2. AN EOQ MODEL WITH EXPONENTIALLY INCREASING DEMAND UNDER TWO LEVELS OF STORAGE.

2.1 INTRODUCTION

In this chapter we consider an EOQ model with two levels of storage (Hartley [50]). The demand is assumed to increase exponentially like the one discussed by Aggarwal and Veena Jain [5]. One can expect exponentially increasing demand in the case of electronic goods particularly computer hardware parts.
When there is a large demand like exponentially increasing demand, the prudent stockist would like to order extra stock. In the classical inventory models, it is assumed that the warehouse owned by the management is adequate to store the stock procured. Some times it may be useful for the management to order such quantities, beyond the storage capacity of the own warehouse (OW). The extra stock i.e. excess of own warehouse capacity can be stored in a rented warehouse (RW) whenever OW is insufficient to keep the extra stock. This type of situation arises in case of seasonal goods like cool drinks, coolers etc. The management prefers to have the rented warehouse when the acquisition costs are higher than the holding costs or when there is an attractive price discounts for bulk orders or when there is large demand for the product under consideration. Usually, the holding cost of the RW is greater than those in OW since the management of RW may provide better storage facilities like lesser rate of deterioration of items compared to that of deterioration of items in OW. In order to reduce the holding cost, the stock kept in RW will be cleared first. Hartley [50] has discussed a lot size model and derived EOQ for the deterministic situation. Sarma [92] called a system with two levels of storage as L_2-system. The general operation of L_2 system has been discussed in (1.14).

It would be interesting to workout the optimal operating policy for the L_2-system with exponentially increasing demand and examine some of the
implications of the model. At first we develop a model in case of infinite horizon and later the case of finite horizon model will be discussed.

2.2 BASIC FEATURES AND NOTATIONS

To develop an EOQ model for the L2-system with exponentially increasing demand, we adopt the following notations.

\[ f(t) = a e^{\alpha t}, \quad 0 < \alpha < 1, \quad a > 0, \] demand is a function of time.

- \( Q_1, Q_2 \) : Economic order quantity for the L1 and L2 systems respectively
- \( C_3 \) : Set-up cost per order
- \( \pi \) : Shortage cost per unit per unit time
- \( T_1, T_2 \) : Replenishment interval for the L1 and L2 systems respectively
- \( H, F \) : Unit cost of holding of units per unit time in OW and RW respectively
- \( W \) : Storage capacity of OW in units
- \( Z \) : Quantity stored in RW
- \( t_w \) : The time at which the RW gets emptied
- \( t_1, t_2 \) : The time points at which shortages start in the L1 and L2 systems respectively.
- \( R_1, R_2 \) : Recorder level (ROL) for the L1 and L2 systems respectively
We assume that

(i) The replenishment rate is infinite and replenishment size is constant.
(ii) Lead time is zero and all shortages are backlogged.
(iii) There is no extra fixed cost associated with the use of RW
(iv) The stock kept in OW is used only after exhausting the goods kept in RW since $F > H$

With the above notations and assumptions, we wish to derive the optimal cycle length for the infinite horizon and to examine the properties of the solution. This section also derives optimal number of replenishments and their intervals for the finite horizon case.

2.3 INFINITE HORIZON MODEL

In this model the horizon length is infinite. Thus, it is enough to obtain the optimal cycle length $T_2^*$ in order to order $Q_2$ units for every $T_2^*$ time units. Here the subscript '2' denotes L$_2$-system.

Let $Q(t)$ be the inventory level at time 't' ($0 < t < T$) and demand rate $f(t)$ assumed to be deterministic and is increasing exponentially with time 't'.

$B_1, B_2$ : Back-order level for the L$_1$ and L$_2$ systems respectively
Further let $f(t) = ae^{at}$, $0 < a < 1$, $a > 0$.

At $t = 0$, a quantity of $Q_2$ units are received of which 'W' units are kept in OW i.e. to its maximum capacity and the remaining units are kept in RW. The items are depleted according to the function $f(t)$ starting from the RW. $t_w$ is the time at which the RW is completely exhausted. Subsequently the OW units are consumed until $t_2$ ($> t_w$) and shortages start from this point. The inventory cycle ends with $T_2$, at the end of $T_2$ another order is placed and backlog is cleared first. Using the method adopted by Donaldson [37], the inventory carried out in RW during $(0, t_w)$ is $\int_0^{t_w} tf(t) \, dt$ and hence the holding cost is equal to $F \int_0^{t_w} tf(t) \, dt$.

