also prove that if \( (\varepsilon) \)-Para \( \alpha \)-Sasakian manifold \( M \) is quasi projectively flat, and then \( M \) is an \( \eta \)-Einstein manifold.

2.2. \( (\varepsilon) \)-Para \( \alpha \)-Sasakian manifolds

For an \( (\varepsilon) \)-Almost paracontact manifold, we have

\[
\phi^2 X = X - \eta(X)\xi, \eta(\xi) = 1, \eta(X) = g(X, \xi)
\]

where \( \phi \) a tensor is field of type \((1, 1)\), \( \xi \) is characteristic vector field and \( \eta \) is the 1-form.

From these conditions, one can deduce that

\[
\phi(\xi) = 0, \eta(\phi(X)) = 0
\]

for any vector field \( X \) on \( M \).

On an \( n \)-dimensional almost paracontact manifold, one can easily obtain

\[
\phi^3 - \phi = 0 \quad \text{Rank (\phi) } = n - 1.
\]

Let \( M \) be a manifold equipped with an almost paracontact structure \((\phi, \xi, \eta)\). Let \( g \) be a semi-Riemannian metric with index \( g = \gamma \) such that

\[
g(\phi X, \phi Y) = g(X, Y) - \varepsilon \eta(X)\eta(Y), \quad X, Y \in TM
\]

Where \( \varepsilon = \pm 1 \) then we say that \( M \) is an \( (\varepsilon) \)-almost paracontact metric manifold equipped with an \( (\varepsilon) \)-almost paracontact metric structure \((\phi, \xi, \eta, g, \varepsilon)\).
**Definition 2.2.1:** A differential manifold \((2n+1)\) called a \((\varepsilon, \alpha)\)-almost paracontact manifold if it admits a 1-1 tensor field \(\phi\), a contravariant vector field \(\xi\), a covariant vector field \(\eta\) and indefinite metric \(g\) which satisfy

\[
\phi^2 X = X - \eta(X)\xi, \quad \eta(\xi) = 1, \quad \eta(\phi(X)) = 0
\]

\[(2.2.2)\]

\[
g(\xi, \xi) = \varepsilon, \quad g(X, \xi) = \varepsilon \eta(X)
\]

\[(2.2.3)\]

\[
g(\phi X, \phi Y) = g(X, Y) - \varepsilon \eta(X) \eta(Y)
\]

\[(2.2.4)\]

\[
g(\phi X, Y) = g(X, \phi Y)
\]

for any vector field \(X, Y\) on \(M\).

The fundamental \((0, 2)\) symmetric tensor of the \((\varepsilon, \alpha)\)-almost paracontact metric structure is defined by

\[
\Phi(X, Y) = g(X, \phi Y)
\]

\[(2.2.4)\]

for all \(X, Y \in TM\) also we get

\[
(\nabla_X \Phi)(Y, Z) = g((\nabla_X \phi)Y, Z) = (\nabla_X \Phi)(Z, Y)
\]

\[(2.2.5)\]

for all \(X, Y, Z \in TM\).

**Definition 2.2.2:** We say that \((\phi, \xi, \eta, \varepsilon, g, \alpha)\) is an \((\varepsilon, \alpha)\)-paracontact metric structure if

\[
2\Phi(X, Y) = (\nabla_X \eta)Y + (\nabla_Y \eta)X
\]

\[(2.2.7)\]

In this case, \(M\) is an \((\varepsilon, \alpha)\)-paracontact manifold.
The condition (2.2.7) is equivalent to $2\Phi = \ell g$

where $\ell$ is the operator of Lie differentiation for $\varepsilon=1$ and $g$ Riemannian.

**Definition 2.2.3:** An $(\varepsilon)$-$\alpha$-almost paracontact metric structure $(\phi, \xi, \eta, g, \varepsilon)$ is called an $(\varepsilon)$-$\alpha$-S-Paracontact metric structure if

(2.2.8) $\nabla_\xi = \varepsilon \alpha \phi$.

A manifold equipped with an $(\varepsilon)$-$\alpha$-S-Paracontact structure is said to be $(\varepsilon)$-$\alpha$-S-Paracontact metric manifold.

Equation (2.2.8) is equivalent to

(2.2.9) $\Phi(X, Y) = g(\phi X, Y) = \frac{\varepsilon}{\alpha} g(\nabla_X \xi, Y) = \frac{1}{\alpha} (\nabla_X \eta) Y$.

**Definition 2.2.4:** An $(\varepsilon)$-$\alpha$-almost paracontact metric structure is called an $(\varepsilon)$-Para $\alpha$-Sasakian structure if

(2.2.10) $(\nabla_X \phi) Y = -\alpha \left\{ g(\phi X, \phi Y) \xi + \varepsilon \eta(Y) \phi^2 X \right\}$ for all $X, Y \in TM$.

where $\nabla$ the Levi-Civita connection with respect to $g$ and $\alpha$ is is a constant. A manifold endowed with an $(\varepsilon)$-Para $\alpha$-Sasakian structure $(\phi, \xi, \eta, g, \varepsilon)$ is called $(\varepsilon)$-Para $\alpha$-sasakian manifold.

For $\varepsilon=1$ and $g$ Riemannian, $M$ is the usual Para Sasakian manifold, For $\varepsilon=-1$, $g$-Lorentzian and $\xi$ replaced $-\xi$, $M$ becomes a Lorentzian Para Sasakian manifold, If $\alpha=1$, Then $M$ is $(\varepsilon)$- Para Sasakian manifold defined by [17].
Lemma 2.2.5: For \((\varepsilon)\)-Para \(\alpha\)-Sasakian manifolds, we have

\[(2.2.11) \quad \nabla_X \xi = \varepsilon \alpha X \text{ or any vector field } X \text{ on } M.\]

Proof: From (2.2.10) of definition 2.2.4 we have

\[\nabla_X(\phi(Y)) - \phi(\nabla_X Y) = -\alpha \left\{ g(\phi X, \phi Y)\xi + \varepsilon \gamma(Y)X^2 \right\} \]

Now taking \(Y = \xi\) in the above equation and using (2.2.1), we get

\[-\phi(\nabla_X \xi) = -\varepsilon \alpha X \]

operating by \(\phi\) to the above equation we get (2.2.11).

Lemma 2.2.6: For \((\varepsilon)\)-Para \(\alpha\)-Sasakian manifold \(M\), we have

\[(2.2.12) \quad (\nabla_X \gamma)Y = \alpha g(Y, \phi X) \]

for any vector \(X, Y\) on \(M\).

