This chapter is concerned with common fixed point theorems for weakly commuting and weakly compatible maps along with E.A./(CLR) properties in complex valued metric spaces. A theorem for weakly commuting maps is proved. Also, some common fixed point theorems for weakly compatible maps along with E.A. and (CLR\(_g\)) properties are proved.

It is divided into five sections. Section 5.1 is concerned with notions of complex valued metric spaces which are central to this paper. Section 5.2 is concerned with some common fixed point theorems for a pair of maps and a theorem for weakly commuting maps. In Section 5.3, some common fixed point results for weakly compatible maps are proved. Fourth section is concerned for weakly compatible maps along with E.A. property. In Section 5.5, some common fixed point theorems for weakly compatible maps along with (CLR\(_g\)) property are proved.

**Introduction**

Azam et al. [10] introduced and studied complex valued metric spaces, wherein some fixed point theorems for mappings satisfying a rational inequality were established. The idea of complex valued metric spaces can be exploited to define complex valued normed spaces and complex valued Hilbert spaces; additionally, it offers numerous research activities in mathematical analysis. Though complex valued metric spaces form a special class of cone metric space, yet this idea is intended to define rational expressions which are not meaningful in cone metric spaces and thus many results of analysis cannot be generalized to cone metric spaces. Indeed, the definition of a cone metric space banks on the underlying Banach space which is not a division Ring. However, in complex valued
metric spaces, we can study improvements of a host of results of analysis involving divisions.

A complex number \( z \in \mathbb{C} \) is an ordered pair of real numbers, whose first co-ordinate is called \( \text{Re}(z) \) and second coordinate is called \( \text{Im}(z) \). Thus a complex-valued metric \( d \) is a function from a set \( X \times X \) into \( \mathbb{C} \), where \( X \) is a nonempty set and \( \mathbb{C} \) is the set of complex numbers.

Let \( \mathbb{C} \) be the set of complex numbers and \( z_1, z_2 \in \mathbb{C} \). Define a partial order \( \preceq \) on \( \mathbb{C} \) as follows:

\( z_1 \preceq z_2 \) if and only if \( \text{Re}(z_1) \leq \text{Re}(z_2) \) and \( \text{Im}(z_1) \leq \text{Im}(z_2) \), that is \( z_1 \preceq z_2 \), if one of the following holds:

(C1) \( \text{Re}(z_1) = \text{Re}(z_2) \) and \( \text{Im}(z_1) = \text{Im}(z_2) \);

(C2) \( \text{Re}(z_1) < \text{Re}(z_2) \) and \( \text{Im}(z_1) = \text{Im}(z_2) \);

(C3) \( \text{Re}(z_1) = \text{Re}(z_2) \) and \( \text{Im}(z_1) < \text{Im}(z_2) \);

(C4) \( \text{Re}(z_1) < \text{Re}(z_2) \) and \( \text{Im}(z_1) < \text{Im}(z_2) \).

In particular, we will write \( z_1 \frac{\preceq}{\prec} z_2 \) if \( z_1 \neq z_2 \) and one of (C2), (C3), and (C4) is satisfied and we will write \( z_1 \frac{\preceq}{\nprec} z_2 \) if only (C4) is satisfied.

Remark 5.1.1. We note that the following statements hold:

(i) \( a, b \in \mathbb{R} \) and \( a \leq b \Rightarrow az \preceq bz \prod z \in \mathbb{C} \).

(ii) \( 0 \preceq z \preceq z \Rightarrow |z| < |z| \forall z \in \mathbb{C} \).

(iii) \( z_1 \preceq z_2 \) and \( z_2 \npreceq z_3 \Rightarrow z_1 \npreceq z_3 \).

Definition 5.1.2. Let \( X \) be a nonempty set. Suppose that the mapping \( d : X \times X \rightarrow \mathbb{C} \) satisfies the following conditions:

(i) \( 0 \preceq d(x, y) \), for all \( x, y \in X \) and \( d(x, y) = 0 \) if and only if \( x = y \);

(ii) \( d(x, y) = d(y, x) \) for all \( x, y \in X \);

(iii) \( d(x, y) \preceq d(x, z) + d(z, y) \), for all \( x, y, z \in X \).

Then \( d \) is called a complex valued metric on \( X \) and \( (X, d) \) is called a complex valued metric space.

Example 5.1.3. Let \( X = \mathbb{C} \). Define the mapping \( d : X \times X \rightarrow \mathbb{C} \) by

\[
d(z_1, z_2) = 2 \cdot |z_1 - z_2|, \quad \text{for all } z_1, z_2 \in X.
\]

Then \( (X, d) \) is a complex valued metric space.
Definition 5.1.4. Let \((X, d)\) be a complex valued metric space. A sequence \(\{x_n\}\) in \(X\) is said to be

(i) convergent to \(x\), if for every \(c \in \mathbb{C}\), with \(0 < c\), there is \(k \in \mathbb{N}\) such that for all \(n > k\), \(d(x_n, x) < c\).

(ii) Cauchy, if for every \(c \in \mathbb{C}\), with \(0 < c\), there is \(k \in \mathbb{N}\) such that for all \(n > k\), \(d(x_n, x_{n+m}) < c\), where \(m \in \mathbb{N}\).

(iii) complete, if every Cauchy sequence in \(X\) converges in \(X\).

Lemma 5.1.5. Let \((X, d)\) be a complex valued metric space and let \(\{x_n\}\) be a sequence in \(X\). Then \(\{x_n\}\) converges to \(x\) if and only if \(|d(x_n, x)| \to 0\) as \(n \to \infty\).

Lemma 5.1.6. Let \((X, d)\) be a complex valued metric space and let \(\{x_n\}\) be a sequence in \(X\). Then \(\{x_n\}\) is a Cauchy sequence if and only if \(|d(x_n, x_{n+m})| \to 0\) as \(n \to \infty\), where \(m \in \mathbb{N}\).

Sessa [98] introduced the notion of weak commutativity as follows:

Definition 5.1.7. Two self maps \(f\) and \(g\) of a metric space \((X, d)\) are said to be weakly commuting if

\[ d(fgx, gfx) \leq d(fx, gx), \text{ for all } x \text{ in } X. \]

In the setting of Complex valued metric space, we introduce notion of commuting mappings as follows:

Definition 5.1.8. Two self maps \(f\) and \(g\) of a complex valued metric space \((X, d)\) are said to be weakly commuting if

\[ d(fgx, gfx) \preceq d(fx, gx), \text{ for all } x \text{ in } X. \]

Example 5.1.9. Let \(X = [0, \infty)\) and define \(d : X \times X \to \mathbb{C}\) by \(d(x, y) = i \cdot x - y\), for all \(x, y \in X\).

Then \((X, d)\) is a complex valued metric space.

Define \(fx = x\) and \(gx = 2x\).

Then, clearly \(d(fgx, gfx) \preceq d(fx, gx)\), for all \(x \in X\).

Thus, \(f\) and \(g\) are weakly commuting.

**Weakly commuting property in complex valued metric spaces**

In this section, we prove some theorems for a rational inequality and a theorem for weakly commuting maps in complex valued metric spaces.
Theorem 5.2.1. Let f and g be self mappings of a complex valued metric space \((X, d)\) satisfying the following:

\[(5.1) \quad d(fx, gy) \leq k \left( \frac{d(x, f x) + d(x, g y) + d(g y, f x)}{1 + d(x, f x)} \right),\]

for all \(x, y \in X\) with \(x \neq y\), \(0 < k < 1\) and \(d(x, f x) + d(x, y) + d(x, g y) \neq 0\). Then \(f\) and \(g\) have a common fixed point.

Further, if \(d(x, f x) + d(x, y) + d(x, g y) = 0\), implies that, \(d(fx, gy) = 0\), then \(f\) and \(g\) have a unique common fixed point.

Proof. Let \(x_0 \in X\). Define a sequence \(\{x_n\}\) in \(X\) by \(x_{2n+1} = f x_{2n}\), \(x_{2n+2} = g x_{2n+1}\), \(n = 0, 1, 2, \ldots\).

Let \(d(x, f x) + d(x, y) + d(x, g y) \neq 0\). From (5.1), we have

\[d(x_{2n+1}, x_{2n+2}) = d(f x_{2n}, g x_{2n+1}) \leq k \left( \frac{d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1})} \right) = k d(x_{2n}, x_{2n+1}).\]

Similarly, we have

\[d(x_{2n}, x_{2n+1}) = d(f x_{2n}, g x_{2n+1}) \leq k \left( \frac{d(x_{2n-1}, x_{2n}) + d(x_{2n-1}, x_{2n+1}) + d(x_{2n}, x_{2n+1})}{1 + d(x_{2n-1}, x_{2n})} \right) = k d(x_{2n}, x_{2n+1}).\]

Consequently, it can be concluded that

\[d(x_n, x_{n+1}) \leq k d(x_{n-1}, x_n) \leq k^2 d(x_{n-2}, x_{n-1}) \leq \ldots \leq k^n d(x_0, x_1).\]

Now, for all \(m > n\),
\[ d(x_m, x_n) \leq d(x_m, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_m, x_{n+1}) \]
\[ \leq k^n d(x_0, x_1) + k^{n+1} d(x_0, x_1) + \ldots + k^{m-1} d(x_0, x_1) \]
\[ \leq \frac{\|1\|}{\|k\|} d(x, x). \]

Therefore, we have \(1 - \frac{\|1\|}{\|k\|} = 0\).

Hence, \(\mathbb{B}(\mathbb{B}, \mathbb{B}) \leq 1 - \frac{\|1\|}{\|k\|} \mathbb{B}(\mathbb{B}, \mathbb{B})\).

\[ \lim_{n \to \infty} \mathbb{B}(\mathbb{B}, \mathbb{B}) = 0. \]

Hence, \(\{x_n\}\) is a Cauchy sequence in \(X\). But \(X\) is complete metric space, so \(\{x_n\}\) is convergent to some point, say \(z\), in \(X\), i.e., \(x_n \to z\) as \(n \to \infty\).

Now, we shall prove that \(z = gz\).

Let, if possible, \(z \neq gz\).