The total inventory units carried out in both the warehouses is $\int_0^{t_w} tf(t) \, dt$, the holding cost in the OW becomes

$$H \left[ \int_0^{t_w} tf(t) \, dt - \int_0^{t_2} tf(t) \, dt \right]$$

Hence the total holding cost in a cycle is given by

$$(F - H) \int_0^{t_2} tf(t) \, dt + H \int_0^{t_w} tf(t) \, dt \quad \ldots \ldots (2.3.1)$$

The shortage cost during $(t_2, T_2)$ is given by
Since the set-up cost is \( C_3 \) and the length of the cycle is \( T_2 \), the total cost per unit per unit time is given by

\[
C_2(T_2, t_2, t_\omega) = \left[ C_3 + (F - H) \int_0^{t_\omega} f(t)\,dt + H \int_0^{t_\omega} f(t)\,dt + \pi \int_{t_2}^{T_2} f(t)\,dt \right] / T_2
\]

\[
(2.3.3)
\]

Using \( f(t) = ae^{at} \), the above cost function is reduced to

\[
C_2(T_2, t_2, t_\omega) = \frac{C_3}{T_2} + \left( \frac{F - H}{T_2} \right) a \left[ \frac{e^{at_\omega}}{a^2} - (at_\omega + 1) \right]
\]

\[
+ \frac{H}{T_2} a \left[ \frac{e^{at_\omega}}{a^2} - (at_\omega + 1) \right]
\]

\[
+ \frac{\pi}{T_2} a \left[ \frac{e^{at_\omega}}{a^2} - (T_2 - t_2) \frac{e^{at_\omega}}{a} - \frac{e^{at_\omega}}{a^2} \right]
\]

\[
(2.3.4)
\]

The optimal values of say \( T_2^* \) of \( T_2 \) and \( t_2^* \) of \( t_2 \) can be obtained by differentiating (2.3.4) partially with respect to \( T_2 \) and \( t_2 \) and equating these equations to zero respectively. The optimal value of \( T_2 \), is the solution of the following equation

\[
\frac{\partial C_2(T_2, t_2, t_\omega)}{\partial T_2} = 0,
\]

which gives

\[
\left( T_2 - \frac{1}{a} \right)e^{at_\omega} = \frac{1}{\pi} \left[ \frac{C_3}{a} + \left( F - H \right) \left( t_\omega - \frac{1}{a} \right)e^{at_\omega} + \frac{F}{a} + \left( H + \pi \right) \left( t_2 - \frac{1}{a} \right)e^{at_\omega} \right]
\]

\[
(2.3.5)
\]
The solution of above equation gives optimal value $T_2^*$ of $T_2$. The above equation could not yield to a closed form solution. By using same argument as in Sarma & Sastry [95], applying Taylor series expansion for $e^x$ and neglecting second and higher order terms, the above equation reduces to

$$T_2^* = \left[ A_1 + A_2 + A_3 \right]^{\frac{1}{2}} \ldots \ldots \text{(2.3.6)}$$

where

$$A_1 = \frac{C_3}{\pi a},$$

$$A_2 = \frac{(F-H)^2}{\pi t_w},$$

$$A_3 = \frac{(H+\pi)^2}{\pi t_2}.$$

Given trial values of $T_2$, $t_2$ and $t_w$, equation (2.3.6) can be iteratively solved. The convergence of the model is usually very fast.

Since the exact quantity consumed during $(t_w, t_2)$ is 'W' we have

$$\int_{t_w}^{t_2} a e^{at} \, dt = W \ldots \ldots \text{(2.3.7)}$$

From the above equation, the optimal value of $t_w$ is given by

$$t_w = \frac{1}{\alpha} \log \left\{ e^{a0_2} - \frac{\alpha W}{a} \right\} \ldots \ldots \text{(2.3.8)}$$

It can be noted from the above equation that, $t_w^0$ is a function of $t_2^0$ and also

$$\frac{dt_w}{dt_2} = e^{\alpha(t_2-t_w)} \ldots \ldots \text{(2.3.9)}$$
Differentiating the cost function (2.3.4) with respect to $t_2$, we get

$$t^0_2 = \frac{\alpha T^0_2 - (F - H)t^0_w}{(H + \pi)}$$

When $F=H$ and $t_w=0$, the above equation reduces to that of the result of Dave [33].