Proof: Consider \(\nabla_X \gamma(Y) = \nabla_X(\gamma(Y)) - \gamma(\nabla_X Y)\)

\[= \varepsilon \nabla_X g(Y, \xi) - \varepsilon g(\nabla_X Y, \xi) \]

\[= \varepsilon g(\nabla_X Y, \xi) + \varepsilon g(Y, \nabla_X \xi) - \varepsilon g(\nabla_X Y, \xi) \]

\[= \varepsilon g(Y, \nabla_X \xi) \]

using (2.2.11) we get (2.2.12).
**Lemma 2.2.7:** For \((\varepsilon)-\text{Para}\ \alpha\text{-Sasakian manifold} \ M\), we have

(2.2.13) \[ R(X, Y)\xi = \alpha^2 [\eta(X)Y - \eta(Y)X] \]

(2.2.14) \[ R(\xi, Y)\xi = \alpha^2 (\phi^2 Y), R(\xi, \xi)\xi = 0 \]

for any vector field \( X, Y \) on \( M \).

**Proof:** From (2.2.11) and the fact that \([X, Y] = \nabla_X Y - \nabla_Y X\), we have

\[ R(X, Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi \]

\[ = \nabla_X (\varepsilon \alpha \phi Y) - \nabla_Y (\varepsilon \alpha \phi X) - \varepsilon \alpha \phi [\nabla_X Y - \nabla_Y X] \]

\[ = \varepsilon [ (\nabla_X \phi) Y + \phi (\nabla_X Y)] - \varepsilon [ (\nabla_Y \phi) X + \phi (\nabla_Y X)] \]

\[ = -\varepsilon \alpha \phi (\nabla_X Y) + \alpha (\nabla_Y X) \]

using (2.2.10), after simplification, we get (2.2.13).

(2.2.14) follows from (2.2.13) by putting \( X = \xi \).

**Lemma 2.2.8:** For \((\varepsilon)-\text{Para}\ \alpha\text{-Sasakian manifold} \ M\), we have

(2.2.15) \[ R(\xi, Y)X = -\varepsilon \alpha^2 g(X, Y)\xi + \alpha^2 \eta(X)Y \]

for any vector fields \( X, Y \) on \( M \).

**Proof:** From the identity \( g(R(\xi, Y)X, Z) = g(R(X, Z)\xi, Y) \)

and from (2.2.13) of Lemma 2.2.7 we have

\[ g(R(\xi, Y)X, Z) = \alpha^2 \{ \eta(X)g(Z, Y) - \eta(Z)g(X, Y) \} \]
after simplification we get (2.2.15).

**Lemma 2.2.9:** For $(\varepsilon)$-Para $\alpha$-Sasakian manifold $M$, we have

\[(2.2.16) \quad R(X, Y, Z, \xi) = \alpha^2 [\eta(X)g(X, Z) - \eta(X)g(Y, Z)]\]

\[(2.2.17) \quad \eta(R(X, Y)Z) = \varepsilon \alpha^2 [\eta(X)g(X, Z) - \eta(X)g(Y, Z)].\]

**Proof:** From the identity $R(X, Y, Z, \xi) = -R(X, Y, \xi, Z)$

using (2.2.13) of Lemma 2.2.7 we get (2.2.16) and consequently we get (2.2.17).

If we put

\[(2.2.18) \quad \alpha^2 R_0(X, Y)W = \alpha^2 g(Y, W)X - \alpha^2 g(X, W)Y \quad \text{where} \quad X, Y, Z \in TM.\]

Then in an $(\varepsilon)$-Para $\alpha$-Sasakian manifold $M$ (2.2.13) and (2.2.15) can be rewritten as

\[(2.2.19) \quad R(X, Y)\xi = -\varepsilon \alpha^2 R_0(X, Y)\xi\]

\[(2.2.20) \quad R(\xi, X) = -\varepsilon \alpha^2 R_0(\xi, X)\]

**Lemma 2.2.10:** In an $(\varepsilon)$-Para $\alpha$-Sasakian manifold $M$, the curvature tensor satisfies

\[(2.2.21) \quad R(X, Y, \phi Z, W) - R(X, Y, Z, \phi W) = \varepsilon \Phi(Y, Z)g(\phi X, \phi W) - \varepsilon \Phi(X, Z)g(\phi Y, \phi W)\]

\[+ \varepsilon \Phi(Y, W)g(\phi X, \phi Z) - \varepsilon \Phi(X, W)g(\phi Y, \phi Z)\]

\[+ \alpha^2 \eta(Z)\eta(Y)g(X, \phi W) - \alpha^2 \eta(X)\eta(Z)g(Y, \phi W)\]

\[+ \alpha^2 \eta(Y)\eta(W)g(X, \phi Z) - \alpha^2 \eta(X)\eta(W)g(Y, \phi Z)\]
(2.2.22) \( R(X, Y, \phi Z, \phi W) - R(X, Y, Z, W) = \)
\[
\left( \Phi(Y, Z) \Phi(X, W) - \Phi(X, Z) \Phi(Y, W) \right) + \left( \Phi(X, Z) \Phi(Y, W) - \Phi(Y, Z) \Phi(X, W) \right) + \alpha^2 \eta(Z) \{ \eta(Y) g(X, W) - \eta(X) g(Y, W) \} - \alpha^2 \eta(W) \{ \eta(Y) g(X, Z) - \eta(X) g(Y, Z) \}
\]

(2.2.23) \( R(X, Y, \phi Z, \phi W) = R(\phi X, \phi Y, Z, W) \)

(2.2.24) \( R(\phi X, \phi Y, \phi Z, \phi W) = R(X, Y, Z, W) + \alpha^2 \eta(Z) \{ \eta(Y) g(X, W) - \eta(X) g(Y, W) \} - \alpha^2 \eta(W) \{ \eta(Y) g(X, Z) - \eta(X) g(Y, Z) \} \).

**Proof:** Writing (2.2.10) equivalently as

(2.2.25) \( (\nabla_Y \Phi)(Z, W) = -\epsilon \alpha \eta(W) g(\phi Y, \phi Z) - \epsilon \alpha \eta(Z) g(\phi Y, \phi W) \)

and differentiating covariantly with to \( X \), we get

(2.2.26) \( -\epsilon (\nabla_X \nabla_Y \Phi)(Z, W) = \Phi(X, Z) g(\phi Y, \phi W) + \alpha \eta(Z) (\nabla_X \Phi)(Y, \phi W) + \alpha \eta(Z) g(\phi(\nabla_X Y), \phi W) + \alpha \eta(Z) (\nabla_X \Phi)(\phi Y, W) + \Phi(X, W) g(\phi Y, \phi Z) + \alpha \eta(W) (\nabla_X \Phi)(Y, \phi Z) + \alpha \eta(W) g(\phi(\nabla_X Y), \phi Z) + \alpha \eta(W) (\nabla_X \Phi)(\phi Y, Z) \)

for all \( X, Y, W \in TM \).
Now using (2.2.26) in the Ricci identity

\[(2.2.27) \quad (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} )\Phi(Z, W) = -\Phi(R(X, Y)Z, W) - \Phi(Z, R(X, Y)W)\]

we obtain (2.2.21). Equation (2.2.22) follows from (2.2.21) and (2.2.16). Equation (2.2.23) follows from (2.2.22). Finally, equation (2.2.24) follows from (2.2.22) and (2.2.23).

