APPENDIX I
(CHAPTER 2)

A1.1. TYPES OF TIDES

The coastal waters respond in different ways to tide producing forces and consequently, various types of tides are formed. The representation of the tidal variations as the sum of several harmonics, each of different period, amplitude and phase is facilitated by harmonic analysis.

A1.2. THE IMPORTANT PARTIAL TIDES

The part of the tide that is solely due to the tide producing force of the sun is termed the solar tide, and that due to the moon, the lunar tide. Lunisolar tides are those derived partly from the lunar forces and partly from the solar forces. Some of the important relevant aspects of partial tides are given below.

A1.3. SPECIES DETAILS

The tidal periods fall into three principal "tidal species" - long period, diurnal and semi-diurnal, represented by 0, 1 and 2, respectively. In shallow waters, third-diurnal, fourth-diurnal and higher order species are also generated in the harmonic analysis, which are represented by 3, 4, etc., respectively. Each tidal species contains "groups" of harmonics which can be separated by analysis of at least a month of
observations. In turn, each group contains "constituents" which can be separated by the analysis of at least a year of observations.

A1.3.1. LONG PERIOD ASTRONOMICAL TIDES

The largest term in the long period species is usually $S_a$, followed by $S_{sa}$. They reflect annual and semiannual variations in the mean sea level caused by seasonal meteorological effects. The two constituents account for the non-uniform changes in the sun's declination and distance (Hicks, 1984). The lunar monthly ($M_m$) and fortnightly ($M_f$) tides are the lunar equivalent to $S_a$ and $S_{sa}$, but there are no radiational or meteorological effects at these periods. The $M_m$ constituent tide expresses the effect of irregularities in the moon's rate of change of distance and speed in orbit and the $M_f$ constituent expresses the effect of departure from a sinusoidal declinational motion. $M_{sf}$ (Lunisolar synodic fortnightly constituent) has a frequency, which is the difference between $M_2$ and $S_2$. The absence of significant energy at $M_{sf}$ is an indication that nonlinear contributions from semi-diurnal tides are unimportant.

A1.3.2. DIURNAL CONSTITUENTS

Diurnal tides are produced mainly by the $K_1$, $O_1$ and $P_1$ constituents and have one high water and one low water each lunar day. The "Lunisolar diurnal constituent" represented by $K_1$ along with "Lunar diurnal constituent" represented by $O_1$ expresses the
effect of the moon's declination. They account for diurnal inequality and at extremes, diurnal tides. The $K_1$ constituent along with the "Solar diurnal constituent" represented by $P_1$ expresses the effect of the sun's declination.

A1.3.3. SEMIDIURNAL CONSTITUENTS

The dominant tidal pattern in most of the oceans is such that it takes 12 hrs. 25 min. for each tidal cycle and since this cycle occupies roughly half a day, it is called semi-diurnal tide (Pugh, 1987). The semi-diurnal tides are produced mainly by the $M_2$, $S_2$ and $N_2$ constituents. These have two nearly equal high waters and low waters in a lunar day. Since tides are always getting higher or lower at any location, due to the spring-neap tide sequence, successive high tides and successive low tides can never be exactly the same at that location. Semi-diurnal tides have a range that increases and decreases cyclically over a 14-day period. The maximum ranges (called spring tides) occur a few days after the full and new moon, whereas the minimum ranges (called neap tides) occur shortly after the times of the first and third quarters. At the times of spring tides the lunar and solar forces combine together, but at neap tides the lunar and solar forces are out of phase and tend to cancel the effect of each other.

The partial tide representing the rotation of the earth with respect to the moon, is called the "Principal lunar semi-diurnal constituent" and is represented by $M_2$. The partial tide
representing the rotation of the earth with respect to the sun, is called the "Principal solar semi-diurnal constituent" and is represented by $S_2'$. The "Larger lunar elliptic semi-diurnal constituent" represented by $N_2$ along with the "Smaller lunar elliptic semi-diurnal constituent" represented by $L_2$ modulates the amplitude and frequency of $M_2$ for the effect of variation in the moon's orbital speed due to its elliptical orbit. The "Lunisolar semi-diurnal constituent" represented by $K_2$ modulates the amplitude and frequency of $M_2$ and $S_2$ for the declinational effect of the moon and sun, respectively.