Now, using the triangular inequality and (5.1), we have

\[ d(z, gz) \leq d(z, x_{2n+1}) + k \left[ \frac{\|z\| + \|2 \| \|x_{2n+1}\| + \|2 \| \|x_{2n+1}\| + \|1 \| \|x_{2n+1}\| + \|2 \| \|x_{2n+1}\|}{\|k\|} \right]. \]

Thus, we have

\[ d(z, gz) = d(z, x_{2n+1}) + k \left[ \frac{\|z\| + \|2 \| \|x_{2n+1}\| + \|2 \| \|x_{2n+1}\| + \|1 \| \|x_{2n+1}\| + \|2 \| \|x_{2n+1}\|}{\|k\|} \right]. \]

\[ \leq d(z, x_{2n+1}) + k \left[ \frac{\|z\| + \|2 \| \|x_{2n+1}\| + \|2 \| \|x_{2n+1}\| + \|1 \| \|x_{2n+1}\| + \|2 \| \|x_{2n+1}\|}{\|k\|} \right]. \]

\[ \leq d(z, x_{2n+1}) + k \left[ \frac{\|z\| + \|2 \| \|x_{2n+1}\| + \|2 \| \|x_{2n+1}\| + \|1 \| \|x_{2n+1}\| + \|2 \| \|x_{2n+1}\|}{\|k\|} \right]. \]

\[ \leq d(z, x_{2n+1}) + k \left[ \frac{\|z\| + \|2 \| \|x_{2n+1}\| + \|2 \| \|x_{2n+1}\| + \|1 \| \|x_{2n+1}\| + \|2 \| \|x_{2n+1}\|}{\|k\|} \right]. \]

Letting \(n \to \infty\), we have

\[ d(z, gz) = 0, \text{ a contradiction.} \]

Hence, we get \(z = gz\), i.e., \(z\) is the fixed point of \(g\).

Similarly, let us suppose that \(z \neq fz\).

Again, using the triangular inequality and (5.1), we have

\[ d(z, fz) \leq d(z, x_{2n+1}) + d(x_{2n+1}, fz) \]
\[ = d(z, x_{2n+1}) + d(fz, x_{2n+1}) \]
\[ = d(z, x_{2n+1}) + d(fz, gx_{2n+1}) \]
\[ \leq d(z, x_{2n+1}) + \left[ \frac{\|z\| + \|2 \| \|x_{2n+1}\| + \|2 \| \|x_{2n+1}\| + \|1 \| \|x_{2n+1}\| + \|2 \| \|x_{2n+1}\|}{\|k\|} \right]. \]

Thus, we have

\[ d(z, gfz) \leq 0, \text{ a contradiction.} \]

Hence, we get \(z = gz\), i.e., \(z\) is the fixed point of \(g\).
Letting $n \to \infty$, we have
\[ d(z, gz) \leq 0, \] a contradiction.

Hence, we get $z = fz$, i.e., $z$ is a fixed point of $f$.

Therefore, we find that $z$ is a common fixed point of $f$ and $g$.

For the uniqueness of $z$, let us suppose that $d(x, fx) + d(x, y) + d(x, gy) = 0$ implies $d(fx, gy) = 0$, and that $w$ is another fixed point of $g$ in $X$.

Then, we have
\[ d(z, fz) + d(z, w) + d(z, gw) = 0 \] implies $d(fz, gw) = 0$.

Therefore, we get
\[ d(z, w) = d(fz, gw) = 0, \] implies that, $z = w$.

Hence $f$ and $g$ have a unique common fixed point.

**Corollary 5.2.2.** Let $f$ be a self map of a complex valued metric space $(X, d)$ satisfying the following:
\[ (5.2) \quad d(fx, fy) \leq k \left[ \frac{d(Ax, Ax) + d(By, By) + d(At, At)}{d(Ax, Ax) + d(By, By) + d(At, At)} \right], \]
for all $x, y$ in $X$ with $x \neq y$, $0 < k < 1$ and $d(x, fx) + d(x, y) + d(x, fy) \neq 0$.

Then $f$ has a fixed point.

Further, if $d(x, fx) + d(x, y) + d(x, fy) = 0$, implies that, $d(fx, fy) = 0$, then $f$ has a unique common fixed point.

**Proof.** By putting $f = g$ in Theorem 5.2.1, we get the Corollary 5.2.2.

**Theorem 5.2.3.** Let $A, B, S$ and $T$ be self mappings of a complex valued metric space $(X, d)$ satisfying the following:

(5.3) $SX \subseteq BX, TX \subseteq AX$,
(5.4) the pairs $(A, S)$ and $(B, T)$ are weakly commuting,
(5.5) for all $x, y$ in $X$, either
\[ d(Sx, Ty) \geq \alpha \left[ \frac{d(Sx, Sx) + d(Ty, Ty) + d(Sx, Ty)}{d(Sx, Sx) + d(Ty, Ty) + d(Sx, Ty)} \right] + \beta d(Ax, By), \]
if $d(Ax, Sx) + d(Ax, By) + d(Ax, Ty) \neq 0$, where $\alpha, \beta < 1$ and $\beta < 1$; or,
\[ d(Sx, Ty) = 0, \] if $d(Ax, Sx) + d(Ax, By) + d(Ax, Ty) = 0$.

If any of $A, B, S$ or $T$ is continuous, then $A, B, S$ and $T$ have a unique common fixed point $z$. Furthermore, $z$ is the unique common fixed point of $A$ and $S$ as well as $B$ and $T$.

**Proof.** Let $x_0 \in X$. Since $SX \subseteq BX$, so there exists a point $x_1$ in $X$ such that $Sx_0 = Bx_1$. Also, since $TX \subseteq AX$, we can choose a point $x_2$ in $X$ such that $T_1 = A_2$.

Continuing this process, we have $Sx_{2n} = Bx_{2n+1}$ and $T_{2n+1} = A_{2n+2}$, for $n = 0, 1, 2, \ldots$
Define \( d_{2n} = d(Sx_{2n}, Tx_{2n+1}) \) and \( d_{2n+1} = d(Sx_{2n+2}, Tx_{2n+1}) \).

Suppose \( d_{2n} \neq 0 \) and \( d_{2n+1} \neq 0 \) for \( n = 1, 2, 3, \ldots \).

From (5.5), we have

\[
d_{2n+1} = d(Sx_{2n+2}, Tx_{2n+1}) \leq \alpha \left[ \frac{d_{2n+1} + d_{2n+2} + d_{2n+2}}{d_{2n+1} + d_{2n+2} + d_{2n+2}} \right] + \beta d(Ax_{2n+2}, Bx_{2n+1})
\]

\[
= \alpha \left[ \frac{d_{2n+1} + d_{2n+2} + d_{2n+2}}{d_{2n+1} + d_{2n+2} + d_{2n+2}} \right] + \beta d(Tx_{2n+1}, Sx_{2n})
\]

\[
= (\alpha + \beta) d_{2n+1} = (\alpha + \beta) d_{2n}.
\]

Thus, we have

\[
\varphi_{2n+1} \leq \varphi + \varphi \varphi_{2n} = k \varphi_{2n}, \text{ where } k = \varphi + \varphi < 1.
\]

In general, we have

\[
\varphi_{2n+1} \leq k \varphi_{2n} \leq k^{2} \varphi_{2n-1} \leq \ldots \leq k^{n+1} \varphi_{1} \leq k^{n+1} \varphi_{1}, \text{ that is,}
\]

\[
\varphi_{2n+1} \leq k^{n+1} \varphi_{1}.
\]

Letting \( n \to \infty \), we have

\[
\varphi_{2n+1} \leq 0, \text{ implies that, } d_{2n+1} = 0.
\]

Therefore, \( d(Sx_{2n+2}, Tx_{2n+1}) \to 0 \) as \( n \to \infty \).

We get the following sequence

(5.6) \( \{Sx_{0}, Tx_{1}, Sx_{2}, Tx_{3}, \ldots, Sx_{2n}, Tx_{2n+1}, \ldots\} \),

which is a Cauchy sequence in the complete complex valued metric space \((X, d)\), and therefore converges to a limit point \( z \) in \( X \).

Therefore, the sequences \( \{Sx_{2n}\} \) = \( \{Bx_{2n+1}\} \) and \( \{Tx_{2n-1}\} \) = \( \{Ax_{2n}\} \), which are the subsequences of (5.6) and hence also converge to the same point \( z \) in \( X \).

Now, suppose that \( A \) is continuous so that the sequences \( \{A^2x_{2n}\} \) and \( \{ASx_{2n}\} \) converge to the same point \( Az \). Since \( A \) and \( S \) are weakly commuting, so we have

\[
d(SAx_{2n}, ASx_{2n}) \leq d(Ax_{2n}, Sx_{2n}), \text{ implies that,}
\]

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\( \mathcal{R}((\mathbb{R}_{2n}, \mathbb{R}_{2n}^{2n})) \leq \mathcal{R}((\mathbb{R}_{2n}, \mathbb{R}_{2n})). \)
Letting $n \to \infty$, we have
\[
\lim_{n \to \infty} \mathbb{F}(\mathbb{F}(x_{2n}, x_{2n+1})) = \mathbb{F}(\mathbb{F}(x_{2n}, x_{2n+1})) = 0, \text{ implies that,}
\]
\[
\lim_{n \to \infty} \mathbb{F}(\mathbb{F}(x_{2n}, x_{2n+1})) = 0, \text{ that is, } \mathbb{F}(x_{2n}) \to z \text{ as } n \to \infty.
\]

Now, we shall show that $A_2 = z$.

Let, if possible, $A_2 \neq z$.