**Remark:** If $\alpha=1$ the result of M. M. Tripathi and colleagues [lemma 5.6] will include as special case of lemma2.2.10.

**Theorem 2.2.11:** An $(\varepsilon)$-Para $\alpha$-Sasakian manifold cannot be flat.

**Proof:** Let $M$ be a flat $(\varepsilon)$-Para $\alpha$-Sasakian manifold. Then, from (2.2.16) we get

\[(2.2.28) \quad \eta(X) g(Y, Z) = \eta(Y) g(X, Z), \quad \text{provided} \quad \alpha\neq 0.\]

From which we obtain

\[(2.2.29) \quad g(\phi X, \phi Z) = 0 \quad \text{for all} \ X, Z \in TM, \ \text{a contradiction}.\]

**Remark:** If $\alpha=1$ the result of M. M. Tripathi and colleagues [Theorem 5.7] will include as special case of Theorem 2.2.11.

**2.3. Recurrent $(\varepsilon)$-Para $\alpha$-Sasakian Manifolds**

A non-flat semi-Riemannian manifold $M$ is said to be recurrent [4] if its Ricci tensor $R$ satisfies the recurrence condition

\[(2.3.1) \quad (\nabla_W R)(X, Y, Z, V) = A(W)R(X, Y, Z, V)\]
where $X, Y, Z, V \in TM$ and $A$ is a 1-form. If $A=0$ in the above equation, then the manifold becomes symmetric in the sense of $\text{car tan} [4]$.

We say that $M$ is proper recurrent if $A \neq 0$.

**Theorem 2.3.1:** An $(\varepsilon)$-Para $\alpha$-Sasakian manifold $M$ cannot be proper recurrent.

**Proof:** Let $M$ be a recurrent $(\varepsilon)$-Para $\alpha$-Sasakian manifold. Then, from (2.3.1), (2.2.16) and (2.2.8) we obtain

\begin{equation}
\in R(X, Y, Z, W) \in \alpha \phi = \alpha \Phi - \eta
\end{equation}

\begin{equation}
= \alpha \phi (X, Y, Z, W) - \eta (Y)
\end{equation}

\begin{equation}
- \alpha \phi (X, Y, Z, W) - \eta (X)
\end{equation}

for all $X, Y, Z, W \in TM$. Putting $Y=\xi$ in the above equation, we get

\begin{equation}
\alpha A (W) \phi (X, \phi Z) = 0, \quad \text{for all } X, Z, W \in TM, \quad \text{a contradiction.}
\end{equation}

**Remark:** The theorem 2.3.1 generalization of the theorem 5.8 is due to M. M. Tripathi and colleagues.

**Theorem 2.3.2:** An $(\varepsilon)$-Para $\alpha$-Sasakian manifold is symmetric if and only if it is of constant curvature $-\varepsilon \alpha^2$.

**Proof:** Let $M$ be a symmetric $(\varepsilon)$-$\alpha$-Para Sasakian manifold. Then, putting $A=0$ in (2.3.2) we obtain

\begin{equation}
\in R(X, Y, Z, W) = \alpha^2 [g(X, Z)\Phi(Y, W) - g(Y, Z)\Phi(X, W)]
\end{equation}
for all $X, Y, Z, W \in TM$. Writing $\phi W$ in place of $W$ in the above equation and using (2.2.3) and (2.2.16) we get

$$(2.3.5) \quad R(X, Y, Z, W) = -\varepsilon \alpha^2 \{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\}.$$ 

This shows that $M$ is a space of constant curvature $-\varepsilon \alpha^2$. The converse is trivial.

**Remark:** The theorem 2.3.2 generalization of the theorem 5.9 is due to M. M. Tripathi and colleagues.

**Corollary 2.3.3:** If an $(\varepsilon)$-Para $\alpha$-Sasakian manifold is of constant curvature, then

$$(2.3.6) \quad \Phi(Y, Z)\Phi(X, W) - \Phi(X, Z)\Phi(Y, W) = -g(\phi Y, \phi Z)g(\phi X, \phi W) + \alpha g(\phi X, \phi Z)g(\phi Y, \phi W)$$

for all $X, Y, Z, W \in TM$.

**Proof:** Obviously, if an $(\varepsilon)$-Para $\alpha$-Sasakian manifold is of constant curvature $k$, then

$$k = -\varepsilon \alpha^2.$$ Therefore, using (2.3.5) in (2.2.22), we get (2.3.6).

**Remark:** The Corollary 2.3.3 generalization of the Corollary 5.10 is due to M. M. Tripathi and colleagues.

**Definition 2.3.4:** A $(\varepsilon)$-Para $\alpha$-Sasakian manifold $(M, g)$ is said to be semi-symmetric space if its curvature tensor $R$ satisfies the condition

$$(2.3.7) \quad R(X, Y) = 0$$
for all vector field $X, Y$ on $M$, where $R(X, Y)$ acts as a derivation on $R$. Symmetric space are obviously semi-symmetric, but the converse need not be true.

**Theorem 2.3.5:** In an $(\varepsilon)$-Para $\alpha$-Sasakian manifold, the condition of semi-symmetry implies the condition of symmetry.

**Proof:** Let $M$ be a symmetric $(\varepsilon)$-Para $\alpha$-Sasakian manifold. Let the condition of being semi-symmetric be true, that is,

$$R(V, U).R = 0, \quad V, U \text{ on } M. \tag{2.3.8}$$

In particular, from the condition $R(\xi, U).R = 0$ we get,

$$0 = [R(\xi, U), R(X, Y)]\xi - R[R(\xi, U)X, Y]\xi - R[X, R(\xi, U)Y]\xi \tag{2.3.9}$$

which in view of (2.2.20) gives

$$o \, \eta(R(X, Y)) = -\varepsilon(\eta(R(X, Y)) \xi) \tag{2.3.10}$$

Equation (2.2.19) then gives

$$R = -\varepsilon R_0 \tag{2.3.11}$$

Therefore $M$ is of constant curvature $-\varepsilon$, and hence symmetric.

**Remark:** The Theorem 2.3.5 generalization of the Theorem 5.11 is due to M. M. Tripathi and colleagues.
**Corollary 2.3.6:** Let $M$ be an $(\epsilon)\alpha$-Para Sasakian manifold. Then the following statements are equivalent:

(i) $M$ is symmetric.

(ii) $M$ is of constant curvature $-\epsilon \alpha^2$.

(iii) $M$ is semi-symmetric.

(iv) $M$ satisfies $R(\xi, U).R=0$

**Proof:** From theorem 2.3.2 and 2.3.5, corollary follows.

### 2.4. Ricci-Symmetric and Ricci-Semi-symmetric $(\epsilon)\alpha$-Para Sasakian Manifolds

In this section, we introduce the notions of Ricci-symmetric and Ricci-semisymmetric, $(\epsilon)\alpha$-Para Sasakian manifolds.