A1.4. TIDAL HEIGHTS

For the surface elevations, the objective of harmonic analysis is to minimise the residuals, $e_i$, in the equation (A1.1) (Foreman et al. 1994; Emery and Thomson, 1998).

$$e_i = y_i - A_0 + \sum_{J=1}^{M} A_j \cos(\sigma_j t_i - \phi_j) \quad \ldots \text{(A1.1)}$$

where $A_j$, $\sigma_j$ and $\phi_j$ are the respective amplitude, frequency and phase of the constituent $j$, and $y_i$ is the observation at time $t_i$. Each equation can be made linear in the new unknowns $C_j$ and $S_j$ by rewriting

$$A_j \cos(\sigma_j t_i - \phi_j) = C_j \cos(\sigma_j t_i) + S_j \sin(\sigma_j t_i) \quad \ldots \text{(A1.2a)}$$

where

$$A_j = (C_j^2 + S_j^2)^{1/2} \quad \text{and} \quad \phi_j = \arctan(S_j/C_j) \quad \ldots \text{(A1.2b)}$$
Assuming the number of observations, $N$, is greater than the number of unknowns, $2M+1$, the system of equations is solved by minimising the residuals by the least square technique. With short records, it is not advisable to include all possible constituents in a harmonic analysis because close frequencies in combination with the presence of non-tidal noise or tide gauge errors can produce a large change in the solutions.

As the potential tidal theory predicts the existence of hundreds of tidal frequencies and as many are so close that several years of data are required to separate neighbours by one cycle, it is not practicable to include all constituents in every analysis (Foreman et al, 1994). In most of the studies concerning harmonic analysis of tides, it is convenient to do annual analysis with 8760 or 8784 hourly values. In tidal analyses, some workers have used even 29 days' data while some others have used 18.5 years of data. It has, however, been established universally, that the best data set for routine predictions is the hourly data for an year, which is relatively easy to procure.

Most harmonic analysis programs require the time series to have a uniform sampling interval. Foreman and Henry (1993) developed a program that can analyse either irregularly sampled data, or observations that were made at only the high or low tide. Franco and Harari (1988) developed programs based on Fourier analysis. These programs, unlike conventional harmonic
analysis techniques, require uniformly sampled series with no data gaps. A disadvantage of this procedure is that Fourier frequencies do not generally correspond to the tidal frequencies, post Fourier processing is required to reassign energy to the tidal constituents. However, this approach is computationally faster because of Fast Fourier Transform techniques. Foreman and Neufeld (1991) developed programs that could extract 500 constituents from a tidal time series that is at least 18.6 years long in duration covering a nodal cycle.
A2.1. HARMONIC ANALYSIS OF THE SEASONAL CYCLE

At any tide gauge station, the seasonal cycle of the mean sea level is usually described as the sum of several harmonics as annual, semi-annual and ter-annual etc., components (Maul et al., 1990; Enfield & Harris, 1995; Emery & Thomson, 1998). The seasonal cycle of other oceanographic and meteorological parameters can also be studied using the same technique. Eventhough parameterising the monthly values for each station as the sum of annual and semi-annual components appears to be adequate, the ter-annual component has also been considered in the present study. The phase $\phi_1$ is represented as the number of months from the beginning of the year to the time at which the annual cycle is maximum. Similarly $\phi_2$ (and $\phi_2 + 6$ months) represents the months when the semi-annual (6 month) cycle peaks, and $\phi_3$ (and $\phi_3 + 4$ months, $\phi_3 + 8$ months) represents the months when the ter-annual (4 months) cycle peaks. For each month 'j' ($j=1$ to 12), an average of the monthly means i.e. the climatological mean from the $N$ years of available data is computed.

$$\text{AVG}(j) = \frac{1}{N} \cdot \left[ \sum_{i=1}^{N} \text{MSL}(j,i) \right] \quad \ldots \text{(A2.1)}$$

where "MSL$(j,i)$" is the monthly mean sea level for month $j$ and year $i$. A least square fit was made in order to determine
the amplitudes $A_1$, $A_2$ & $A_3$ and the phases $\phi_1$, $\phi_2$ & $\phi_3$ of the annual, semi-annual and ter-annual cycles.