Now, using the triangle inequality and (5.5), we get
\[
d(A_2, z) \leq d(A_2, \mathbb{F}(x_{2n})) + d(\mathbb{F}(x_{2n}), x_{2n+1}) + d(x_{2n+1}, z)
\]
\[
\leq d(A_2, \mathbb{F}(x_{2n})) + \alpha\left[\frac{n_0 + n_1 + \cdots + n_{2n} + n_{2n+1} + \cdots n_{2n+2} + \cdots n_{2n+1}}{n_0 + n_1 + \cdots + n_{2n} + n_{2n+1} + \cdots n_{2n+2} + \cdots n_{2n+1}}\right]
\]
\[
+ \beta d(A_2, B) + d(x_{2n}, B) + d(x_{2n+1}, B) + d(x_{2n+1}, z).
\]

Thus, we have
\[
d(A_2, z) \leq d(A_2, B) + \alpha\frac{n_0 + n_1 + \cdots + n_{2n} + n_{2n+1} + \cdots n_{2n+2} + \cdots n_{2n+1}}{n_0 + n_1 + \cdots + n_{2n} + n_{2n+1} + \cdots n_{2n+2} + \cdots n_{2n+1}} + \beta d(A_2, z)
\]
\[
\leq d(A_2, B) + \frac{n_0 + n_1 + \cdots + n_{2n} + n_{2n+1} + \cdots n_{2n+2} + \cdots n_{2n+1}}{n_0 + n_1 + \cdots + n_{2n} + n_{2n+1} + \cdots n_{2n+2} + \cdots n_{2n+1}} + \beta d(A_2, z)
\]
\[
\leq d(A_2, z) + n_0 + n_1 + \cdots + n_{2n} + n_{2n+1} + \cdots n_{2n+2} + \cdots n_{2n+1} + \beta d(A_2, z).
\]

Thus, we have
\[
d(A_2, z) \leq d(A_2, z) + \alpha\frac{n_0 + n_1 + \cdots + n_{2n} + n_{2n+1} + \cdots n_{2n+2} + \cdots n_{2n+1}}{n_0 + n_1 + \cdots + n_{2n} + n_{2n+1} + \cdots n_{2n+2} + \cdots n_{2n+1}} + \beta d(A_2, z)
\]
\[
\leq d(A_2, z) + \frac{n_0 + n_1 + \cdots + n_{2n} + n_{2n+1} + \cdots n_{2n+2} + \cdots n_{2n+1}}{n_0 + n_1 + \cdots + n_{2n} + n_{2n+1} + \cdots n_{2n+2} + \cdots n_{2n+1}} + \beta d(A_2, z)
\]
\[
\leq d(A_2, z) + \frac{n_0 + n_1 + \cdots + n_{2n} + n_{2n+1} + \cdots n_{2n+2} + \cdots n_{2n+1}}{n_0 + n_1 + \cdots + n_{2n} + n_{2n+1} + \cdots n_{2n+2} + \cdots n_{2n+1}} + \beta d(A_2, z).
\]

Hence $A_2 = z$.

Now, we shall prove that $S_2 = z$.

Again, using the triangle inequality and (5.5), we have
\[
d(S_2, z) \leq d(S_2, x_{2n+1}) + d(x_{2n+1}, z)
\]
\[
\leq \alpha\frac{n_0 + n_1 + \cdots + n_{2n} + n_{2n+1} + \cdots n_{2n+2} + \cdots n_{2n+1}}{n_0 + n_1 + \cdots + n_{2n} + n_{2n+1} + \cdots n_{2n+2} + \cdots n_{2n+1}} + \beta d(S_2, B)
\]
\[
+ d(x_{2n+1}, z).
\]

Thus, we have
\[
d(S_2, z) \leq \alpha\frac{n_0 + n_1 + \cdots + n_{2n} + n_{2n+1} + \cdots n_{2n+2} + \cdots n_{2n+1}}{n_0 + n_1 + \cdots + n_{2n} + n_{2n+1} + \cdots n_{2n+2} + \cdots n_{2n+1}} + \beta d(S_2, B)
\]
\[
+ d(x_{2n+1}, z).
\]
\[
+ \infty (2^2, 1, 0) \,.
\]

Letting \( n \to \infty \), we have
\[
d(Sz, z) \leq \alpha \left( \frac{\sum \ldots \sum + \sum \ldots \sum + \sum \ldots \sum}{\sum \ldots \sum + \sum \ldots \sum + \sum \ldots \sum} \right) + \beta \left( \sum \sum \sum \sum \right) + \infty (2, 1, 0)
\]
\[
= 0, \text{ implies that, } Sz = z.
\]

Now, since \( SX \subseteq BX \), so there exists a point \( w \) in \( X \) such that \( Bw = z \).
Thus, we have
\[
d(z, Tw) = d(Sz, Tw)
\]
\[
\leq \alpha \left( \frac{\sum \ldots \sum + \sum \ldots \sum + \sum \ldots \sum}{\sum \ldots \sum + \sum \ldots \sum + \sum \ldots \sum} \right) + \beta \left( \sum \sum \sum \sum \right), \text{ implies that,}
\]
\[
\infty (2, 1, 0) \leq 0, \text{ that is, } Tw = z.
\]

Since \( B \) and \( T \) are weakly commuting, so we have
\[
d(TBw, BTw) \leq d(Bw, Tw), \text{ that is,}
\]
\[
d(TBw, BTw) \leq d(Bw, Tw), \text{ implies that,}
\]
\[
d(Tz, Bz) \leq d(z, z) = 0, \text{ that is, } Tz = Bz.
\]

Now, we shall prove that \( Tz = z \). Let, if possible, \( Tz \neq z \).

From (5.5), we have
\[
d(z, Tz) = d(Sz, Tz)
\]
\[
\leq \alpha \left( \frac{\sum \ldots \sum + \sum \ldots \sum + \sum \ldots \sum}{\sum \ldots \sum + \sum \ldots \sum + \sum \ldots \sum} \right) + \beta \left( \sum \sum \sum \sum \right), \text{ implies that,}
\]
\[
\infty (2, 1, 0) \leq 0, \text{ that is, } Tw = z.
\]

Hence \( Tz = z = Bz \) and \( Sz = z = Az \).
So, \( z \) is the common fixed point of \( A, B, S \) and \( T \).

Now, if one of the mappings \( B, S \) or \( T \) is continuous instead of \( A \), then one can show that \( A, B, S \) and \( T \) have a common fixed point.

To show that \( z \) is unique, let \( u \) be another common fixed point of \( A \) and \( S \).

From (5.5), we have
\[
d(u, z) = d(Su, Tz)
\]
\[
\leq \alpha \left( \frac{\sum \ldots \sum + \sum \ldots \sum + \sum \ldots \sum}{\sum \ldots \sum + \sum \ldots \sum + \sum \ldots \sum} \right) + \beta \left( \sum \sum \sum \sum \right), \text{ that is,}
\]
\[
d(u, z) \leq \alpha \left( \frac{\sum \ldots \sum + \sum \ldots \sum + \sum \ldots \sum}{\sum \ldots \sum + \sum \ldots \sum + \sum \ldots \sum} \right) + \beta \left( \sum \sum \sum \sum \right), \text{ implies that,}
\]

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Thus, we have, $u = z$, i.e., $A$ and $S$ have a unique common fixed point. In the same way, it can be shown that $z$ is the unique common fixed point of $B$ and $T$.

**Weakly compatible maps in complex valued metric spaces**

Jungck [47] introduced the notion of weakly compatible mappings as follows:

Two self maps are said to be weakly compatible, if they commute at their coincidence points.

Now, we show the existence of common fixed point for two, three and four self maps.

**Theorem 5.3.1.** Let $f$ and $g$ be self maps of a complex valued metric space $(X, d)$ satisfying the following:

(5.7) $\forall x \in X$, $fX \subseteq gX$,

(5.8) $d(fx, fy) \leq A \, d(gx, gy) + B \, \frac{d(gx, gx) + d(gy, gy)}{2} + C \, \frac{d(gx, gx) + d(gy, gy)}{2} + D \, \frac{d(gx, gx) + d(gy, gy)}{2} + E \, \frac{d(gx, gx) + d(gy, gy)}{2}$, for all $x, y \in X$, where $A, B, C, D$ and $E$ are non-negative constants with $A + B + C + D + E < 1$,

(5.9) $gX$ is a complete subspace of $X$.

Then $f$ and $g$ have a coincidence point.

Moreover, if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point.

**Proof.** Let $x_0 \in X$. From (5.7), we can construct sequences $\{x_n\}$ and $\{y_n\}$ in $X$ by $y_n = g(x_{n-1}) = fx_n$, $n = 0, 1, 2, \ldots$.

From (5.8), we have

$$d(y_{n+1}, y_n) = d(fx_{n+1}, fx_n) = A \, d(gx_{n+1}, gx_n) + B \, \frac{d(gx_{n+1}, gx_{n+1}) + d(gx_n, gx_n)}{2} + C \, \frac{d(gx_{n+1}, gx_{n+1}) + d(gx_n, gx_n)}{2} + D \, \frac{d(gx_{n+1}, gx_{n+1}) + d(gx_n, gx_n)}{2} + E \, \frac{d(gx_{n+1}, gx_{n+1}) + d(gx_n, gx_n)}{2}$$

Here, $A, B, C, D$ and $E$ are non-negative constants with $A + B + C + D + E < 1$.
1 + \Phi \left( \mu, \beta \sigma \right) - 1 + \Phi \left( \mu, \beta \sigma \right) - 1
\[ = A \, d(y_n, y_{n-1}) + B \frac{\bar{d}(g(x_{n-1}), y_{n-1})}{1+\bar{d}(g(x_{n-1}), y_{n-1})} + D \frac{\bar{d}(g(x_{n-1}), y_{n})}{1+\bar{d}(g(x_{n-1}), y_{n})} \]

Thus, we have

\[ k \leq \frac{d(x_n, x_{n-1})}{d(x_{n-1}, x_{n})} \]

Since \( 1 + d(x_n, x_{n-1}) > d(x_{n-1}, x_{n-2}) \), we have

\[ (1 - B) \, d(x_{n-1}, x_{n}) \leq (A + D) \, d(x_{n-1}, x_{n-2}) \], that is,

\[ d(x_{n-1}, x_{n}) \leq \frac{k}{1-k} d(x_{n-1}, x_{n-2}) \]

Consequently, it can be concluded that

\[ d(x_{n-1}, x_n) \leq k \, d(x_{n-2}, x_{n-1}) \]

\[ \ldots \]

\[ \leq k^n \, d(x_0, x_1) \]

Now, for all \( m > n \),

\[ d(x_m, x_n) \leq d(x_m, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_n, x_{n+1}) \]

\[ \leq k^n d(x_0, x_1) + k^{n+1} d(x_0, x_1) + \ldots + k^{m-1} d(x_0, x_1) \]

\[ \leq \sum_{i=n}^{m-1} k^i d(x_0, x_1) \]

Therefore, we have

\[ \lim_{n \to \infty} d(x_n, x_{n+1}) = 0 \]

Hence, \( \{x_n\} \) is a Cauchy sequence in \( gX \).

But \( gX \) is a complete subspace of \( X \), so there is a point \( u \) in \( gX \) such that \( x_n \to u \) as \( n \to \infty \).