**Lemma 2.4.1:** In an $(2n+1)$-dimensional $(\epsilon)\alpha$-Para Sasakian manifold we get

(2.4.1) \[ R(\xi, Y, Z, \xi) = \alpha^2 g(\phi Y, \phi Z) \]

Consequently,

(2.4.2) \[ \sum_{i=1}^{2n} R(e_i, Y, Z, e_i) = \sum_{i=1}^{2n} R(\phi e_i, Y, Z, \phi e_i) = S(Y, Z) - \alpha^2 g(\phi Y, \phi Z) \]

**Proof:** From (2.2.15) and (2.2.2) we get

\[ R(\xi, Y, Z, \xi) = -\alpha^2 [g(Y, Z) - \epsilon \eta(Z)\eta(Y)] \]

using (2.2.3), we get (2.4.1).
In a \((2n+1)\)-dimensional \((\varepsilon)\)-Para \(\alpha\)-Sasakian manifold \(M\), if \(\{e_1, \ldots, e_{2n}, \xi\} \) is a local orthonormal basis of vector fields in \(M\), then \(\{\phi e_1, \ldots, \phi e_{2n}, \xi\} \) is also a local orthonormal basis. Therefore

\[
\sum_{i=1}^{2n} R(e_i, Y, Z, e_i) + R(\xi, Y, Z, \xi) = S(Y, Z)
\]

using (2.4.1), we get (2.4.2).

**Lemma 2.4.2:** In an \((2n+1)\)-dimensional \((\varepsilon)\)-Para \(\alpha\)-Sasakian manifold \(M\) the Ricci tensor \(S\) satisfies

\[
(2.4.3) \quad S(\phi Y, \phi Z) = S(Y, Z) + 2n\alpha^2 \eta(Y) \eta(Z) \quad \text{for all } Y, Z \in \mathcal{T}M,
\]

consequently,

\[
(2.4.4) \quad S(\phi Y, Z) = S(Y, \phi Z)
\]

\[
(2.4.5) \quad S(Y, \xi) = -2n\alpha^2 \eta(Y)
\]

\[
(2.4.6) \quad S(\xi, \xi) = -2n\alpha^2
\]

\[
(2.4.7) \quad \sum_{i=1}^{2n} S(e_i, e_i) = \sum_{i=1}^{2n} S(\phi e_i, \phi e_i) = r + 2n\alpha^2 \quad \text{where } r \text{ is Sealer curvature.}
\]

\[
(2.4.8) \quad QY = -2n \in \alpha^2 Y \quad \text{for any vector fields } X, Y, Z \text{ on } M \text{ and } Q \text{ is the Ricci operator given by } S(X, Y) = g(QX, Y).
\]
Proof: Let \( \{e_i\} \), \( i=1, 2, 3\ldots 2n+1 \) be the orthonormal basis at each point of the tangent space \( M \). Then putting \( X=W=e_i \) in (2.2.24), we have

\[
R(\phi e_i, \phi Y, \phi Z, \phi e_i) = R(e_i, Y, Z, e_i)
\]

\[
+\alpha^2 \eta(Z)\{\eta(Y)g(e_i, e_i) - \eta(e_i)g(Y, e_i)\}
\]

\[
+\alpha^2 \eta(e_i)\{\eta(Y)g(e_i, Z) - \eta(e_i)g(Y, Z)\}
\]

using (2.4.2) and after simplification gives (2.4.3). Replacing \( Z \) by \( \phi Z \) in (2.4.3), we get (2.4.4). Putting \( Z=\xi \) in (2.4.3), we get (2.4.5). Replacing \( Y=\xi \) in (2.4.5), we get (2.4.6).

From (2.4.6) we get (2.4.7). Also from (2.4.3), we get (2.4.8).

A semi- Riemannian manifold \( M \) is said to be Ricci- recurrent [11] if its Ricci tensor \( S \) satisfies the condition

\[
(\nabla_X S)(Y, Z) = A(X)S(Y, Z)
\]

where \( X, Y, Z \) on \( M \), and \( A \) is a 1-form. If \( A=0 \) in the above equation, then the manifold becomes Ricci- symmetric. We say that \( M \) is proper Ricci-recurrent, if \( A\neq0 \).

Theorem 2.4.2: An \((\epsilon)\)-Para \( \alpha \)-Sasakian manifold cannot be proper Ricci recurrent.

Proof: Let \( M \) be an \((2n+1)\)-dimensional \((\epsilon)\)-Para \( \alpha \)-Sasakian manifold. If possible, Let \( M \) is proper Ricci-recurrent. Then

\[
(\nabla_X S)(Y, \xi) = A(X)S(Y, \xi) = -2n\alpha^2 A(X)\eta(Y)
\]

but we have
(2.4.11) \[
(\nabla_X S)(Y, \xi) = -2n\alpha^2 (\nabla_X \eta) Y - \epsilon S(Y, \phi X)
\]

using (2.4.11) in (2.4.10), we get

(2.4.12) \[
\epsilon S(\phi X, Y) + 2n\alpha^2 \Phi(X, Y) = 2n\alpha A(X) \eta(Y)
\]

putting \(Y=\xi\) in the above equation, we get \(A(X)=0\), a contradiction.

**Remark:** The Theorem 2.4.2 generalization of the Theorem 5.14 is due to M. M. Tripathi and colleagues.

**Definition 2.4.3:** A \((\epsilon)\)-Para \(\alpha\)-Sasakian manifold \(M\) is said to be Ricci-semisymmetric if its Ricci tensor \(S\) satisfies the condition

(2.4.13) \[
R(X, Y).S=0
\]

for any vector fields \(X, Y\) on \(M\) where \(R(X, Y)\) acts as a derivation on \(S\).

**Theorem 2.4.4:** For an \((2n+1)\)-dimensional \((\epsilon)\)-Para \(\alpha\)-Sasakian manifold \(M\), the following three statements are equivalent:

(i) \(M\) is an Einstein manifold.

(ii) \(M\) is Ricci-symmetric.

(iii) \(M\) is Ricci-semi-symmetric.

**Proof:** Obviously, the statement (i) implies each of the statements (ii) and (iii).

Let (ii) be true.

Then putting \(A=0\) in (2.4.12) we get
Replacing $X$ by $\phi X$ in the above equation, we get

(2.4.15) \[ S = -2n \in \alpha^2 g \] which show that the statement (i) is true.

Let (iii) be true. In particular,

(2.4.16) \[ (R(\xi, X), S)(Y, \xi) = 0 \] implies that

(2.4.17) \[ S(R(\xi, X) Y, \xi) + S(Y, R(\xi, X) \xi) = 0 \]

which in view (2.2.15) and (2.4.5) we have

(2.4.18) \[ S(X, Y) = -2n \in \alpha^2 g(X, Y) \]

(2.4.19) \[ r = -2n(2n+1) \in \alpha^2 \] where $r$ is scalar curvature of $M$.