The optimal reorder level can be obtained by using $t^0_2$ as

$$R^0_2 = \frac{a}{\alpha} \left( e^{\alpha t^0_2} - 1 \right)$$

It is to be noted that $R^0_2 > w$ is a necessary condition for the feasibility of two levels of storage.

$$\therefore Q^0_2 = \frac{a}{\alpha} \left( e^{\alpha T^0_2} - 1 \right)$$

and

$$B^0_2 = Q^0_2 - R^0_2$$

2.3.1 PARTICULAR CASES:

The model proposed in the previous section has the following particular cases.

(i) By setting $F=H$ and $t^0_w = 0$ in (2.3.4) and (2.3.6) the effect of $Rw$ is nullified and we get the cost function for the $L_1$ - system as

$$C_i(T_1, t_1) = \frac{C_0}{T_1} + \frac{H}{T_1} \left[ \frac{e^{\alpha t_1}}{\alpha} (t_1 - 1) + \frac{1}{\alpha^2} \right] + \frac{\pi a}{T_1} \left[ \frac{e^{\alpha t_1}}{\alpha^2} - (T_1 - t_1) \frac{e^{\alpha t_1}}{\alpha} - \frac{e^{\alpha t_1}}{\alpha^2} \right]$$
The optimal replenishment interval is given by

\[
\left( T_i^0 - \frac{1}{\alpha} \right) e^{aT_i^0} = e^{aT_i^0} \left[ \frac{1}{\alpha} \left( \frac{C_i \alpha + H (H + \pi)}{\alpha} \left( e^\alpha (t_i - \frac{1}{\alpha}) \right) \right) + \frac{1}{\alpha^2} \right]^{\frac{1}{2}}
\]  

\[\text{(2.3.14)}\]

Taking first order approximation for \( e^x \) in the above equation, (2.3.15) reduces to

\[
T_i^0 = \left[ \frac{C_i \alpha + (H + \pi)}{\pi} T_i^0 \right]^{\frac{1}{2}}
\]

\[\text{(2.3.15)}\]

The first order approximation can be justified as the values of \( \alpha \) and \( T \) lies in between 0 and 1, and we have

\[
t_i^0 = \frac{\pi T_i^0}{(H + \pi)}
\]

\[\text{(2.3.16)}\]

The optimum ROL is

\[
R_1^0 = \frac{a}{\alpha} \left[ e^{\alpha t_i^0} - 1 \right]
\]

\[\text{(2.3.17)}\]

\[
Q_1^0 = \frac{a}{\alpha} \left[ e^{\alpha t_i^0} - 1 \right]
\]

\[\text{(2.3.18)}\]

and

\[
B_1^0 = Q_1^0 - R_1^0
\]

(ii) When \( \alpha = 0 \) i.e. the demand is uniform, we get from equation (2.3.16) for \( L_1 \) – system:
When \( \alpha = 0 \) and \( F > H \), the replenishment interval becomes

\[
T^0_2 = \left[ N_1 + N_2 + N_3 \right]^{\frac{1}{2}}
\]  

\[
\ldots \ldots (2.3.20)
\]

where

\[
N_1 = \frac{C_1}{\pi a},
\]

\[
N_2 = \frac{(F - H)(aT^0_2 - W)^2}{\pi a^2}
\]

\[
N_3 = \frac{\left( \pi aT^0_2 - (F - H)(aT^0_2 - W) \right)^2}{\pi (H + \pi)a^2}
\]

To handle this model, at first we start with \( L_1 \) system by setting \( F = H \) and \( t_w = 0 \) and determine \( t_1 \) and \( T_1 \). If the EOQ obtained with this model is less than \( W \), then the optimal policy is obtained. Otherwise \( L_2 \)-system is used. Once \( L_2 \) - system is used, the cost of \( L_2 \)-system is compared with the threshold cost of \( L_1 \) and \( L_2 \) systems i.e. the solution which requires ordering exactly \( W \) units.

Keeping in view of this discussion we now illustrate the solution method with the following sequence of steps.