Let \( v \in g^{-1}u \). Then \( gv = u \).

Now, we shall prove that \( fv = u \).

Putting \( x = v \) and \( y = x_{n+1} \) in (5.8), we get

\[ d(fv, fx_{n+1}) \leq A \, d(gv, gx_{n+1}) + B \frac{\bar{d}(g(x_{n+1}), y_{n+1})}{1+\bar{d}(g(x_{n+1}), y_{n+1})} + C \frac{\bar{d}(g(x_{n+1}), y_{n})}{1+\bar{d}(g(x_{n+1}), y_{n})} + D \frac{\bar{d}(g(x_{n+1}), y_{n})}{1+\bar{d}(g(x_{n+1}), y_{n})} + E \frac{\bar{d}(g(x_{n+1}), y_{n})}{1+\bar{d}(g(x_{n+1}), y_{n})} \]

Letting \( n \to \infty \), we have

\[ d(fv, u) \leq A \, d(u, u) + B \frac{\bar{d}(g(x_{n+1}), y_{n+1})}{1+\bar{d}(g(x_{n+1}), y_{n+1})} + C \frac{\bar{d}(g(x_{n+1}), y_{n})}{1+\bar{d}(g(x_{n+1}), y_{n})} + D \frac{\bar{d}(g(x_{n+1}), y_{n})}{1+\bar{d}(g(x_{n+1}), y_{n})} + E \frac{\bar{d}(g(x_{n+1}), y_{n})}{1+\bar{d}(g(x_{n+1}), y_{n})} \]

Hence, \( \{y_n\} \) is a Cauchy sequence in \( gX \).
1 + \bar{\xi}(\sigma, \nu)
\[ + E \frac{\mathcal{H}(\delta, \delta) \mathcal{H}(\delta, \delta)}{1 + \mathcal{H}(\delta, \delta)}, \text{ that is,} \]

\[ \mathcal{H}(\delta, \delta) \leq 0, \text{ implies that, } f v = u. \]

Thus, \( f v = u = g v \), and hence \( v \) is the coincidence point of \( f \) and \( g \).

Now, since \( f \) and \( g \) are weakly compatible, so, \( u = f v = g v \), implies that, \( f u = f g v = g f v = gu \).

Now, we claim that \( gu = u \).

Let, if possible, \( gu \neq u \).

From (5.8), we have

\[ d(u, gu) = d(fv, fu) \]

\[ \leq A d(gv, gu) + B \frac{\mathcal{H}(\delta, \delta) \mathcal{H}(\delta, \delta)}{1 + \mathcal{H}(\delta, \delta)} + C \frac{\mathcal{H}(\delta, \delta) \mathcal{H}(\delta, \delta)}{1 + \mathcal{H}(\delta, \delta)} + D \frac{\mathcal{H}(\delta, \delta) \mathcal{H}(\delta, \delta)}{1 + \mathcal{H}(\delta, \delta)}, \text{ that is,} \]

\[ \mathcal{H}(\delta, \delta) \leq A \frac{\mathcal{H}(\delta, \delta) \mathcal{H}(\delta, \delta)}{1 + \mathcal{H}(\delta, \delta)} + C \frac{\mathcal{H}(\delta, \delta) \mathcal{H}(\delta, \delta)}{1 + \mathcal{H}(\delta, \delta)}. \]

Since \( 1 + \mathcal{H}(\delta, \delta) > \mathcal{H}(\delta, \delta) \), we have

\[ \mathcal{H}(\delta, \delta) \leq (A + C) \mathcal{H}(\delta, \delta), \text{ implies that, } A + C \geq 1, \text{ a contradiction.} \]

Hence, \( gu = u = fu \).

Therefore, \( u \) is the common fixed point of \( f \) and \( g \).

For the uniqueness, let \( w \) be another common fixed point of \( f \) and \( g \) such that \( w \neq u \).

From (5.8), we have

\[ d(w, u) = d(fw, fu) \]

\[ \leq A d(gw, gu) + B \frac{\mathcal{H}(\delta, \delta) \mathcal{H}(\delta, \delta)}{1 + \mathcal{H}(\delta, \delta)} + C \frac{\mathcal{H}(\delta, \delta) \mathcal{H}(\delta, \delta)}{1 + \mathcal{H}(\delta, \delta)} + D \frac{\mathcal{H}(\delta, \delta) \mathcal{H}(\delta, \delta)}{1 + \mathcal{H}(\delta, \delta)}, \text{ that is,} \]

\[ \mathcal{H}(\delta, \delta) \leq A \frac{\mathcal{H}(\delta, \delta) \mathcal{H}(\delta, \delta)}{1 + \mathcal{H}(\delta, \delta)} + C \frac{\mathcal{H}(\delta, \delta) \mathcal{H}(\delta, \delta)}{1 + \mathcal{H}(\delta, \delta)}. \]

Since \( 1 + \mathcal{H}(\delta, \delta) > \mathcal{H}(\delta, \delta) \), we have

\[ \mathcal{H}(\delta, \delta) \leq (A + C) \mathcal{H}(\delta, \delta), \text{ implies that, } A + C \geq 1, \text{ a contradiction.} \]

Hence \( f \) and \( g \) have a unique common fixed point.

**Corollary 5.3.2.** Let \( f \) and \( g \) be self mappings of a complex valued metric space \((X, d)\) satisfying (5.7), (5.9) and the following:
\[(5.10) \quad d(fx, fy) \leq A d(gx, gy) + B \frac{d(gx, gy)}{1+B(gx, gy)} + C \frac{d(gx, gy)}{1+C(gx, gy)} + D \frac{d(gx, gy)}{1+D(gx, gy)}, \]

for all \(x, y \in X\), where \(A, B, C, D\) are non-negative constants with \(A + B + C + D < 1\).

Then \(f\) and \(g\) have a coincidence point.

Moreover, if \(f\) and \(g\) are weakly compatible, then \(f\) and \(g\) have a unique common fixed point.

**Proof.** By putting \(E = 0\) in Theorem 5.3.1, we get the Corollary 5.3.2.

**Corollary 5.3.3.** Let \(f\) and \(g\) be self mappings of a complex valued metric space \((X, d)\) satisfying (5.7), (5.9) and the following:

\[(5.11) \quad d(fx, fy) \leq A d(gx, gy), \quad \text{for all} \quad x, y \in X, \quad \text{where} \quad 0 \leq A < 1.\]

Then \(f\) and \(g\) have a coincidence point.

Moreover, if \(f\) and \(g\) are weakly compatible, then \(f\) and \(g\) have a unique common fixed point.

**Proof.** By putting \(B = C = D = E = 0\) in Theorem 5.3.1, we get the Corollary 5.3.3.

**Example 5.3.4.** Let \(X = [0, 1]\) and define \(d : X \times X \rightarrow \mathbb{C}\) by \(d(x, y) = i \frac{y - x}{|y - x|}\), for all \(x, y \in X\). Then \((X, d)\) is a complex valued metric space.

Define the functions \(f, g : X \rightarrow X\) by \(fx = \frac{x}{3}\) and \(gx = \frac{x}{2}\).

Clearly \(fX = [0, \frac{1}{3}] \subseteq [0, \frac{1}{2}] = gX\).

Also \(f\) and \(g\) are weakly compatible.

For \(A = \frac{1}{3} < 1\), we have

\[d(fx, fy) \leq A d(gx, gy), \quad \text{for all} \quad x, y \in X.\]

Also 0 is the unique common fixed point of \(f\) and \(g\).

Hence all the conditions of Corollary 5.3.3 are satisfied.

**Theorem 5.3.5.** Let \(S, T\) and \(f\) be three self maps of a complex valued metric space \((X, d)\) satisfying the following:

\[(5.12) \quad SX \cup TX \subseteq fX, \]

\[(5.13) \quad d(Sx, Ty) \leq h d(fx, fy), \quad \text{for all} \quad x, y \in X, \quad \text{where} \quad 0 \leq h < 1, \]

\[(5.14) \quad fX \text{ is complete subspace of } X. \]

Then \(S, T\) and \(f\) have a unique coincidence point.
Moreover, if (S, f) and (T, f) are weakly compatible, then S, T and f have a unique common fixed point.

**Proof.** Let x₀ ∈ X. Choose a point x₁ in X such that fx₁ = Sx₀. This can be done, since SX ⊆ fX. Similarly, choose a point x₂ in X such that fx₂ = Tx₁. Continuing this process and having chosen xₙ in X, we obtain xₙ₊₁ in X such that

\[ fx_{2k+1} = Sx_{2k}, \quad fx_{2k+2} = Tx_{2k+1}, \quad k = 0, 1, 2, \ldots. \]

From (5.13), we have

\[ d(fx_{2k+1}, fx_{2k+2}) = d(Sx_{2k}, Tx_{2k+1}) \]

\[ \leq h d(fx_{2k}, fx_{2k+1}). \]

Similarly,

\[ d(fx_{2k+2}, fx_{2k+3}) \leq h d(fx_{2k+1}, fx_{2k+2}). \]

Now, by induction, we obtain for each k = 0, 1, 2, \ldots,

\[ d(fx_{2k+2}, fx_{2k+3}) \leq h d(fx_0, fx_1). \]

Let \( y_n = fx_n, \ n = 0, 1, 2, \ldots. \)

Now, for all n, we have

\[ d(y_{n+1}, y_{n+2}) \leq h d(y_n, y_{n+1}) \]

\[ \leq h^2 d(y_{n-1}, y_n) \leq \ldots \leq h^{n+1} d(y_0, y_1). \]

Now, for any m > n,

\[ d(y_m, y_n) \leq d(y_{n+1}, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots + d(y_{m-1}, y_m) \]

\[ \leq [h^n + h^{n+1} + \ldots + h^{m-1}] d(y_0, y_1) \]

\[ \leq \frac{h^n}{1-h} d(y_0, y_1). \]

Therefore, we have

\[ \lim_{n \to \infty} \frac{h^n}{1-h} = 0, \]

which implies that \( \{y_n\} \) is a Cauchy sequence. Since fX is complete, there exists u, v in X such that \( y_n \to v = fu. \)

Choose a natural number N such that

\[ d(y_n, v) < \frac{\varepsilon}{2}, \text{ for all } n \geq N. \]

Hence, for all \( n \geq N, \) using triangle inequality and (5.13), we have

\[ d(fu, Su) \leq d(fu, y_{2n+2}) + d(y_{2n+2}, Su) \]

\[ = d(v, y_{2n+2}) + d(Tx_{2n+1}, Su) \]

\[ \leq d(v, y_{2n+2}) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]
Thus, \( d(fu, Su) \leq d(v, y_{2n+2}) + h \ d(fx_{2n+1}, fu) \leq d(v, y_{2n+2}) + h \ d(y_{2n+1}, v) \) for all \( m \geq 1 \), that is,
\[
\frac{d(fu, Su)}{2} \leq \frac{d(v, y_{2n+2})}{2} + \frac{h \ d(fx_{2n+1}, fu)}{2} + \frac{h \ d(y_{2n+1}, v)}{2}
\]

But, since \( m \) was arbitrary, so
\[
d((fu, Su), (v, y_{2n+2})) = 0, \text{ implies that, } fu = Su.
\]

Similarly, by using
\[
d(fu, Tu) \leq d(fu, y_{2n+1}) + d(y_{2n+1}, Tu),
\]
one can show that \( fu = Tu \), it implies that, \( v \) is a common point of coincidence of \( S, T \) and \( f \), that is, \( v = fu = Su = Tu \).