Which show that the statement (i) is true. This completes the proof.

**Remark:** If $\alpha=1$ the result of M. M. Tripathi and colleagues [Theorem 5.15] will include as special case of Theorem 2.4.4.

**Theorem 2.4.5:** A Ricci-semi-symmetric ($\in$)-Para $\alpha$-Sasakian manifold the scalar curvature $r$ of $M$ is constant and is given by (2.4.19).

**2.5. Generalized Recurrent ($\in$)-Para $\alpha$-Sasakian Manifolds**

**Definition 2.5.1:** A ($\in$)-Para $\alpha$-Sasakian manifold $M$ is said to be a generalized recurrent manifold if the curvature tensor $R$ of $M$ satisfies
(2.5.1) \( (\nabla_X R)(Y, Z) W = A(X) R(Y, Z) W + B(X) \{ g(Z, W) Y - g(Y, W) Z \} \)

where \( A \) and \( B \) are associated 1-forms and \( X, Y, Z, W \) are any vector field on \( M \).

**Lemma 2.5.2:** For a generalized recurrent \((\varepsilon)-\text{Para} \alpha\)-Sasakian manifold \( M \), we have

\begin{equation}
(\nabla_X R)(\xi, Z) \xi = 0
\end{equation}

for any vector fields \( X, Z \) on \( M \).

**Proof:** We know that

\[
(\nabla_X R)(\xi, Z) \xi = \nabla_X (R(\xi, Z) \xi) - R(\nabla_X \xi, Z) \xi - R(\xi, \nabla_X Z) \xi - R(\xi, Z) \nabla_X \xi
\]

\[
= \nabla_X \{ -\varepsilon \alpha^2 g(Z, \xi) \xi + \alpha^2 \eta(\xi) Z \} - R(\varepsilon \alpha X, Z) \xi
\]

\[
-\{ -\varepsilon \alpha^2 g(\nabla_X Z, \xi) \xi + \alpha^2 \eta(\xi) \nabla_X Z \} - \varepsilon \{ -\alpha^2 g(\xi, \phi X) \xi \}
\]

Now using (2.2.2) and (2.2.12) in the above equation after simplification, we get (2.5.2).

**Example 2.5.3:** We consider the 3-dimensional manifold \( M^3 = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\} \)

where \( (x, y, z) \) are the standard coordinates in \( \mathbb{R}^3 \). Let \( \{e_1, e_2, e_3\} \) be a linearly independent global frame on \( M \) given by

\[
e_1 = e^x \frac{\partial}{\partial y}, \quad e_2 = e^x \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right), \quad e_3 = \alpha \frac{\partial}{\partial z}
\]

are linearly independent at each point of \( M \) where \( \alpha \) is non zero constant. Let \( g \) be the Para Sasakian metric defined by

\[
g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0
\]
Let $e_3 = \xi$ and the $(\varepsilon)$-Para $\alpha$-Sasakian metric $g$ is thus given by

$$g = 2e^{-2z}(dx)^2 + e^{-2z}(dy)^2 + \frac{\varepsilon}{\alpha^2}(dz)^2 - 2e^{-2z}dx \wedge dy$$

where $\varepsilon = \pm 1$. If $\varepsilon = -1$ then $\varepsilon$-Lorentzian metric $g$ becomes a Riemannian positive definite metric on $M$ so that in this case the characteristic vector field $\xi$ becomes a space like and If $\varepsilon = 1$, then it becomes a light like.

Let $\eta$ be the 1-form defined by $\eta(X) = \varepsilon g(X, \xi)$

for any vector field $X$ on $M^3$. Let $\phi$ be the tensor field of type $(1, 1)$ defined by

$$\phi(e_1) = -e_1, \phi(e_2) = -e_2, \phi(e_3) = 0$$

Using the linearity properties of $g$ and $\phi$, one can deduce,

$$\phi^2 X = X - \eta(X)\xi, \eta(\xi) = 1, g(\xi, \xi) = \varepsilon$$

$$g(\phi X, \phi Y) = g(X, Y) - \varepsilon \eta(X)\eta(Y)$$

also $\eta(e_1) = 0, \eta(e_2) = 0, \eta(e_3) = 1$
for any vector field $X, Y$ on $M$. Let $\nabla$ be the Levi-civita connection with respect to $g$. then we have  
\[ [e_1, e_2] = 0, [e_1, e_3] = -\varepsilon \alpha e_1, [e_2, e_3] = -\varepsilon \alpha e_2 \]

Using Koszule’s formula for Levi-civita connection $\nabla$ with respect to $g$, that is

\[ 2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \]

\[ -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]) \]

One can easily calculate  
\[ \nabla_{e_1} e_3 = -\varepsilon \alpha e_1, \nabla_{e_2} e_3 = 0, \nabla_{e_2} e_1 = -\varepsilon \alpha e_2 \]

\[ \nabla_{e_2} e_2 = \varepsilon \alpha e_3, \nabla_{e_1} e_2 = 0, \nabla_{e_2} e_1 = 0 \]

\[ \nabla_{e_1} e_1 = \varepsilon \alpha e_3, \nabla_{e_3} e_2 = 0, \nabla_{e_3} e_1 = 0 \]

With these information, the structure $(\phi, \xi, \eta, g, \varepsilon)$ satisfies (2.2.10) and (2.2.11). Hence

$M^3 (\phi, \xi, \eta, g, \varepsilon)$ defines a $(\varepsilon)$-Para $\alpha$-Sasakian manifold.

Hence the structure $(\phi, \xi, \eta, g, \varepsilon)$ is a $(\varepsilon)$-Para $\alpha$-Sasakian manifold. Now using the above results, we obtain

\[ R(e_1, e_2)e_3 = 0, R(e_2, e_3)e_3 = -\varepsilon^2 \alpha^2 e_2, R(e_1, e_3)e_3 = -\varepsilon^2 \alpha^2 e_1 \]

\[ R(e_1, e_2)e_2 = -\varepsilon^2 \alpha^2 e_1, R(e_2, e_3)e_2 = \varepsilon^2 \alpha^2 e_3, R(e_1, e_2)e_1 = \varepsilon^2 \alpha^2 e_2 \]

\[ R(e_3, e_1)e_1 = -\varepsilon^2 \alpha^2 e_3, R(e_2, e_1)e_1 = -\varepsilon^2 \alpha^2 e_2, R(e_3, e_2)e_2 = -\varepsilon^2 \alpha^2 e_3 \]

From which it follows that
\[(\nabla_X R)(\xi, Z)\xi = 0\].

Thus lemma 2.5.2 is verified.