$$AVG(j) = A_1 \cdot \cos \left[ \frac{2\pi}{12} (t-\phi_1) \right] + A_2 \cdot \cos \left[ \frac{2\pi}{6} (t-\phi_2) \right] +$$

$$A_3 \cdot \cos \left[ \frac{2\pi}{4} (t-\phi_3) \right]$$

...(A2.2)

for $j = 1$ to $12$

and $t = j - 0.5$ to account for avg($j$) being the average of the $j$th month.

The RMS error ($\sigma$) is obtained following Wyrtki and Leslie (1980)

$$\sigma^2 = \frac{1}{12} \sum \left[ AVG(j) - CALC(j) \right]^2$$

...(A2.3)

where 'CALC($j$)' is the value for the month $j$ determined from the parameters of the Fourier expansion. CALC($j$) has been determined using the annual plus semi-annual as well as for annual plus semi-annual plus ter-annual harmonic parameters.

A2.2. ATMOSPHERIC PRESSURE CORRECTIONS

The pressure correction factor to be applied to the observed sea level (to account for the inverted barometer effect causing the sea level to depress by about 1 cm for an increase of 1 mb in atmospheric pressure and vice-versa) has not been considered for the tropical region in many of the earlier studies. This
correction factor becomes important nearer to the limits of the tropics. Two types of corrections are required to be applied to account for isostasy. The first correction is due to the mean pressure change over the oceans, caused by a shift in the air mass towards Siberia in winter (Pattullo et al., 1955; Lisitzin, 1974; Hendricks et al., 1996). This global mean pressure change varies from 1012 mb in December to 1014 mb in July. To compute the contribution of the atmospheric pressure variations to the changes in sea level, data on the global and the local monthly mean atmospheric pressure at sea level are needed. The global monthly mean atmospheric pressure at sea level has been generated by Pattullo et al. (1955), who have also defined a procedure for correcting the sea level for atmospheric pressure changes. The procedure is as follows:

The mean monthly atmospheric pressure in the general vicinity of the tide gauge station \(P_g\), and the mean pressure over all the oceans for the same month \(P_o\) are made use of for this correction. \(C' = P_g - P_o\) then represents the amount by which the water surface in this area is depressed relative to the mean global sea level. The recorded sea level has to be raised by this amount to correct for the effect of the atmospheric pressure. It is more convenient to use the correction factor \(C = C' - \bar{C}'\), whose mean annual value is zero. This procedure for isostatic adjustment for atmospheric pressure changes would not be valid for quick changes in atmospheric pressure, because there would be no time for the water to move.
The second correction is due to local atmospheric pressure. The corrected series is generated by adding the atmospheric pressure (in mb) to the sea level (in cm) (Brown et al., 1985; Varadarajulu and Bangarupapa, 1984; Ramp et al., 1997).

A2.3. ALONGSHORE COMPONENT OF CURRENT AND SEA LEVEL

The theoretical background for examining this relationship has been discussed in detail by Shetye and Almeida (1985). The following description has been adopted from the same.

Pattullo et al. (1955) examined the effect of (1) atmospheric pressure and (2) "steric" fluctuations on the recorded cycle. The contribution to the sea level change due to atmospheric pressure change (inverted barometer effect) depends on the difference between the local pressure and the mean global sea surface pressure. The annual cycle of this contribution could be important for stations located at higher latitudes.

The most important finding of Pattullo et al. (1955) is that the variations in the recorded monthly mean sea level at a location agree very well with the "steric" departures \( Z_\alpha \) in the vicinity of the location

\[
Z_\alpha = \frac{1}{g} \int_{P_a}^{P_0} \Delta \alpha \, dP
\]  

....(A2.4)
and \[ \Delta \alpha = \frac{1}{\rho(T,S,P)} - \frac{1}{\rho(T,S)} \] ....(A2.5)

\( \rho \) is the density, \( T \) and \( S \) are the temperature and salinity, \( \bar{T} \) and \( \bar{S} \) being their respective annual means, \( g \) is the acceleration due to gravity, \( P_{a} \) is the atmospheric pressure, and \( P_{o} \) is the pressure at a depth where all seasonal effects are assumed to be small.