**Step 1:**

For the given hypothetical parameters, obtain \( t^0_1 \) and \( T^0_1 \) for \( L_1 \) - system using (2.3.17) and (2.3.16) and compute \( R^0_1 \) using (2.3.18). Find the
corresponding costs, $C_1 (R_1^0)$ by making use of equation (2.3.14). If $R_1^0 < W$, then the optimal cost is $C_1 (R_1^0)$, otherwise go to step 2.

**Step 2:**

If $R_1^0 > W$, find the optimal cost with $R_1^0 = W$ in equation (2.3.14) and denote this cost as $C_1 (W)$. We call this as threshold cost and denote the system as $I^{(W)}$. Go to step 3.

**Step 3:**

Using equations (2.3.6), (2.3.8) and (2.3.10), obtain optimal values of $T_2^0$, $i_{w}^0$ and $i_{2}^0$ respectively. Determine $R_2^0$ and corresponding $C_2 (R_2^0)$. Compare this cost with threshold cost obtained in step 2 i.e. if $C_2 (R_2^0) < C_1 (W)$, then the $L_2$ - system is optimum, otherwise $L_1$ - system is optimum.

### 2.4 SENSITIVITY ANALYSIS OF THE MODEL

As an illustration to the above developed model, hypothetical values for the parameters are given below.

$$a=200, \alpha = 0.01, H=1, F=2, \Pi=3, W=50 \text{ and } C_3=100.$$  

All the parameters are expressed in consistent units. With these parameters the step-wise procedure gives the following results.
(i) \( L_1 \) - system

\[ t_1^0 = 0.6124, T_1^0 = 0.8165, R_1^0 = 123 (> W) \]
\[ Q_1^0 = 164, B_1^0 = 41 \]

and the optimal cost is Rs.184.

(ii) Threshold cost of \( L_1 \)-system:

\[ t_1^0 = 0.1084, T_1^0 = 0.4270, R_1^0 = 50 (= W), Q_1^0 = 86, B_1^0 = 36 \]

and the optimal cost is Rs.308.

(iii) \( L_2 \) - System

\[ t_w^0 = 0.1148, t_2^0 = 0.5135, T_2^0 = 0.7229, R_2^0 = 103, Q_2^0 = 145, B_2^0 = 42 \]

and the associated minimum cost is Rs.195.

From the above results we note that minimum cost is occurred for \( L_1 \)-system. However, \( L_1 \)-system is not feasible as \( R_1^0 (= 123) > W \). If the ROL is limited to \( W \), the associated cost is Rs.308. If we opt for \( L_2 \)-system, the ROL is 103 units and the optimal EOQ is 145 units. The associated minimum cost is Rs.195. Hence, we recommend \( L_2 \)-system by keeping 95 units in RW. The optimal backorder level found to be 42 units. In this case the use of \( L_2 \)-system is economical. This may not be always economical and hence the working of the model is seen for different values of \( F \) i.e. cost of holding in RW and are shown in table. 2.1.
TABLE 2.1: SENSITIVITY OF THE MODEL TO CHANGES IN F

<table>
<thead>
<tr>
<th>F</th>
<th>( t_w^0 )</th>
<th>( t'^0 )</th>
<th>( T^0 )</th>
<th>Optimal Quantity</th>
<th>Optimal Cost</th>
<th>System</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.11485</td>
<td>0.51347</td>
<td>0.72291</td>
<td>103</td>
<td>195</td>
<td>L_2</td>
</tr>
<tr>
<td>3</td>
<td>0.09138</td>
<td>0.45956</td>
<td>0.67367</td>
<td>92</td>
<td>203</td>
<td>L_2</td>
</tr>
<tr>
<td>4</td>
<td>0.07628</td>
<td>0.42489</td>
<td>0.64283</td>
<td>85</td>
<td>209</td>
<td>L_2</td>
</tr>
<tr>
<td>5</td>
<td>0.06565</td>
<td>0.40047</td>
<td>0.62156</td>
<td>80</td>
<td>213</td>
<td>L_2</td>
</tr>
<tr>
<td>6</td>
<td>0.05771</td>
<td>0.38223</td>
<td>0.60582</td>
<td>77</td>
<td>217</td>
<td>L_2</td>
</tr>
<tr>
<td>7</td>
<td>0.05153</td>
<td>0.36805</td>
<td>0.59381</td>
<td>74</td>
<td>220</td>
<td>L_2</td>
</tr>
<tr>
<td>8</td>
<td>0.04658</td>
<td>0.35667</td>
<td>0.58425</td>
<td>71</td>
<td>222</td>
<td>L_2</td>
</tr>
<tr>
<td>9</td>
<td>0.04252</td>
<td>0.34734</td>
<td>0.57649</td>
<td>70</td>
<td>224</td>
<td>L_2</td>
</tr>
<tr>
<td>10</td>
<td>0.03912</td>
<td>0.33953</td>
<td>0.57006</td>
<td>68</td>
<td>226</td>
<td>L_1^{w}</td>
</tr>
<tr>
<td>15</td>
<td>0.0280</td>
<td>0.31401</td>
<td>0.54936</td>
<td>63</td>
<td>232</td>
<td>L_1</td>
</tr>
<tr>
<td>20</td>
<td>-</td>
<td>0.6124</td>
<td>0.8165</td>
<td>123</td>
<td>184</td>
<td>L_1</td>
</tr>
</tbody>
</table>