**Example 2.5.4:** We consider the 3-dimensional manifold \(M^3 = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}\) where \((x, y, z)\) are the standard coordinates in \(\mathbb{R}^3\). Let \(\{e_1, e_2, e_3\}\) be a linearly independent global frame on \(M\) given by

\[
e_1 = e^z \frac{\partial}{\partial y}, e_2 = e^z \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right), e_3 = e^z \frac{\partial}{\partial z}
\]

are linearly independent at each point of \(M\) where \(\alpha\) is constant. Let \(g\) be the Para Sasakian metric defined by

\[
g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0
\]

\[
g(e_1, e_1) = g(e_2, e_2) = 1, g(e_3, e_3) = \varepsilon
\]

Let \(e_3 = \xi\) and the \(\varepsilon\)-Para \(\alpha\)-Sasakian metric \(g\) is thus given by

\[
g = e^{-2z} \left\{ 2(dx)^2 + (dy)^2 + \varepsilon (dz)^2 - 2dx \wedge dy \right\}
\]

\[
g = \begin{pmatrix}
2e^{-2z} & -e^{-2z} & 0 \\
-e^{-2z} & e^{-2z} & 0 \\
0 & 0 & \varepsilon e^{-2z}
\end{pmatrix}
\]

where \(\varepsilon = \pm 1\), If \(\varepsilon = -1\) then \(\varepsilon\)-Lorentzian metric \(g\) becomes a Riemannian positive definite metric on \(M\) so that in this case the characteristic vector field \(\xi\) becomes a space like and If \(\varepsilon = 1\), then it becomes a light like.
Let $\eta$ be the 1-form defined by $\eta(X) = \varepsilon g(X, \xi)$ for any vector field $X$ on $M^3$. Let $\phi$ be the tensor field of type $(1, 1)$ defined by $\phi(e_1) = -e_1, \phi(e_2) = -e_2, \phi(e_3) = 0$

Using the linearity properties of $g$ and $\phi$, one can deduce,

$\phi^2 X = X - \eta(X)\xi, \eta(\xi) = 1, g(\xi, \xi) = \varepsilon$

$g(\phi X, \phi Y) = g(X, Y) - \varepsilon \eta(X)\eta(Y)$

also $\eta(e_1) = 0, \eta(e_2) = 0, \eta(e_3) = 1$

for any vector field $X, Y$ on $M$. Let $\nabla$ be the Levi-civita connection with respect to $g$. Then we have $[e_1, e_2] = 0, [e_1, e_3] = -\varepsilon e^2 e_1, [e_2, e_3] = -\varepsilon e^2 e_2$

Using Koszule’s formula for Levi-civita connection $\nabla$ with respect to $g$, that is

$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y)$

$-g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y])$

One can easily calculate $\nabla_{e_1} e_3 = -\varepsilon e^2 e_1, \nabla_{e_3} e_3 = 0, \nabla_{e_2} e_3 = -\varepsilon e^2 e_2$

$\nabla_{e_2} e_2 = \varepsilon e^2 e_3, \nabla_{e_1} e_2 = 0, \nabla_{e_2} e_1 = 0$

$\nabla_{e_1} e_1 = \varepsilon e^2 e_3, \nabla_{e_3} e_2 = 0, \nabla_{e_3} e_1 = 0$
With these information the structure \((\phi, \xi, \eta, g, \epsilon)\) defines a 3-dimensional \((\epsilon)\)-Para \(\alpha\)-Sasakian manifold \(M\) with \(\alpha = e^\zeta\) and satisfies (2.2.10) and (2.2.11). This completes the example.

We assume that \(M\) is a generalised recurrent \((\epsilon)\)-Para \(\alpha\)-Sasakian manifold. then (2.5.1) of definition 5.1 holds. Now put \(Y=W=\xi\) in (2.5.1), we find

\[
(2.5.3) \quad (\nabla_X R)(\xi, Z)\xi = A(X)R(\xi, Z)\xi - B(X)\{g(Z, \xi)\xi - g(\xi, \xi)Z\}
\]

By virtue of (2.5.2) of Lemma 5.2 and above equation (2.5.3), we find

\[
\alpha^2 A(X) + \epsilon B(X) = 0
\]

for any vector field \(X\) on \(M\). Hence we state

**Theorem 2.5.3:** A generalised recurrent \((\epsilon)\)-Para \(\alpha\)-Sasakian manifold \(M\) satisfies

\[
\alpha^2 A + \epsilon B = 0 \quad \text{where } \epsilon = \pm 1.
\]

2.6. A \((\epsilon)\)-Para \(\alpha\)-Sasakian Manifolds With \(C=0\)

The Weyl’s conformal curvature tensor \(C\) of type \((1, 3)\) on \(M\) is defined by

\[
(2.6.1) \quad C(X, Y)Z = R(X, Y)Z + \frac{1}{2n+1}[S(X, Z)Y - S(Y, Z)X + g(X, Z)QY - g(Y, Z)QX]
\]

\[
+ (2n+1)\alpha^2 \left\{ \eta(Y)X - \eta(X)Y \right\} + \frac{\epsilon}{2n} \left\{ \eta(Y)X - \eta(X)Y \right\}
\]

where \(S(X, Y) = g(QX, Y)\) and \(X, Y, Z\) are any vector fields on \(M\).
Theorem 2.6.1: A conformally flat \( (\varepsilon) \)-Para \( \alpha \)-Sasakian manifold \( M \) \((n>1)\) is \( \eta \)-Einstein manifold.

Proof: Suppose \( M \) is conformally flat. Then, \( C=0 \) on \( M \) so that (2.6.1) takes the form

\[
R(X,Y)Z = -\frac{1}{2n+1}[S(X,Z)Y - S(Y,Z)X + g(X,Z)QY - g(Y,Z)QX] \\
-\frac{1}{2n(2n+1)}[g(X,Z)Y - g(Y,Z)X]
\]

Set \( Z=\xi \) in (2.6.2) and then using (2.2.13) and (2.4.5) we get

\[
\varepsilon \left[ \eta(X)QY - \eta(Y)QX \right] = -2n\alpha^2 \left\{ \eta(Y)X - \eta(X)Y \right\} \\
+ (2n+1)\alpha^2 \left\{ \eta(Y)X - \eta(X)Y \right\} + \frac{\varepsilon}{2n} \left\{ \eta(Y)X - \eta(X)Y \right\}
\]

which after simplification gives

\[
\eta(X)QY - \eta(Y)QX = \left\{ \varepsilon \alpha^2 + \frac{1}{2n} \right\} \left[ \eta(Y)X - \eta(X)Y \right]
\]

Putting \( Y=\xi \) in (2.6.3) and using (2.4.6) we have

\[
QX = -\left\{ \varepsilon \alpha^2 + \frac{1}{2n} \right\} X - \left\{ (2n-1)\alpha^2 - \frac{1}{2n} \right\} \eta(X)\xi
\]

from which, we have

\[
S(X,Y) = -\left\{ \varepsilon \alpha^2 + \frac{1}{2n} \right\} g(X,Y) - \left\{ (2n-1)\alpha^2 - \frac{1}{2n} \right\} \eta(X)\eta(Y)
\]
which prove the theorem.