Pattullo et al. (1955) did not examine the implications of the above result to the circulation in the vicinity of the tide gauge. This aspect was examined by Shetye and Almeida (1985) as the mass field (which determines \( Z_{\alpha} \), the sea surface topography, and the geostrophic velocity fields are interrelated.

A coastal current, the motion of which is restricted up to a distance \( R \) from the coast (\( R \) being of the order of a Rossby radius of deformation - approximately 100 kms), flows southward along a north-south coastline (Fig. A2.1). Assuming that the pressure gradient normal to the coast is in geostrophic equilibrium with the velocity field, we get

\[ f\nu^{S}_{1} = g \frac{\partial^{2} s}{\partial x^{2}} \] ... (A2.6)

where \( f \) and \( g \) are the Coriolis parameter and the acceleration due to gravity, \( \nu^{S}_{1} \) is the alongshore component of
Fig. A2.1. An idealized coastal current. The coastline stretches along the north-south direction. The current is southward, and the motion extends up to a distance $R$ from the coast. $Z_s(x,y)$ is the sea surface. $X$, $Y$ and $Z$ are the eastward, northward and upward axes, respectively. The sea surface tilts down by $Z_c$ at the coast. The thermocline tilts upward towards the coast. $P_0$ is the pressure at the level of no motion.

(from Shetye and Almeida, 1985)
the surface geostrophic velocity, and $Z_s(x,y)$ is the topography of the ocean surface. Variations in $Z_s$ near a coast can be determined from sea level data. We can write $Z_s$ in terms of variations in dynamic height normal to the coast,

$$ f v_s = \frac{\partial}{\partial x} \left[ \int_{p_0}^{p_a} \delta \, dp \right] \quad \text{...(A2.7)} $$

where

$$ \delta = \frac{1}{\rho(T,S,P)} - \frac{1}{\rho(0,35,P)} \quad \text{...(A2.8)} $$

is the specific volume anomaly. $P_0$ is the pressure at the depth of no motion. The term in square brackets is proportional to $Z_\alpha$. The observation made by Pattullo et al. (1955) that variations in $Z_\alpha$ closely match sea level changes, when viewed in isolation, bears no particular implication to the dynamics of the currents in the vicinity of the tide gauge. A static ocean, heated or cooled uniformly at the surface, will show variations in $Z_\alpha$ which will match the sea level variations. If we impose the restriction that the changes in mass field are mainly due to advection, the Pattullo et al. (1955) result becomes a powerful tool to monitor coastal geostrophic currents.

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Approximating equation (A2.6), we obtain

\[ f v_I^S = \frac{g}{R} \left[ z_c - z_o \right] \] ....(A2.9)

where \( z_c \) is the sea level at the coast (point A in Fig. A2.1), \( z_o \) is the sea level at the point B, at a distance \( R \) which is the offshore boundary of the coastal current. The variations in \( z_\alpha \) would be small in comparison to those in \( z_c \) because point B is located in a regime which is quiescent in comparison to that at A. Under these conditions, variations in \( z_\alpha, z_c \) and \( v_I^S \) will match.

Along the west coast, the alongshore component of current is taken as positive if the flow is northward and along the east coast, the alongshore component is taken as positive if the flow is southward. This choice is made to ensure that the sign of sea level change and that of change in \( v_I^S \) would be the same, if geostrophic balance as given in equation A2.9 holds.
A3.1. FILTERING

To remove the tides from the records and to focus attention on other periodicities, numerical tapering is required. "Tapering" denotes operations in the time domain and "filtering" denotes equivalent operations in the frequency domain. Traditionally tides were suppressed by using a Cosine-Lanczos taper on the hourly tide gauge readings (Mooers and Smith, 1968).