Now, we show that \( f, S \) and \( T \) have a unique point of coincidence. For this, assume that there exists another point \( w \) in \( X \) such that \( w = fz = Sz = Tz \) for some \( z \) in \( X \).

From (5.13), we have
\[
d(v, w) = d(Su, Tz) \leq h \ d(fu, fz) = h \ d(v, w), \text{ implies that, } v = w.
\]

Now, since \( (S, f) \) and \( (T, f) \) are weakly compatible, we have \( Sv = Sfu = fSu = fv \) and \( Tv = Tfu = fTu = fv \).

It implies that \( Sv = Tv = fz = Sz = Tz \) for some \( z \) in \( X \).

Therefore, \( v = t \), by uniqueness.

Hence \( v \) is a common fixed point of \( S, T \) and \( f \).

**Theorem 5.3.6.** Let \( S, T \) and \( f \) be three self maps of a complex valued metric space \((X, d)\) satisfying (5.12), (5.14) and the following:
\[
d(Sx, Ty) \leq h \ [d(fx, Sx) + d(fy, Ty)], \text{ for all } x, y \in X,
\]
where \( 0 \leq h < \frac{1}{2} \).

Then \( S, T \) and \( f \) have a unique coincidence point. Moreover, if \( (S, f) \) and \( (T, f) \) are weakly compatible, then \( S, T \) and \( f \) have a unique common fixed point.

**Proof.** Let \( x_0 \in X \). Define a sequence of points in \( X \), as in Theorem 5.3.5, given by the rule:
\[
fx_{2k+1} = Sx_{2k}, \quad fx_{2k+2} = Tx_{2k+1}, \ k = 0, 1, 2, \ldots
\]

From (5.15), we have
\[
d(fx_{2k+1}, fx_{2k+2}) = d(Sx_{2k}, Tx_{2k+1}) \leq h \ [d(fx_{2k}, Sx_{2k}) + d(fx_{2k+1}, Tx_{2k+1})] \]
= h [d(f_{2k}, f_{2k+1}) + d(f_{2k+1}, f_{2k+2})], that is,

\[ d(f_{2k}, f_{2k+1}) \leq \frac{1}{1-\beta} d(f_{2k}, f_{2k+1}). \]

Similarly, it can be shown that

\[ d(f_{2k+1}, f_{2k+2}) \leq \frac{1}{1-\beta} d(f_{2k+1}, f_{2k+2}) \]

\[ = p d(f_{2k+1}, f_{2k+2}), \quad p = \frac{1}{1-\beta} < 1. \]

Now, by induction, we obtain for each \( k = 0, 1, 2, \ldots \)

\[ d(f_{2k+1}, f_{2k+2}) \leq p d(f_{2k}, f_{2k+1}) \]

Let \( y_n = f_{2n}, \ n= 0, 1, 2, \ldots \)

\[ \leq p^2 d(f_{2k}, f_{2k}) \leq \ldots \leq p^{2k+1} d(f_0, f_1). \]

Now, for all \( n \), we have

\[ d(y_{n+1}, y_{n+2}) \leq p d(y_n, y_{n+1}) \]

\[ \leq p^2 d(y_{n-1}, y_n) \leq \ldots \leq p^{n+1} d(y_0, y_1). \]

Hence, for any \( m > n \),

\[ d(y_m, y_n) \leq [p^n + p^{n+1} + \ldots + p^{m-1}] d(y_0, y_1) \]

\[ \leq \frac{n}{1-\beta} d(y_0, y_1). \]

Therefore, we have

\[ \leq \frac{n}{1-\beta} d(y_0, y_1). \]

Hence, \( \frac{1}{n} \left( \beta, \beta \right) \leq \frac{1}{1-\beta} \left( \beta, 1 \right) \).

\[ \lim_{\beta \to \infty} \left( \frac{1}{\beta}, \frac{1}{\beta} \right) = 0, \]

which implies that \( \{y_n\} \) is a Cauchy sequence.

Since \( fX \) is complete, there exists \( u, v \) in \( X \) such that \( y_n \to v = fu \).

Choose a natural number \( N \) such that

\[ d(y_{n+1}, y_n) < \frac{1}{n} \left( \frac{1}{\beta}, \frac{1}{\beta} \right) \quad \text{and} \quad d(y_{n+1}, v) < \frac{1}{n} \left( \frac{1}{\beta}, \frac{1}{\beta} \right), \quad \text{for all} \ n \geq N. \]

Hence, for all \( n \geq N \), using triangle inequality and (5.15), we have

\[ d(fu, Su) \leq d(fu, y_{2n+2}) + d(y_{2n+2}, Su) \]

\[ = d(v, y_{2n+2}) + d(Tx_{2n+1}, Su) \]

\[ \leq d(v, y_{2n+2}) + h [d(fu, Su) + d(fx_{2n+1}, Tx_{2n+1})], \] that is,

\[ d(fu, Su) \leq \frac{1}{1-\beta} d(v, y_{2n+2}) + \frac{\beta}{1-\beta} d(y_{2n+1}, y_{2n+2}) \]

\[ \leq \frac{1}{1-\beta} + \frac{\beta}{1-\beta} = c. \]

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Thus, \( d(fu, Su) \leq \frac{d}{d} \) for all \( m \geq 1 \), that is, 
\[ d(fu, Su) \leq \frac{d}{d}. \]

But, since \( m \) was arbitrary, so

\[ d(fu, Su) = 0, \] implies that, \( fu = Su. \)

Similarly, by using

\[ d(fu, Tu) \leq \frac{d}{d} (d(fu, y_{2n+1}) + d(y_{2n+1}, Tu), \]

one can show that \( fu = Tu \), it implies that, \( v \) is a common point of coincidence of \( S, T \) and \( f \), that is, \( v = fu = Su = Tu. \)

Now, we show that \( f, S \) and \( T \) have a unique point of coincidence. For this, assume that there exists another point \( w \) in \( X \) such that \( w = fz = Sz = Tz \) for some \( z \) in \( X. \)

From (5.15), we have

\[ d(v, w) = d(Su, Tz) \leq h[d(fu, Su) + d(fz, Tz)] = h[d(v, v) + d(w, w)] = 0, \] that is,

\[ d(fu, Su) = 0, \] implies that, \( v = w. \)

Now, since \( (S, f) \) and \( (T, f) \) are weakly compatible, we have

\[ Sv = Sfu = fSu = fv \text{ and } Tv = Tfu = fTu = fv. \]

It implies that \( Sv = Tv = fv = t \) (say). Then \( w \) is a point of coincidence of \( S, T \) and \( f. \)

Therefore, \( v = t, \) by uniqueness.

Hence \( v \) is a common fixed point of \( S, T \) and \( f. \)

**Theorem 5.3.7.** Let \( S, T \) and \( f \) be three self maps of a complex valued metric space \( (X, d) \) satisfying (5.12), (5.14) and the following:

\[ d(Sx, Ty) \leq h[d(fy, Sx) + d(fx, Ty)], \] for all \( x, y \) in \( X, \)

where \( 0 \leq h < 1. \)

Then \( S, T \) and \( f \) have a unique coincidence point. Moreover, if \( (S, f) \) and \( (T, f) \) are weakly compatible, then \( S, T \) and \( f \) have a unique common fixed point.

**Proof.** Let \( x_0 \in X. \) Define a sequence of points in \( X, \) as in Theorem 5.3.5, given by the rule:

\[ fx_{2k+1} = Sx_{2k}, \] \[ fx_{2k+2} = Tx_{2k+1}, k = 0, 1, 2, \ldots. \]

From (5.16), we have

\[ d(fx_{2k+1}, fx_{2k+2}) = d(Sx_{2k}, Tx_{2k+1}) \leq h[d(fx_{2k+1}, Sx_{2k}) + d(fx_{2k}, Tx_{2k+1})]\]
\[ h \left[ d(f_{2k+1}, f_{2k+1}) + d(f_{2k}, f_{2k+2}) \right] \]
\[ \leq h \left[ d(f_{2k}, f_{2k+1}) + d(f_{2k+1}, T_{2k+2}) \right], \text{that is,} \]
\[ h \left[ d(f_{2k}, f_{2k+1}) + d(f_{2k+1}, T_{2k+2}) \right] \]
\[ \leq h \left[ d(f_{2k}, f_{2k+1}) \right]. \]

Similarly, it can be shown that
\[ d(f_{2k+1}, f_{2k+2}) \leq p \left( d(f_{2k+1}, f_{2k+2}) \right), \]
where \( p = \frac{1}{1 + \epsilon} < 1. \)

Now, by induction, we obtain for each \( k = 0, 1, 2, \ldots \)
\[ d(f_{2k+1}, f_{2k+2}) \leq p \left( d(f_{2k}, f_{2k+1}) \right) \]
\[ \leq p^2 \left( d(f_{2k-1}, f_{2k}) \right) \leq \ldots \leq p^k \left( d(f_{0}, f_{1}) \right). \]

Let \( y_n = f_{2n}, n = 0, 1, 2, \ldots \)
\[ \leq p^2 \left( d(f_{2n-1}, f_{2n}) \right) \leq \ldots \leq p^{2k} \left( d(f_{0}, f_{1}) \right). \]

Now, for all \( n, \) we have
\[ d(y_{n+1}, y_{n+2}) \leq p d(y_n, y_{n+1}) \]
\[ \leq p^2 d(y_{n-1}, y_n) \leq \ldots \leq p^{n+1} d(y_0, y_1). \]

Now, for any \( m > n, \)
\[ d(y_m, y_n) \leq d(y_m, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots + d(y_{m-1}, y_m) \]
\[ \leq \left[ p^n + p^{n+1} + \ldots + p^{m-1} \right] d(y_0, y_1) \]
\[ \leq \frac{p^n}{1-p} d(y_0, y_1). \]

Therefore, we have
\[ \leq \frac{p^n}{1-p} \]
Hence,
\[ \frac{\rho(\bar{1}, \bar{1})}{\rho(\bar{0}, \bar{1})} \leq \frac{\rho(\bar{0}, \bar{1})}{\rho(\bar{0}, \bar{1})}. \]

\[ \lim_{\bar{n} \to \infty} \frac{\rho(\bar{1}, \bar{1})}{\rho(\bar{0}, \bar{1})} = 0. \]

which implies that \( \{y_n\} \) is a Cauchy sequence.