Contracting (2.6.4), we have the following expression for the scalar curvature $r$ of $M$.

\[
(2.6.5) \quad r = \frac{-2n(4n-1)\epsilon \alpha^2 - 2n + 1}{2n}
\]

provided $\epsilon \neq -1$.

If $\epsilon = 1$ then from (2.6.5) we have

\[
r = \frac{-2n(4n-1)\alpha^2 - 2n + 1}{2n}
\]

**Theorem 2.6.2:** In a conformally flat $(\epsilon)$-Para $\alpha$-Sasakian manifold $M$ $(n>1)$, the scalar curvature $r$ of $M$ is constant and given by (2.6.5).

### 2.7. Projective Curvature Tensor

**Definition 2.7.1:** A $(2n+1)$-dimensional $(n>1)$, $(\epsilon)$-Para $\alpha$-Sasakian manifold is said to be Quasi projectively flat, $\xi$-projectively flat and $\phi$-projectively flat if it satisfies

\[
g(P(X, Y, Z), \phi W)=0, \quad P(X, Y, \xi)=0, \quad g(P(\phi X, \phi Y)\phi Z, \phi W)=0
\]

where $P(X, Y, Z)$ is the projective curvature on $(\epsilon)$-Para $\alpha$-Sasakian manifold defined by

\[
(2.7.1) \quad P(X, Y, Z) = R(X, Y, Z) - \frac{1}{2n}[g(Y, Z)QX - g(X, Z)QY]
\]

for any vector field $X, Y, Z$ on $M$, $Q$ is Ricci operator on $M$, such that

\[
g(QX,Y)=S(X, Y)
\]

where $S$ is Ricci curvature tensor on $M$. 

52
**Theorem 2.7.2:** A $(\varepsilon)$-Para $\alpha$-Sasakian manifold is $\xi$-projectively flat.

**Proof:** Putting $Z=\xi$ in (2.7.1) we get

\[(2.7.2) \quad P(X, Y, \xi) = R(X, Y, \xi) - \frac{1}{2n} [g(Y, \xi)QX - g(X, \xi)QY] \]

using (2.2.13) and (2.4.8) in (2.7.2) we get $P(X, Y, \xi) = 0$.

**Theorem 2.7.3:** A $(\varepsilon)$-Para $\alpha$-Sasakian manifold is $\xi$-projectively flat if and only if it is Einstein Sasakian.

**Proof:** Using (2.2.2) in (2.7.1) we get

\[(2.7.3) \quad g(P(X, Y)\xi, W) = R(X, Y, \xi, W) - \frac{\varepsilon}{2n} [\eta(Y)S(X, W) - \eta(X)S(Y, W)] \]

for all $X, Y, W$ on $M$. For a local orthonormal basis $\{e_1, \ldots, e_{2n}, \xi\}$ of vector fields in $M$, from (2.7.3) we get

\[(2.7.4) \quad \sum_{i=1}^{2n} g(P(e_i, Y)\xi, e_i) = \sum_{i=1}^{2n} R(e_i, Y, \xi, e_i) - \frac{\varepsilon}{2n} \sum_{i=1}^{2n} \eta(Y)S(e_i, e_i) \]

for $Y$ on $M$. Using (2.4.2) from lemma 4.1 and (2.4.7) from lemma 4.2 in (2.7.4) we get

\[(2.7.5) \quad \sum_{i=1}^{2n} g(P(e_i, Y)\xi, e_i) = S(Y, \xi) - \frac{\varepsilon (2n\alpha^2 + r)}{2n} \eta(Y) \]

If $M$ is projectively flat then from (2.7.5) we get

\[(2.7.6) \quad S(Y, \xi) = \frac{\varepsilon (2n\alpha^2 + r)}{2n} \eta(Y) \]
Putting \( Y = \xi \) in (2.7.6) and using (2.4.6), (2.2.1) we get

\[
(2.7.7) \quad r = -2n\alpha^2 \left( 1 + \frac{2n}{\epsilon} \right).
\]

In view of (2.7.7), equation (2.7.6) becomes (2.4.5). Since \( M \) is \( \xi \)-projectively flat, putting \( Y = \xi \) in (2.7.3) and using (2.2.13), (2.4.5), (2.2.1) we obtain (2.4.18), which shows that \( M \) is Einstein. If \( M \) is \( \xi \)-projectively flat and using (2.4.18) in (2.7.3), we obtain (2.2.13), which shows that \( M \) is \( (\infty) \)-Para \( \alpha \)-Sasakian.

Conversely, if \( M \) is Einstein Sasakian, then in view of (2.2.13) and (2.4.18) from (2.7.1) we get \( M \) is \( \xi \)-projectively flat.

**Theorem 2.7.4:** A \( (\infty) \)-Para \( \alpha \)-Sasakian manifold \( M \), we have \( P(X, Y, Z, \xi) = 0 \)

where \( P(X, Y, Z, W) = g(P(X, Y, Z), W) \).

**Proof:** Using (2.2.2), (2.2.17) and (2.4.5), we obtain

\[
(2.7.8) \quad P(X, Y, Z, \xi) = \epsilon \eta(R(X, Y, Z)) - \left[ -\alpha^2 g(Y, Z)\eta(X) + \alpha^2 g(X, Z)\eta(Y) \right]
\]

using (2.2.17) in (2.7.8) we get \( P(X, Y, Z, \xi) = 0 \).

This proves the Theorem.

We suppose \( P(X, Y) Z = 0 \) then from (2.7.1) we get

\[
(2.7.9) \quad R(X, Y)Z = \frac{k}{2n} \left[ g(Y, Z)X - g(X, Z)Y \right]
\]

From (2.7.9) we get
Putting $X=W=\xi$ in (2.7.10) and using (2.2.1), (2.2.2) and (2.2.17) we get

\[
\left(\alpha^2 + \frac{k}{2n}\right)[g(Y, Z) - \epsilon \eta(Y)\eta(Z)] = 0
\]

This shows that either $k = -2n \in \alpha^2$ or $g(Y, Z) = \epsilon \eta(Y)\eta(Z)$. But if $g(Y, Z) = \epsilon \eta(Y)\eta(Z)$ then we get $g(\phi Y, \phi Z) = 0$ which is not possible. Then $k = -2n \in \alpha^2$. Now putting $k = -2n \in \alpha^2$ in (2.7.9) we get

\[
R(X, Y)Z \in \alpha^2[g(Y, Z)X - g(X, Z)Y]
\]

Therefore, the manifold is of constant scalar curvature $-\epsilon \alpha^2$. Hence we can state

**Theorem 2.7.5:** A projectively flat $(\in)$-Para $\alpha$-Sasakian manifold is locally isometric to a sphere, $S^{(2n+1)}(c)$ where $c = -\epsilon \alpha^2$ with $\epsilon = -1$.