Breaker (1986) used a Godin low passed digital filter to remove diurnal and semi-diurnal tidal components. The low pass filter, which is most simple, is a 24-hour running mean which is convolved over the hourly time series. This boxcar filter should be avoided as it fails to remove 3.5% of the $M_2$ amplitude, 8.2% of $O_1$ and 5.4% of $N_2$ (Godin, 1972), as well as energy from all constituents except $S_2$ and its harmonics. A 25 hour running mean filter will remove more of the $M_2$ signal, but pass significant portions of $K_1$, $S_2$ and other constituents.

Godin (1972) suggested that the energy leakage problem may be avoided by running average filters in succession. His $A_{24}A_{24}A_{25}$ filter is applied by initially convolving the time series with a 24 hour running mean. A second 24 hour running mean is then applied to the output of the first filter and a final 25 hour running mean is applied to the output of the
second. Although negligible tidal energy remains after these passes, considerable energy at low frequencies is also removed (Thompson, 1983; Foreman et al., 1994). The amplitude of the original signal is reduced to one-half at a period of about 4 days, and the half power period is close to 3 days. At a period of 8 days, about 10% of the amplitude is removed. However, the moving average filter is easy to program and is useful where signals at periods less than 10 days are not needed (Breaker, 1986). Walters and Heston (1982) found that the Godin filter, in addition to removing tidal variations, also attenuates variations in the 2 to 4 day range.

A3.2. DETRENDING

For detrending, the first exercise is to find the equation of the straight line that best fits the points (Spiegel, 1981). The equation could be used to find corresponding 'y' values for points on the x-axis (should be between \( x_1 \) and \( x_n \)). The least squares method can be used to find the equation of the straight line that best fits the experimental points. We will fit the points \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\) to the straight line \( y = mx + c \)

where \( m \) (the slope) and \( c \) (the y intercept) are given by the following formulae:

\[
m = \frac{N \times \text{SUM}_{xy} - \text{SUM}_x \times \text{SUM}_y}{N \times \text{SUM}_{x^2} - (\text{SUM}_x)^2}
\]

\[\ldots\text{(A3.1)}\]
\[
\begin{align*}
\text{c} &= \frac{\sum_{i=1}^{n} x_i^2 * \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i * \sum_{i=1}^{n} x_i y_i}{N* \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2} \\
&= \text{.......(A3.2)}
\end{align*}
\]

\(N\) is the total number of points
\(\sum_{x,y}\) is sum of \(x*y\) product for each point: i.e.,
\[x_1*y_1 + x_2*y_2 + \ldots + x_n*y_n\]
\(\sum_{x}\) is the sum of the \(x\)'s, i.e., \(x_1 + x_2 + \ldots + x_n\)
\(\sum_{y}\) is the sum of the \(y\)'s, i.e., \(y_1 + y_2 + \ldots + y_n\)
\(\sum_{x^2}\) is the sum of all \(x\)'s squared, i.e.,
\[x_1^2 + x_2^2 + \ldots + x_n^2\]

A3.3. DATA WINDOWS

Fourier series apply to infinite-duration periodic data sets. If we examine only a finite size record of data (this period is called the data window), the Fourier analysis implicitly assumes that the data is periodic and thus repeats itself both before and after our limited period of measurement.

Even after detrending, the sharp edges of the data window cause what is known as leakage, where spectral estimates from any one frequency are contaminated with some spectral amplitude leaking in from neighbouring frequencies. To reduce leakage, a modified data window with smoother edges is recommended. Although a variety of smoothers can be used, a common one
utilizes sine or cosine squared terms near the beginning and ending 10% of the period of record, and is known as a bell taper:

\[
W(k) = \begin{cases} 
\sin^2\left(\frac{5\pi k}{N}\right) & \text{for } 0.0 \leq k \leq 0.1N \\
1 & \text{elsewhere} \\
\sin^2\left(\frac{5\pi k}{N}\right) & \text{for } 0.9N \leq k \leq N
\end{cases}
\] ....(A3.3)

When the window weight, \(W(k)\), is multiplied by the time series, \(A(k)\), the result yields a modified time series with fluctuations that decrease in amplitude at the beginning and at the end of the series. The Fourier transform can then be performed on this modified time series.