> OBSERVATIONS

We observe from the above table that the cost increases as F increases and ROL decreases which is obvious. It is clear from the table that when F=0, the optimal ROL is exactly equal to OW capacity, which means any additional demand should be treated as backordered.
The other important parameters that can possibly influence the choice between \( L_1 \) and \( L_2 \) system are the \( \alpha \), shortage cost \( \pi \), and the storage capacity \( W \). Keeping \( F=2 \) and \( \pi =3 \) and other parameters are unchanged the sensitivity of the model with respect of \( \alpha \) has been demonstrated in table 2.2.

**TABLE 2.2: SENSITIVITY OF THE MODEL WITH RESPECT TO ‘\( \alpha \)’**

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( t^0_w )</th>
<th>( t^0_2 )</th>
<th>( T^0_2 )</th>
<th>( R^0_2 )</th>
<th>Optimal Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.11485</td>
<td>0.51347</td>
<td>0.72291</td>
<td>103</td>
<td>195</td>
</tr>
<tr>
<td>0.02</td>
<td>0.11517</td>
<td>0.51324</td>
<td>0.72271</td>
<td>103</td>
<td>203</td>
</tr>
<tr>
<td>0.03</td>
<td>0.11548</td>
<td>0.51301</td>
<td>0.72251</td>
<td>103</td>
<td>209</td>
</tr>
<tr>
<td>0.04</td>
<td>0.11580</td>
<td>0.51278</td>
<td>0.72230</td>
<td>104</td>
<td>213</td>
</tr>
<tr>
<td>0.05</td>
<td>0.11612</td>
<td>0.51255</td>
<td>0.72210</td>
<td>104</td>
<td>217</td>
</tr>
<tr>
<td>0.06</td>
<td>0.11644</td>
<td>0.51232</td>
<td>0.72191</td>
<td>104</td>
<td>220</td>
</tr>
<tr>
<td>0.07</td>
<td>0.11675</td>
<td>0.51209</td>
<td>0.72171</td>
<td>104</td>
<td>222</td>
</tr>
<tr>
<td>0.08</td>
<td>0.11707</td>
<td>0.51186</td>
<td>0.72151</td>
<td>105</td>
<td>224</td>
</tr>
<tr>
<td>0.09</td>
<td>0.11738</td>
<td>0.51164</td>
<td>0.72131</td>
<td>105</td>
<td>226</td>
</tr>
<tr>
<td>0.10</td>
<td>0.11769</td>
<td>0.51141</td>
<td>0.72111</td>
<td>105</td>
<td>232</td>
</tr>
</tbody>
</table>

**OBSERVATIONS**

From the above figures we note that when \( \alpha \) increases the model requires \( L_2 \)-system with marginal increase in ROL and in minimum cost.
This is quite expected result, since the large demand always leads to high ordering quantity which in turn requires RW.

The performance of the model with respect to changes in W has been depicted in the following table.