Since \( f_1 \) is complete, there exists \( u, v \) in \( X \) such that \( y_n \to v = f u. \)

Choose a natural number \( N \) such that
\[ \frac{\rho(\bar{1}, \bar{1})}{\rho(\bar{0}, \bar{1})} \]
\[ d(y_{n+1}, y_n) \leq \frac{\rho(\bar{1}, \bar{1})}{\rho(\bar{0}, \bar{1})}, \text{for all } n \geq N. \]

Hence, for all \( n \geq N, \) using triangle inequality and (5.16), we have
\[ d(u, Su) \leq d(u, y_{2n+1}) + d(y_{2n+1}, Su) \]
\[ = d(v, y_{2n+1}) + d(T_{2n+1}, Su) \]
\[ \leq d(v, y_{2n+1}) + h \left[ d(fu, T_{2n+1}) + d(fx_{2n+1}, Su) \right]. \]
implies that, 
\[ d(fu, Su) \leq d(v, y_{2n+1}) + h \left[ d(v, y_{2n+2}) + d(y_{2n+1}, v) + d(v, Su) \right], \]
that is,
\[ d(fu, Su) \leq \frac{1}{1-h} \left[ d(v, y_{2n+1}) + h \left[ d(v, y_{2n+2}) + d(y_{2n+1}, v) + d(v, Su) \right] \right]. \]

Thus, we have
\[ d(fu, Su) \leq \frac{3}{h} \quad \text{for all } m \geq 1, \]
that is,
\[ \frac{(\| \), (\edge, \edge)}{\} \leq \frac{1}{m} \]
But, since \( m \) was arbitrary, so
\[ \frac{(\| \), (\edge, \edge)}{\} = 0, \] implies that, \( fu = Su. \)

Similarly, by using
\[ d(fu, Tu) \leq d(fu, y_{2n+1}) + d(y_{2n+1}, Tu), \]
one can show that \( fu = Tu, \) it implies that, \( v \) is a common point of coincidence of \( S, T \) and \( f, \) that is, \( v = fu = Su = Tu. \)

Now, we show that \( f, S \) and \( T \) have a unique point of coincidence. For this, assume that there exists another point \( w \) in \( X \) such that \( w = fz = Sz = Tz \) for some \( z \) in \( X. \)

From (5.16), we have \( d(v, w) = d(Su, Tz) \)
\[ \leq h \left[ d(fz, Su) + d(fu, Tz) \right] \]
\[ = h \left[ d(w, v) + d(v, w) \right] = 2h \left[ d(v, w) \right], \] that is,
\[ \frac{(\| \), (\edge, \edge)}{\} \leq 2h \frac{(\| \), (\edge, \edge)}{\}, \] implies that, \( v = w. \)

Now, since \( (S, f) \) and \((T, f)\) are weakly compatible, we have \( Sv = Sfu = fSu = fv \) and \( Tv = Tfu = fTu = fv. \)
It implies that \( Sv = Tv = fv = t \) (say). Then \( w \) is a point of coincidence of \( S, T \) and \( f. \)

Therefore, \( v = t, \) by uniqueness.
Hence \( v \) is a common fixed point of \( S, T \) and \( f. \)

Example 5.3.8. Let \( X = [0, 1] \) and let \( d : X \times X \to \mathbb{C} \) by \( d(x, y) = \frac{1}{2} \left[ |x - y| - \frac{1}{2} \right], \) for all \( x, y \) in \( X. \)
Then \( (X, d) \) is a complex valued metric space.

Define the functions \( S, T, f : X \to X \) by \( Sx = \frac{1}{2} = Tx \) and \( fx = \frac{1}{4}. \)
Clearly $\text{SX} \cup \text{TX} = [0, \frac{1}{4}] \subseteq [0, \frac{1}{2}] = f\text{X}$.

Also $(S, f)$ and $(T, f)$ are weakly compatible.

Now,

$$d(Sx, Ty) = 4 - \frac{4}{3} - \frac{4}{3}, \quad d(fx, fy) = 4 - \frac{4}{3} - \frac{4}{3}. \quad (2)$$

Clearly, for $h = \frac{2}{3} < 1$,

$$d(Sx, Ty) \leq h d(fx, fy).$$

Also 0 is the common fixed point of $S$, $T$, and $f$.

Hence all the conditions of Theorem 5.3.5 are satisfied.

**Theorem 5.3.9.** Let $A$, $B$, $S$ and $T$ be four self mappings of a complex valued metric space $(X, d)$ satisfying the followings:

(5.17) $\text{SX} \subseteq \text{BX}$, $\text{TX} \subseteq \text{AX},$

(5.18) $d(Sx, Ty) \leq \alpha m(x, y) + \beta M(x, y)$, where

$m(x, y) = d(By, Ty) + 1 + \beta (\frac{3}{2}, \frac{3}{2})$ and $M(x, y) = \max\{d(Ax, By), d(Ax, Sx), d(By, Ty)\},$

for each $x$, $y$ in $X$, and $\alpha > 0$, $\beta > 0$, $\frac{1}{3} + \beta < 1$.

Suppose that one of $AX$, $BX$, $SX$ and $TX$ is a complete subspace of $X$ and the pair $(A, S)$ and $(B, T)$ are weakly compatible.

Then $A$, $B$, $S$ and $T$ have a unique common fixed point.

**Proof.** Let $x_0 \in X$. Since $\text{SX} \subseteq \text{BX}$ and $\text{TX} \subseteq \text{AX}$, define for each $n \geq 0$, the sequence $\{y_n\}$ in $X$ by $y_{2n+1} = Sx_{2n} = Bx_{2n+1}$ and $y_{2n+2} = Tx_{2n+1} = Ax_{2n+2}$.

Suppose that $y_{2n} = y_{2n+1}$ for some $n$.

Then by (5.18), we have, $y_{2n+1} = y_{2n+2}$, and so, $y_m = y_{2n}$ for every $m > 2n$. Thus, the sequence $\{y_n\}$ is Cauchy.

The same conclusion holds if $y_{2n+1} = y_{2n+2}$ for some $n$.

Assume that $y_n \neq y_{n+1}$ for all $n$.

Putting $x = x_{2n}$ and $y = x_{2n-1}$ in (5.18), we have

$$d(Sx_{2n}, Tx_{2n-1}) \leq \alpha m(x_{2n}, x_{2n-1}) + \beta M(x_{2n}, x_{2n-1}),$$

where

$$m(x_{2n}, x_{2n-1}) = d(Bx_{2n-1}, Tx_{2n-1}) + 1 + \beta (\frac{3}{2}, \frac{3}{2})$$

and

$$M(x_{2n}, x_{2n-1}) = \max\{d(Ax_{2n}, Bx_{2n-1}), d(Ax_{2n}, Sx_{2n}), d(Bx_{2n-1}, Tx_{2n-1})\},$$

$$= \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\}.$$
Now, if $M(x_{2n}, x_{2n-1}) = d(y_{2n}, y_{2n+1})$, it follows that

$$(5.20) \quad d(y_{2n-1}, y_{2n}) \left(1 + d(y_{2n}, y_{2n+1})\right) \leq d(y_{2n}, y_{2n+1}) \left(1 + d(y_{2n}, y_{2n+1})\right).$$

Thus, we have

$$m(x_{2n}, x_{2n-1}) = d(y_{2n-1}, y_{2n}) \frac{1 + \varphi(\bar{y}_{2n}, \bar{y}_{2n+1})}{1 + \varphi(\bar{y}_{2n}, \bar{y}_{2n-1})} \leq d(y_{2n}, y_{2n+1}).$$

So, we obtain

$$d(y_{2n}, y_{2n+1}) \leq \alpha d(y_{2n}, y_{2n+1}) + \beta d(y_{2n}, y_{2n+1}) = (\alpha + \beta) d(y_{2n}, y_{2n+1}).$$

Thus, we have

$$\varphi(\bar{y}_{2n}, \bar{y}_{2n+1}) \leq \alpha + \beta = \varphi(\bar{y}_{2n}, \bar{y}_{2n+1}),$$

Consequently, we must have

$$M(x_{2n}, x_{2n-1}) = d(y_{2n-1}, y_{2n}) \text{ and } m(x_{2n}, x_{2n-1}) \leq d(y_{2n-1}, y_{2n}).$$

Thus, we have

$$d(y_{2n}, y_{2n+1}) \leq (\alpha + \beta) d(y_{2n-1}, y_{2n}).$$

On putting $x = x_{2n-2}$ and $y = x_{2n-1}$ in (5.18), we have

$$d(Sx_{2n-2}, Tx_{2n-1}) \leq \alpha m(x_{2n-2}, x_{2n-1}) + \beta M(x_{2n-2}, x_{2n-1}),$$

where

$$m(x_{2n-2}, x_{2n-1}) = d(Bx_{2n-1}, Tx_{2n-1}) \frac{1 + \varphi(\bar{y}_{2n-2}, \bar{y}_{2n-2})}{1 + \varphi(\bar{y}_{2n-2}, \bar{y}_{2n-1})} = d(y_{2n-1}, y_{2n}),$$

and

$$M(x_{2n-2}, x_{2n-1}) = \max \{d(Ax_{2n-2}, Bx_{2n-1}), d(Ax_{2n-2}, Sx_{2n-2}), d(Bx_{2n-1}, Tx_{2n-1})\} = \max \{d(y_{2n-2}, y_{2n-1}), d(y_{2n-2}, y_{2n-1}), d(y_{2n-1}, y_{2n})\}.$$