The process of detrending, despiking (removing erroneous data points), filtering and bell tapering is known as conditioning the data. Conditioning should be used with caution, because anytime data is modified, errors or biases can be introduced. It is best to do as little conditioning as is necessary.

A3.4. DISCRETE FOURIER TRANSFORM

Any continuous function can be described by an infinite Fourier series, namely the sum of an infinite number of sine and cosine terms. In the case of discrete time series with a finite number of points, we are required to have only a finite number of sine and cosine terms to fit our points exactly.
Using Euler's notation, i.e., \( \exp(ix) = \cos(x) + i\cdot\sin(x) \), (where \( i \) is the square root of \(-1\)) as a shorthand notation for the sines and cosines, we can write the discrete Fourier series representation of \( A(k) \) as:

Inverse Transform

\[
A(k) = \sum_{n=0}^{N-1} F_A(n) e^{i2\pi nk/N} \quad \ldots \ldots \quad (A3.4)
\]

where \( n \) is the frequency, and \( F_A(n) \) is the discrete Fourier transform. We see that a time series with \( N \) data points (indexed from \( k=0 \) to \( n-1 \)) needs not more than \( N \) different frequencies to describe it.

There are a number of ways to describe frequency:

- \( n \) = number of cycles (per time period \( P \))
- \( \bar{n} \) = cycles per second = \( n/P \)
- \( f \) = radians per second = \( 2\pi n/P = 2\pi n/(N\Delta t) \)

A frequency of zero (\( n=0 \)) denotes a mean value. The fundamental frequency, where \( n=1 \), means that exactly one wave fills the whole time period, \( P \). Higher frequencies correspond to the harmonics of the fundamental frequency. For example, \( n=5 \) means that exactly 5 waves fill the period \( P \).
$F_A(n)$ is a complex number, where the real part represents the amplitude of the cosine waves and the imaginary part is the sine wave amplitude. It is a function of frequency because the waves of different frequencies must be multiplied by different amplitudes to reconstruct the original time series. If the original time series, $A(k)$, is known, then these coefficients can be found from:

**Forward Transform**

$$F_A(n) = \sum_{k=0}^{N-1} \left( \frac{A(k)}{N} \right) e^{-i2\pi nk/N} \quad \ldots \text{(A3.5)}$$

This is the same as

$$F_A(n) = \frac{1}{N} \sum_{k=0}^{N-1} A(k) \cos(2\pi nk/N) - i/N \sum_{k=0}^{N-1} A(k) \sin(2\pi nk/N) \quad \ldots \text{(A3.6)}$$

The two equations (A3.4) and (A3.5) are called the Fourier transform pairs. The second equation performs the forward transform, creating a representation of the signal in phase space (another name for the frequency or spectral domain). This process is also known as Fourier decomposition. The first equation performs the inverse transform, converting from frequencies back into physical space.

If we have a total of $N$ data points, then the highest frequency that can be resolved in the fourier transform is $n_f = N/2$, which is called the Nyquist Frequency.
A3.5. DISCRETE ENERGY SPECTRUM

The square of the norm of the complex Fourier transform for any frequency \( n \) is:

\[
|F_A(n)|^2 = |F_{\text{real}}(n)|^2 + |F_{\text{imag}}(n)|^2 \quad \ldots (A3.7)
\]

When \( |F_A(n)|^2 \) is summed over frequencies \( n = 1 \) to \( N-1 \), the result yields the total biased variance of the original time series:

\[
\sigma_A^2 = \frac{1}{N} \sum_{k=0}^{N-1} (A_k - \bar{A})^2 = \sum_{n=1}^{N-1} |F_A(n)|^2 \quad \ldots (A3.8)
\]

Thus, we can interpret \( |F_A(n)|^2 \) as the portion of the variance explained by waves of frequency \( n \). We also notice that the sum over frequencies does not include \( n=0 \), because \( |F_A(0)| \) is the mean value and does not contribute any information about the variation of the signal about the mean. We define \( G_A(n) = |F_A(n)|^2 \). The ratio \( G_A(n)/\sigma_A^2 \) represents the fraction of the variance explained by component \( n \), and is very much like the correlation coefficient squared, \( r^2 \).