**Theorem 2.7.6:** A projectively flat $(\in)$-Para $\alpha$-Sasakian manifold is locally hyperbolic sphere, $\epsilon = 1$ where $c = -\epsilon \alpha^2$ with $\epsilon = 1$.

**Theorem 2.7.7:** A $(\in)$-Para $\alpha$-Sasakian manifold $M$ is quasi-projectively flat.

**Proof:** From (2.7.1) we get

\[
g(P(X, Y, Z), \phi W) = g(R(X, Y, Z), \phi W)
\]
\[-\frac{1}{2n}[g(Y, Z)g(QX, \phi W) - g(X, Z)g(QY, \phi W)]\]

using (2.2.5), (2.3.4) and (2.4.8) in (2.7.11) we get \( g(P(X, Y, Z), \phi W) = 0 \).

**Theorem 2.7.8:** A \((\varepsilon)-\)Para \(\alpha\)-Sasakian manifold \(M\) is quasi projectively flat if and only if

\[(2.7.12) \quad R(X, Y, Z, W) g(Y, Z) g(X, W) g(X, Z) g(Y, W) = -\varepsilon \alpha^2 [g(Y, Z) g(X, \phi W) - g(X, Z) g(Y, \phi W)]\]

for all \(X, Y, Z, W \in TM\).

**Proof:** If \(M\) is quasi projectively flat, using (2.4.18) in

\[g(P(X, Y)Z, W) g(R(X, Y)Z, W) \phi = \phi - \phi - \phi\]

we obtain (2.7.12). The converse is straightforward.

**Theorem 2.7.9:** A \((\varepsilon)-\)Para \(\alpha\)-Sasakian manifold \(M\) of dimension \((2n+1), n>0\), with \(\alpha\) nonzero constant if manifold is quasi projectively flat then \(M\) is an \(\eta\)-Einstein manifold.

**Proof:** (2.7.1) may be written as

\[(2.7.13) \quad g(P(X, Y)Z, \phi W) = g(R(X, Y)Z, \phi W) - \frac{1}{2n}[g(Y, Z)S(X, \phi W) - g(X, Z)S(Y, \phi W)]\]

Suppose manifold quasi projectively flat then one can write (2.7.13) as

\[(2.7.14) \quad g(R(X, Y)Z, \phi W) = \frac{1}{2n}[g(Y, Z)S(X, \phi W) - g(X, Z)S(Y, \phi W)]\]
Setting $Y=Z=\xi$ in (2.7.14) and using (2.2.2), (2.4.18), and (2.4.5) we get

(2.7.15) \[ S(X, \phi W) = -\alpha^2 g(X, \phi W) \]

Taking $X=\phi X$ in (2.7.15) and using (2.4.3), we obtain

\[ S(X, W) = -\alpha^2 g(X, W) + \alpha^2 (\epsilon - 2n) \eta(X) \eta(Y) \]

which prove the theorem.

**Theorem 2.7.10:** A $(\epsilon)$-Para $\alpha$-Sasakian manifold $M$ is $\phi$-projectively flat if and only if $M$ satisfies

(2.7.16) \[ R(\phi X, \phi Y, \phi Z, \phi W) = \alpha^2 \left[ g(\phi Y, \phi Z) g(\phi X, \phi W) - g(\phi X, \phi Z) g(\phi Y, \phi W) \right] \]

for all $X, Y, Z, W \in TM$.

**Proof:** Let $M$ be a $(2n+1)$-dimensional $(\epsilon)$-Para $\alpha$-Sasakian manifold. From (2.7.1) we have

(2.7.17) \[ g(P(\phi X, \phi Y) \phi Z, \phi W) = R(\phi X, \phi Y, \phi Z, \phi W) \]

\[ - \frac{1}{2n} \left[ g(\phi Y, \phi Z) S(\phi X, \phi W) - g(\phi X, \phi Z) S(\phi Y, \phi W) \right] \]

for all $X, Y, Z, W \in TM$. For an orthonormal basis of vector fields $\{e_1, \ldots, e_{2n}, \xi\}$ in $M$, from (2.7.17) it follows that

\[ \sum_{i=1}^{2n} g(P(\phi e_i, \phi Y) \phi Z, \phi e_i) = \sum_{i=1}^{2n} R(\phi e_i, \phi Y, \phi Z, \phi e_i) \]
\[-\frac{1}{2n}\sum_{i=1}^{2n}[g(\phi Y, \phi Z)S(\phi e_i, \phi e_i) - g(\phi e_i, \phi Z)S(\phi Y, \phi e_i)]\]

which in view of (2.4.2), (2.4.7) and using

\[
\sum_{i=1}^{2n}[g(\phi e_i, \phi Z)S(\phi Y, \phi e_i)] = S(\phi Y, \phi Z) \text{ in above we get}
\]

\[
(2.7.18) \quad \sum_{i=1}^{2n} g(P(\phi e_i, \phi Y) \phi Z, \phi e_i) = \left(1 + \frac{1}{2n}\right) S(\phi Y, \phi Z) - \left(\frac{r}{2n} + 2\alpha^2\right)g(\phi Y, \phi Z)
\]

for all \(Y, Z \in TM\). If \(M\) is \(\phi\)-projectively flat then from (2.7.18) we get

\[
(2.7.19) \quad S(\phi Y, \phi Z) = \left(\frac{r + 4n\alpha^2}{2n + 1}\right)g(\phi Y, \phi Z)
\]

using (2.4.7) and \(\sum_{i=1}^{2n} g(\phi e_i, \phi e_i) = 2n\) in above equation we get

\[
(2.7.20) \quad r = 2n\alpha^2(2n - 1)
\]

In view of (2.7.20), equation (2.7.19) becomes

\[
(2.7.21) \quad S(\phi Y, \phi Z) = 2n\alpha^2 g(\phi Y, \phi Z)
\]

Since \(M\) is \(\phi\)-projectively flat, in view (2.7.21), equation (2.7.17) yields (2.7.16). The converse is obvious.

**Theorem 2.7.11:** Let \(M\) be an \((2n+1)\)-dimensional, \(\phi\)-projectively flat \((\varepsilon)\)-Para \(\alpha\)-Sasakian manifold then \(M\) is an \(\eta\)-Einstein manifold.
Proof. Using (2.4.3) and (2.2.3) in equation (2.7.19) we get

\[ S(Y, Z) + 2n\alpha^2 \eta(Y)\eta(Z) = \left( \frac{r + 4n\alpha^2}{2n+1} \right) [g(Y, Z) - \epsilon \eta(Y)\eta(Z)] \]

Finally, we obtain

\[ S(Y, Z) = \left( \frac{r + 4n\alpha^2}{2n+1} \right) g(Y, Z) - \left[ \frac{\epsilon (r + 4n\alpha^2)}{2n+1} + 2n\alpha^2 \right] \eta(Y)\eta(Z) \]

Therefore M is an $\eta$-Einstein manifold. The proof is complete.

References