**TABLE 2.3: SENSITIVITY OF THE MODEL TO CHANGES IN 'W'**

<table>
<thead>
<tr>
<th>W</th>
<th>$t^0_1$</th>
<th>$t^0_2$</th>
<th>$T^0_2$</th>
<th>$R^0_2$</th>
<th>Optimal Cost</th>
<th>System</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>0.15662</td>
<td>0.48509</td>
<td>0.69812</td>
<td>97</td>
<td>200</td>
<td>$L_2$</td>
</tr>
<tr>
<td>50</td>
<td>0.11485</td>
<td>0.51347</td>
<td>0.72291</td>
<td>103</td>
<td>195</td>
<td>$L_2$</td>
</tr>
<tr>
<td>75</td>
<td>0.07436</td>
<td>0.54478</td>
<td>0.75115</td>
<td>109</td>
<td>191</td>
<td>$L_2$</td>
</tr>
<tr>
<td>100</td>
<td>0.03498</td>
<td>0.57889</td>
<td>0.78351</td>
<td>116</td>
<td>187</td>
<td>$L_2$</td>
</tr>
<tr>
<td>110</td>
<td>0.02441</td>
<td>0.58869</td>
<td>0.79306</td>
<td>121</td>
<td>188</td>
<td>$L_2$</td>
</tr>
<tr>
<td>120</td>
<td>0.00939</td>
<td>0.60309</td>
<td>0.80725</td>
<td>124</td>
<td>187</td>
<td>$L_2$</td>
</tr>
<tr>
<td>125</td>
<td>0.00194</td>
<td>0.61044</td>
<td>0.81457</td>
<td>126</td>
<td>187</td>
<td>$L_2$</td>
</tr>
<tr>
<td>130</td>
<td>-</td>
<td>0.28138</td>
<td>0.52176</td>
<td>56</td>
<td>240</td>
<td>$L_1^a$</td>
</tr>
<tr>
<td>-</td>
<td>0.61237</td>
<td>0.81649</td>
<td>123</td>
<td>184</td>
<td>$L_1$</td>
<td></td>
</tr>
</tbody>
</table>
OBSERVATIONS

From the above table we note that when $W = 25$ the model requires $L_2$-system with ROL 97 units and the corresponding cost is Rs. 200. It can be observed that as $W$ increases the use of RW decreases i.e. $t_w$ decreases and the corresponding optimal cost also decreases. This is an expected result since high value of $W$ will always leads to less usage of RW.

The performance of the model with respect to the changes in shortage cost is depicted in table 2.4.

Table 2.4: SENSITIVITY OF THE MODEL WITH RESPECT TO SHORTAGE COST ($\pi$)

<table>
<thead>
<tr>
<th>$\pi$</th>
<th>$t_w^0$</th>
<th>$t^0$</th>
<th>$T^0$</th>
<th>$R$</th>
<th>Optimal Cost</th>
<th>System</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.10307</td>
<td>0.48642</td>
<td>0.78116</td>
<td>97</td>
<td>182</td>
<td>$L_2$</td>
</tr>
<tr>
<td>3</td>
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<td>0.51347</td>
<td>0.72291</td>
<td>103</td>
<td>195</td>
<td></td>
</tr>
<tr>
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<td>0.52922</td>
<td>0.69195</td>
<td>106</td>
<td>203</td>
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<td>0.55227</td>
<td>0.64999</td>
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<td>0.55647</td>
<td>0.64273</td>
<td>111</td>
<td>216</td>
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<td>0.55982</td>
<td>0.63702</td>
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<td>219</td>
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</tr>
<tr>
<td>10</td>
<td>0.13621</td>
<td>0.56255</td>
<td>0.63243</td>
<td>113</td>
<td>221</td>
<td></td>
</tr>
</tbody>
</table>

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> OBSERVATIONS

The above table demonstrates that the largest value of shortage cost leads to increase of $t^0, t^c, \text{ROL}$ and the optimal cost. This is also an expected result since, when shortage cost is high the prudent stockist would opt for RW and he tries to maintain less duration of time for the shortages. It is also to be noted that the optimal cost is marginally increasing as the shortage cost increases.

The discussion made so far is mainly meant for determining the EOQ for an infinite horizon model when the demand increases exponentially. The usage of RW requires when there is a large demand so that the replenishment interval decreases. The effect of such intervals is more important in the finite horizon model compared to infinite – horizon system. Donaldson [37] and Silver [102] showed that the economic replenishment policy always requires in an unequal replenishment intervals in case of linear trend in demand.