The discrete spectral intensity (or energy), \( E_A(n) \), is defined as \( E_A(n) = 2|F_A(n)|^2 \), for \( n=1 \) to \( n_T \), when \( N \) is odd. When \( N \) is even, \( E_A(n) = 2|F_A(n)|^2 \) is used for frequencies from \( n = 1 \) to \( (n_T-1) \), along with \( E_A(n) = |F_A(n)|^2 \) at the Nyquist frequency. This presentation is called the discrete variance(or energy spectrum).
Theoretical concepts assume that there is a spectral energy density, $S_A(n)$ that can be integrated over $n$ to yield the total variance.

$$\sigma_A^2 = \int S_A(n) \, dn \quad \text{......(A3.9)}$$

The spectral energy density has units of $A$ squared per unit frequency.

We can approximate the spectral energy density by

$$S_A(n) = E(n)/(\Delta n) \quad \text{......(A3.10)}$$

Where $\Delta n$ is the difference between neighbouring frequencies, when $n$ is used to represent frequency, $\Delta n = 1$.

A3.6. CROSS-SPECTRA

Cross-spectrum analysis relates the spectra of two variables. We have defined $G_A = |F_A(n)|^2$ as the spectral energy for variable $A$ and frequency, $n$. We rewrite this definition as

$$G_A = F_A^* \cdot F_A^*, \text{ where } F_A^* \text{ is the complex conjugate of } F_A, \text{ and where the dependance on } n \text{ is still implied.}$$

To demonstrate this last definition, let $F_A = F_{Ar} + i F_{Ai}$, where subscripts $r$ and $i$ denote real and imaginary parts, respectively. Thus, the complex conjugate is simply $F_A^* =
The expression for the spectral energy can now be written as:

$$G_A = F_A^*F_A = (F_{Ar} - iF_{Ai})(F_{Ar} + iF_{Ai})$$

$$= F_{Ar}^2 + iF_{Ai}F_{Ar} - iF_{Ai}F_{Ar} - i^2F_{Ai}^2$$

$$= F_{Ar}^2 + F_{Ai}^2 = |F_A(n)|^2 \quad \ldots(A3.11)$$

Similarly, we define the spectral intensity, \(G_B = F_B^*F_B\), for different variable \(B\). We can now define the cross-spectrum between \(A\) and \(B\) by

$$G_{AB} = F_A^*F_B = F_{Ar}F_{Br} + iF_{Ai}F_{Bi} - iF_{Ai}F_{Br} - i^2F_{Ai}F_{Bi} \quad \ldots(A3.12)$$

Upon collecting the real and imaginary parts, the real part is defined as the cospectrum, \(C_{o}\), and the imaginary part is called the quadrature spectrum, \(Q\):

$$G_{AB} = C_{o} - iQ \quad \ldots(A3.13)$$

where

$$C_{o} = F_{Ar}F_{Br} + F_{Ai}F_{Bi} \quad \text{and} \quad Q = F_{Ai}F_{Br} - F_{Ar}F_{Bi} \quad \ldots(A3.14)$$

Although not explicitly written in the equations above, \(F_A\) and \(F_B\) are functions of \(n\), making both the cospectrum and quadrature spectrum functions of \(n\) too: \(C_{o}(n)\) and \(Q(n)\). The sum over frequency of all cospectral amplitudes, \(C_{o}\), equals the covariance between \(A\) and \(B\).
Two additional spectra can be constructed from the quadrature and co-spectra.

An amplitude spectrum, $Am$, defined as

$$Am = G_{AB}^* \cdot G_{AB} = Q^2 + Co^2$$  \(\text{...(A3.15)}\)

and

Phase spectrum, $\phi$, defined as

$$\tan \phi = Q/Co$$  \(\text{...(A3.16)}\)

This can be interpreted as the phase difference between the two series A and B that yields the greatest correlation for any frequency, $n$. In other words, the phase gives the time displacement of one record with respect to the other as functions of frequency. The phase difference can give useful information even when the spectrums of the separate time series are relatively featureless (Hamon, 1962).