Now, if $M(x_{2n-2}, x_{2n-1}) = d(y_{2n-2}, y_{2n-1})$, it follows that

$$d(y_{2n-1}, y_{2n}) \leq \alpha d(y_{2n-1}, y_{2n}) + \beta d(y_{2n-1}, y_{2n}) = (\alpha + \beta) d(y_{2n-1}, y_{2n}).$$

Thus, we have

$$\varphi(\bar{y}_{2n-1}, \bar{y}_{2n}) \leq \alpha + \beta = \varphi(\bar{y}_{2n-1}, \bar{y}_{2n}),$$

Then, we must have $M(x_{2n-2}, x_{2n-1}) = d(y_{2n-2}, y_{2n-1})$, and hence,

$$d(y_{2n-1}, y_{2n}) \leq \alpha d(y_{2n-1}, y_{2n}) + \beta d(y_{2n-2}, y_{2n-1}),$$

that is,

$$d(y_{2n-1}, y_{2n}) \leq \frac{\alpha}{1 - \alpha} d(y_{2n-2}, y_{2n-1}).$$

Thus, we have

$$d(y_{2n-1}, y_{2n}) \leq \frac{\alpha}{1 - \alpha} \varphi(\bar{y}_{2n-1}, \bar{y}_{2n}) \leq \frac{\alpha}{1 - \alpha} \varphi(\bar{y}_{2n-2}, \bar{y}_{2n-2}).$$

Define $k = \max \{\varphi + \varphi, \frac{\alpha}{1 - \alpha}\}$.
Consequently, it can be concluded that
\[ d(y_m, y_{m+1}) \leq k \cdot d(y_{m-1}, y_n) \]
\[ \leq k^2 \cdot d(y_{m-2}, y_{n-1}) \]
\[ \vdots \]
\[ \leq k^n \cdot d(y_0, y_1). \]
Now, for all \( m > n \), we have
\[ d(y_m, y_n) \leq d(y_m, y_{m+1}) + d(y_{m+1}, y_{m+2}) + \ldots + d(y_n, y_{m-1}) \]
\[ \leq k^n \cdot d(y_0, y_1) + k^{n+1} \cdot d(y_0, y_1) + \ldots + k^{m-1} \cdot d(y_0, y_1) \]
\[ \leq \frac{1 - k^{m-n}}{1 - k} \cdot d(y_0, y_1). \]
Therefore, we have
\[ \lim_{m \to \infty} d(y_m, y_n) = 0. \]
Hence, \( \{y_n\} \) is a Cauchy sequence.

Now, suppose that \( A(X) \) is complete. Note that \( \{y_{2n}\} \) is contained in \( A(X) \) and has a limit in \( A(X) \), say \( u \), that is, \( \lim_{n \to \infty} y_{2n} = u \). Let \( v \in A^{-1}u \). Then \( Av = u \).

Now, we shall prove that \( Sv = u \).

Let, if possible, \( Sv \neq u \).

Putting \( x = v \) and \( y = x_{2n-1} \) in (5.18), we have
\[ d(Sv, Tx_{2n-1}) \leq \alpha \cdot m(v, x_{2n-1}) + \beta \cdot M(v, x_{2n-1}), \]
with
\[ m(v, x_{2n-1}) = d(y_{2n-1}, y_{2n}) \]
\[ = d(y_{2n-1}, y_{2n}) \frac{1 - k^{2n-1}}{1 - k} \]
and
\[ M(v, x_{2n-1}) = \max \{d(u, y_{2n-1}), d(u, Sv), d(y_{2n-1}, y_{2n}) \}. \]

As the sequence \( \{y_{2n-1}\} \) is convergent to \( u \), therefore
\[ \lim_{n \to \infty} d(u, y_{2n-1}) = \lim_{n \to \infty} d(y_{2n-1}, y_{2n}) = 0. \]
Thus, letting \( n \to \infty \) in (5.21), we have
\[ d(Sv, u) \leq \alpha \cdot 0 + \beta \cdot d(Sv, u), \]
that is,
\[ d(Sv, u) \leq \beta \cdot d(Sv, u), \]
and
\[ d(Sv, u) \leq \beta \cdot d(Sv, u), \]
a contradiction to \( \beta < 1 \).
Hence \( Sv = u = Av \).

Now, since \( SX \subseteq BX \), \( Sv = u \) implies that \( u \in BX \).
Let \( w \in B^{-1}u \). Then \( Bw = u \). By using the same arguments as above, one can easily verify that, \( Tw = u = Bw \), that is, \( w \) is the coincidence point of the pair \((B, T)\). The same result holds, if we assume that \(BX\) is complete.

Now, if \(TX\) is complete, then by (5.17), \( u \in TX \subseteq AX \).

Now, since the pairs \((A, S)\) and \((B, T)\) are weakly compatible, so

\[
\begin{align*}
    u &= Sv = Av = Tw = Bw, \\
    Au &= ASv = SAv = Su, \\
    Bu &= BTw = TBw = Tu.
\end{align*}
\]  

(5.22)

Now, we claim that \( Tu = u \).

Let, if possible, \( Tu \neq u \).

From (5.18), we have

\[
\begin{align*}
    d(u, Tu) &= d(Sv, Tu) \\
    &\geq \alpha m(v, u) + \beta M(v, u),
\end{align*}
\]  

(5.23)

with

\[
m(v, u) = d(Bu, Tu) \leq 0,
\]

and

\[
M(v, u) = \max\{d(Av, Bu), d(Av, Sv), d(Bu, Tu)\} = d(u, Tu).
\]

Thus, from (5.23), we have

\[
d(u, Tu) \geq \alpha 0 + \beta d(u, Tu), \text{ that is,}
\]

\[
d(u, Tu) \leq \beta d(u, Tu), \text{ a contradiction to } \beta < 1.
\]

Therefore, \( Tu = u \).

Similarly, one can prove that \( Su = u \), and then, \( u \) is a common fixed point of \( A, B, S \) and \( T \).

For the uniqueness, let \( z \) be another common fixed point of \( A, B, S \) and \( T \).

Now, we claim that, \( u = z \).

Let, if possible, \( u \neq z \).

From (5.18), we have

\[
\begin{align*}
    d(u, z) &= d(Su, Tz) \\
    &\geq \alpha m(u, z) + \beta M(u, z),
\end{align*}
\]  

(5.24)

with

\[
m(u, z) = d(Bz, Tz) \leq 0,
\]

\[
M(u, z) = \max\{d(Av, Bu), d(Av, Sv), d(Bu, Tu)\} = d(u, Tu).
\]

Therefore, \( Tu = u \).

Similarly, one can prove that \( Su = u \), and then, \( u \) is a common fixed point of \( A, B, S \) and \( T \).

For the uniqueness, let \( z \) be another common fixed point of \( A, B, S \) and \( T \).

Now, we claim that, \( u = z \).
and
\[ M(u, z) = \max\{d(Au, Bz), d(Au, Su), d(Bz, Tz)\} = d(u, z). \]

Thus, from (5.24), we have
\[ d(u, z) \leq \alpha 0 + \beta d(u, z), \]
that is,
\[ d(u, z) \leq \beta d(u, z), \]
a contradiction to \( \beta < 1 \).

Therefore, \( u = z \).

Hence \( A, B, S \) and \( T \) have a unique common fixed point.

**Corollary 5.3.10.** Let \( B \) and \( S \) be two self maps of a complex valued metric space \((X, d)\)
satisfying the following:

(5.19) \( SX \subseteq BX \),

(5.20) \( d(Sx, Sy) \leq \alpha m(x, y) + \beta M(x, y) \),

with \( m(x, y) = d(By, Sy) \) and
\[ M(x, y) = \max\{d(Bx, By), d(Bx, Sx), d(By, Sy)\}, \]
for each \( x, y \) in \( X \), and \( \alpha > 0, \beta > 0, \alpha + \beta < 1 \).

If one of \( SX \) or \( BX \) is complete subspace of \( X \), then the pair \((B, S)\) have a coincidence point. Moreover, if \( B \) and \( S \) are weakly compatible, then \( B \) and \( S \) have a unique common fixed point.

**Proof.** On putting \( A = B \) and \( S = T \) in Theorem 5.3.9, we get the corollary 5.3.10.

**E. A. property in complex valued metric spaces along with weakly compatible maps**

Here, we prove some common fixed point theorems using E.A. property along with weakly compatible maps.

**Theorem 5.4.1.** Let \( f \) and \( g \) be self mappings of a complex valued metric space \((X, d)\)
satisfying (5.7), (5.8) and the followings:

(5.21) \( f \) and \( g \) are weakly compatible,

(5.22) \( f \) and \( g \) satisfy E. A. property,

(5.23) \( gX \) is a closed subset of \( X \).

Then \( f \) and \( g \) have a unique common fixed point.

**Proof.** Since \( f \) and \( g \) satisfy the E. A. property, there exists a sequence \( \{x_n\} \) in \( X \) such that

\[ \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = z, \]
for some \( z \) in \( X \).
Now, \( gX \) is closed subset of \( X \), therefore

\[
\lim_{n \to \infty} g_{x_n} = g_{a},
\]

for some \( a \) in \( X \).

So, from (5.24), we have

\[
\lim_{n \to \infty} g_{x_n} = g_{a}.
\]

We claim that \( f_a = g_a \).

From (5.8), we have

\[
d(f_a, f_{x_n}) \leq A d(g_a, g_{x_n}) + B + D + E + C,
\]

Letting \( n \to \infty \), we have

\[
d(f_a, g_a) \leq A d(g_a, g_{a}) + B + D + E + C
\]

\[
= 0, \text{ implies that,}
\]

\[
\phi(f_a, g_a) \leq 0, \text{ that is, } f_a = g_a.
\]

Now, we show that \( f_a \) is the common fixed point of \( f \) and \( g \). Let, if possible, \( f_a \neq f f_a \).

Since \( f \) and \( g \) are weakly compatible, \( g f_a = f g_a \), implies that, \( f f_a = f g_a = g f_a = g g_a \).