In the subsequent section the researcher makes use of Silver-Meal Heuristic model to obtain the optimal replenishment intervals for two levels of storage when demand is increasing exponentially with finite horizon model.
2.5 FINITE HORIZON MODEL

Let 'U' be the length of the inventory cycle. The problem is to obtain the optimal number of replenishments i.e. m and their durations. Shortages are allowed in every cycle except in the last cycle since the policy would terminate at 'U'. The model determines the duration of (m - 1) cycles in which shortages are backlogged and forcing the duration of the mth cycle so as to terminate at 'U' which precludes shortages. Adopting the notations used in section 2.3 for the L2-system, define T_{2i} = duration of the ith cycle, t_{2i} is the time at which shortages start in the ith cycle and t_{w} is the time at which the 'RW' is emptied in the ith cycle. Moreover, using the terminating condition at 'U', T_{2m} = t_{2m} = U.

The problem is to determine the optimal values of m, t_{w}, t_{2i} and T_{2i} for all i = 1, 2, ..., m when demand is exponentially increasing. The pertinent cost function is minimised and is given by

\[ C_2(t_{w}, t_{2i}, T_{2i}) = mC_3 + \sum_{i=1}^{m} K_i(t_{w}, t_{2i}, T_{2i}) + k_m(t_{w}, U) \]  

(2.5.1)

where

\[ K_i(t_{w}, t_{2i}, T_{2i}) = \frac{C_i}{T_{2i}} \left( t_{wi} \frac{1}{\alpha} \frac{e^{aw}}{\alpha} + \frac{1}{\alpha^2} \right) + \frac{H}{T_{2i}} \left( t_{2i} \frac{1}{\alpha} \frac{e^{aw}}{\alpha} + \frac{1}{\alpha^2} \right) + \frac{1}{T_{2i}} \left( t_{2i} \frac{e^{aw}}{\alpha} - \frac{e^{aw}}{\alpha^2} \right) \]

(2.5.2)

and
\[
Km(t_{\infty}, U) = C_3 + (F - H)\alpha \left[ \left( t_{\infty} - \frac{1}{\alpha} \right) e^{\frac{\alpha t_{\infty}}{\alpha}} + \frac{1}{\alpha^2} \right] + Ha \left[ \left( t_{2i} - \frac{1}{\alpha} \right) e^{\frac{\alpha t_{2i}}{\alpha}} + \frac{1}{\alpha^2} \right]
\] .... (2.5.3)

Dave [33] has used the Silver-Meal Heuristic for determining optimal number of replenishments and established that the penalty of using Heuristic instead of analytical solution is very meagre. It is worth to note that the successive replenishment durations will decrease as one approach to the horizon.

We now develop a step-wise procedure for the Silver-Meal Heuristic in case of exponentially increasing demand with two-levels of storage.

2.5.1 STEP-WISE PROCEDURE FOR SILVER-MEAL HEURISTIC

(i) Obtain the value of \( T^0_{2l} \) or \( T^0_{1l} \) for the first cycle, depending on the aptness of the \( L_2 \) or \( L_1 \) – system. Obtain \( t^0_{2l} \) or \( t^0_{1l} \) and \( t^0_{w} \), using (2.3.10), (2.3.17) and (2.3.8).

(ii) Revise the demand at \( T^0_{2l} \) (or \( T^0_{1l} \)) and obtain second replenishment interval \( T^0_{22} \) (or \( T^0_{12} \)).

(iii) Repeat (i) and (ii) until \( \sum_{i=1}^{m} T^0_{2i} \geq U \left( \sum_{i=1}^{m} T^0_{1i} \geq U \right) \) for some value of \( m \).
(iv) Redefine $T_{2m}^0 = U - \sum_{i=1}^{m} T_{2i}^0$ as the length of the $m$th cycle. This gives the series of optimal cycle lengths with $L_2$ (or $L_1$) system.

(v) Determine the optimal cost using (2.5.1) and (2.5.2). For getting optimal cost for $L_1$-system, substitute $F=H$ in equations (2.5.1) and (2.5.2) if necessary.

The above procedure differs from that of Dave [33] by way of determining additional parameters $t_s^0$ and $t_u^0$ for the $L_2$-system. The following numerical illustration demonstrates the working of the proposed model.

2.5.2 SENSITIVITY ANALYSIS OF THE MODEL

Consider the following parameter values in appropriate units.

$a=200, \alpha=0.01, F=2, H=1, \pi=3, W=50, C_3 = 100$ and $U=5$.

Using the step-wise method given (2.4.1), the following sequence of time points and the corresponding replenishment lengths are obtained and they are shown in the following table.
## Replenishment Cycle

<table>
<thead>
<tr>
<th>Cycle</th>
<th>Duration of the Cycle</th>
<th>Time at which order is placed</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.72291</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0.72098</td>
<td>0.72291</td>
</tr>
<tr>
<td>3</td>
<td>0.71906</td>
<td>1.44389</td>
</tr>
<tr>
<td>4</td>
<td>0.71528</td>
<td>2.16295</td>
</tr>
<tr>
<td>5</td>
<td>0.71341</td>
<td>2.87823</td>
</tr>
<tr>
<td>6</td>
<td>0.70971</td>
<td>3.59164</td>
</tr>
<tr>
<td>7</td>
<td>0.69864</td>
<td>4.30135</td>
</tr>
<tr>
<td>8</td>
<td>0.57349</td>
<td>4.87484</td>
</tr>
</tbody>
</table>

Starting with $a=200$, the duration of first cycle is found to be 0.72291. Now at this point the demand function becomes as $(200 e^{0.01(0.72291)}) = 201$. Using $a=201$ and keeping $\alpha$ fixed, the duration of the second cycle is computed. The same method is repeated and the cumulative duration is compared with the horizon $U$.

As per the Heuristic procedure, the duration of the $8^{th}$ cycle is found to be 0.57349 but the forcing the sum of the durations equal to $U (=5)$. The $8^{th}$ cycle duration is found to be only 0.12516. Hence $m^0=8$. For computing minimum cost and for checking feasibility conditions of the $L_2$-system the following points are worth to note.
In all the replenishment cycles the L2-system is found to be optimal with the hypothetical parametric values proposed in this illustration. As expected the successive cycle lengths are decreasing as one move close to the horizon. The results of other decision variables namely re-order levels and the optimal costs are computed and furnished in table 2.5.

**TABLE 2.5: RESULTS OF THE SILVER-MEAL HEURISTIC FOR THE L2-SYSTEM**

<table>
<thead>
<tr>
<th>Cycle</th>
<th>$t_w$</th>
<th>$t^0$</th>
<th>$T^0$</th>
<th>ROL</th>
<th>Average cost per cycle</th>
<th>System</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.11485</td>
<td>0.51347</td>
<td>0.72291</td>
<td>103</td>
<td>195</td>
<td>L2</td>
</tr>
<tr>
<td>2</td>
<td>0.11476</td>
<td>0.51205</td>
<td>0.72098</td>
<td>103</td>
<td>196</td>
<td>L2</td>
</tr>
<tr>
<td>3</td>
<td>0.11468</td>
<td>0.51063</td>
<td>0.71906</td>
<td>103</td>
<td>196</td>
<td>L2</td>
</tr>
<tr>
<td>4</td>
<td>0.11451</td>
<td>0.50783</td>
<td>0.71528</td>
<td>104</td>
<td>197</td>
<td>L2</td>
</tr>
<tr>
<td>5</td>
<td>0.11443</td>
<td>0.50645</td>
<td>0.71341</td>
<td>104</td>
<td>198</td>
<td>L2</td>
</tr>
<tr>
<td>6</td>
<td>0.11426</td>
<td>0.50372</td>
<td>0.70931</td>
<td>105</td>
<td>199</td>
<td>L2</td>
</tr>
<tr>
<td>7</td>
<td>0.11418</td>
<td>0.50237</td>
<td>0.69864</td>
<td>105</td>
<td>197</td>
<td>L2</td>
</tr>
<tr>
<td>8</td>
<td>0.09974</td>
<td>-</td>
<td>0.12516</td>
<td>37</td>
<td>74</td>
<td>L2</td>
</tr>
</tbody>
</table>
> OBSERVATIONS

The optimal total cost for all the 8 cycles is Rs.1452 and the cost per unit time during the horizon is \( \frac{1452}{5} = 290.4 \). This value is essential for the comparison of costs over different horizons.

DISCUSSION

In this chapter, two models are discussed for obtaining EOQ for deterministic inventory system in which the demand is suppose to increase exponentially with time for infinite and finite horizons. When there is a larger demand the use of RW arises. The model also determines the optimal length of replenishment cycle for the \( L_2 \) – system. To obtain this, the researcher has used the Silver-Meal Heuristic procedure for decision making in the \( L_2 \)-system. The novelty of the model compared to the earlier work lies in the determination of the duration for which the RW is used in each cycle. In the next chapter, the researcher develops an EOQ model for deteriorating items with exponentially increasing demand under two levels of storage.