From (5.8), we have

\[
d(f f_a, f_a) \leq A d(f g_a, g_a) + B + D + E + C
\]

\[
= A d(f f_a, f_a) + C,
\]

\[
\phi(f f_a, f_a) \leq A \phi(f f_a, f_a) + C, \text{ that is,}
\]

\[
\phi(f f_a, f_a) \leq (A + C) \phi(f f_a, f_a), \text{ implies that, } A + C \geq 1,
\]

a contradiction.

Hence \( f f_a = f_a = g f_a \).

Thus, \( f_a \) is the common fixed point of \( f \) and \( g \).

Finally, we show that the common fixed point is unique.

For this, let \( u \) and \( v \) be two common fixed points of \( f \) and \( g \) such that \( u \neq v \)

\[
d(v, u) = d(f v, f u)
\]

\[
\leq A d(g v, g u) + B + D + E + C
\]

\[
= A \phi(g v, g u) + C + D + E + C, \text{ that is,}
\]

\[
\phi(g v, g u) = 0, \text{ which contradicts the uniqueness of the common fixed point.}
\]
\[ + D \frac{\bar{\alpha}(\bar{\beta}, \bar{\beta})}{(\bar{\beta}, \bar{\beta})} + E \frac{\bar{\beta}(\bar{\beta}, \bar{\beta})}{(\bar{\beta}, \bar{\beta})} = A d(v, u) + C \frac{1 + \bar{\beta}(\bar{\beta}, \bar{\beta})}{(\bar{\beta}, \bar{\beta})}, \text{ that is,} \]
\[ \bar{\beta}(\bar{\beta}, \bar{\beta}) \leq A \bar{\beta}(\bar{\beta}, \bar{\beta}) + C \frac{1 + \bar{\beta}(\bar{\beta}, \bar{\beta})}{1 + \bar{\beta}(\bar{\beta}, \bar{\beta})}. \]

Since \( 1 + \bar{\beta}(\bar{\beta}, \bar{\beta}) > \bar{\beta}(\bar{\beta}, \bar{\beta}) \), we have
\[ \bar{\beta}(\bar{\beta}, \bar{\beta}) \leq (A + C) \bar{\beta}(\bar{\beta}, \bar{\beta}), \text{ implies that, } A + C \geq 1, \text{ a contradiction.} \]
Hence \( f \) and \( g \) have a unique common fixed point.

**Corollary 5.4.2.** Let \( f \) and \( g \) be self mappings of a complex valued metric space \( (X, d) \) satisfying (5.7), (5.21), (5.22) and the following:
(5.25) \[ d(fx, fy) \leq A d(gx, gy) + C \frac{1 + \bar{\beta}(\bar{\beta}, \bar{\beta})}{1 + \bar{\beta}(\bar{\beta}, \bar{\beta})}, \text{ for all } x, y \in X, \]
where \( A \) and \( C \) are non-negative constants with \( A + C < 1 \).
Then \( f \) and \( g \) have a unique common fixed point.

**Proof.** By putting \( B = D = E = 0 \) in Theorem 5.4.1, we get the Corollary 5.4.2.

**Corollary 5.4.3.** Let \( f \) and \( g \) be self mappings of a complex valued metric space \( (X, d) \) satisfying (5.7), (5.21), (5.22) and the following:
(5.26) \[ d(fx, fy) \leq C \frac{1 + \bar{\beta}(\bar{\beta}, \bar{\beta})}{1 + \bar{\beta}(\bar{\beta}, \bar{\beta})}, \text{ for all } x, y \in X, \]
where \( C \) is a non-negative constant with \( C < 1 \).
Then \( f \) and \( g \) have a unique common fixed point.

**Proof.** By putting \( A = 0 \) in Corollary 5.4.2, we get the Corollary 5.4.3.

**Corollary 5.4.4.** Let \( f \) and \( g \) be self mappings of a complex valued metric space \( (X, d) \) satisfying (5.7), (5.21), (5.22) and the following:
(5.27) \[ d(fx, fy) \leq A d(gx, gy), \text{ for all } x, y \in X, \]
where \( A \) is a non-negative constant with \( A < 1 \).
Then \( f \) and \( g \) have a unique common fixed point.

**Proof.** By putting \( C = 0 \) in Corollary 5.4.2, we get the Corollary 5.4.4.

**Example 5.4.5.** Let \( X = [0, 1] \) and define \( d : X \times X \to \mathbb{C} \) by \( d(x, y) = i \frac{\bar{x} - \bar{y}}{x - y} \), for all \( x, y \in X \).
Then \( (X, d) \) is a complex valued metric space.

Define the functions \( f, g : X \to X \) by \( fx = \frac{n}{6} \) and \( gx = \frac{n}{2} \).

Clearly \( fX = [0, \frac{1}{6}] \subseteq [0, \frac{1}{2}] = gX. \)
Also, \( f \) and \( g \) are weakly compatible.
Consider the sequence \( \{x_n\} = \{\frac{1}{n}\}, n \in \mathbb{N} \).

Since \( \lim_{n \to \infty} x_n = \lim_{n \to \infty} g_n = 0 \), where \( 0 \in X \), so \( f \) and \( g \) satisfy E. A. property. Also, for \( A = \frac{1}{3} < 1 \), we have

\[
d(fx, fy) \leq A \cdot d(gx, gy), \quad \text{for all } x, y \in X.
\]

Here 0 is the unique common fixed point of \( f \) and \( g \).

Hence all the conditions of Corollary 5.4.4 are satisfied.

**Theorem 5.4.6.** Let \( A, B, S \) and \( T \) be self mappings of a complex valued metric space \((X, d)\) satisfying (5.17), (5.18) and the followings:

1. (5.28) pairs \((A, S)\) and \((B, T)\) are weakly compatible,
2. (5.29) pair \((A, S)\) or \((B, T)\) satisfy the E.A. property.

If any one of \( AX, BX, SX \) and \( TX \) is a complete subspace of \( X \), then \( A, B, S \) and \( T \) have a unique common fixed point.

**Proof.** Suppose that \((A, S)\) satisfies the E.A. property. Then there exists a sequence \( \{x_n\} \) in \( X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = z \), for some \( z \in X \).

Since \( SX \subseteq BX \), there exists a sequence \( \{y_n\} \) in \( X \) such that \( Sx_n = By_n \). Hence \( \lim_{n \to \infty} By_n = z \).

We shall show that \( \lim_{n \to \infty} Ty_n = z \).

Let, if possible, \( \lim_{n \to \infty} Ty_n = t \neq z \).

From (5.18), we have

\[
d(Sx_n, Ty_n) \leq \alpha \cdot m(x_n, y_n) + \beta \cdot M(x_n, y_n).
\]

Letting \( n \to \infty \), we have

\[
d(z, t) \leq \alpha \cdot \lim_{n \to \infty} m(x_n, y_n) + \beta \cdot \lim_{n \to \infty} M(x_n, y_n),
\]

with

\[
\lim_{n \to \infty} m(x_n, y_n) = \lim_{n \to \infty} d(By_n, Ty_n) \leq \beta \cdot \lim_{n \to \infty} \max \{d(Ax_n, By_n), d(Ax_n, Sx_n), d(By_n, Ty_n)\}.
\]

and

\[
\lim_{n \to \infty} M(x_n, y_n) = \lim_{n \to \infty} \max \{d(Ax_n, By_n), d(Ax_n, Sx_n), d(By_n, Ty_n)\} = \max \{0, 0, d(z, t)\} = d(z, t).
\]

Thus, from (5.30), we have

\[
d(z, t) \leq \alpha d(z, t) + \beta d(z, t) = \alpha + \beta \cdot d(z, t), \quad \text{that is,}
\]

\[
d(z, t) \leq \alpha + \beta \cdot d(z, t), \quad \text{a contradiction to } \alpha + \beta < 1.
\]

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Therefore, \( t = z \), that is, \( \lim_{n \to \infty} T y_n = z \).

Suppose that \( B X \) is a complete subspace of \( X \). Then \( z = B u \) for some \( u \) in \( X \).

Subsequently, we have

\[
\lim_{n \to \infty} T y_n = \lim_{n \to \infty} S x_n = \lim_{n \to \infty} A x_n = \lim_{n \to \infty} B y_n = z = B u.
\]

Now, we shall show that \( T u = B u \).

Let, if possible, \( T u \not= B u \).

From (5.18), we have

\[
d(S x_n, T u) \leq \alpha m(x_n, u) + \beta M(x_n, u).
\]

Letting \( n \to \infty \), we have

\[
(5.31) \quad d(z, T u) \leq \alpha \lim_{n \to \infty} m(x_n, u) + \beta \lim_{n \to \infty} m(x_n, u),
\]

with

\[
\lim_{n \to \infty} m(x_n, u) = \lim_{n \to \infty} d(B u, T u) = d(B u, T u) = d(z, T u)
\]

and

\[
\lim_{n \to \infty} M(x_n, u) = \lim_{n \to \infty} \max \{ d(A x_n, B u), d(A x_n, S x_n), d(B u, T u) \} = \max \{ d(B u, T u) \} = d(B u, T u) = d(z, T u).
\]

Thus, from (5.31), we have

\[
d(z, T u) \leq \alpha \ d z, T u + \beta \ d z, T u = \alpha + \beta \ d z, T u , \text{ that is,}
\]

\[
d(z, T u) \leq \alpha + \beta \ d(z, T u) , \text{ a contradiction to } \alpha + \beta < 1.
\]

Therefore, \( T u = B u \).

Since \( B \) and \( T \) are weakly compatible, therefore, \( B T u = T B u \), implies that,

\( T T u = T B u = B B u \).

Since \( T X \subseteq A X \), there exists \( v \in X \), such that, \( T u = A v \).

Now, we claim that \( A v = S v \).

Let, if possible, \( A v \not= S v \).

From (5.18), we have

\[
(5.32) \quad d(S v, T u) \leq \alpha m(v, u) + \beta M(v, u),
\]

with

\[
m(v, u) = d(B u, T u) = 0,
\]

and

\[
M(v, u) = \max \{ d(A v, B u), d(A v, S v), d(B u, T u) \}
\]

\[
= \max \{ 0, d(A v, S v) \} = d(A v, S v) = d(T u, S v).
\]

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Thus, from (5.32), we have

\[ d(Sv, Tu) \leq \alpha 0 + \beta d(Sv, Tu), \]

that is,

\[ d(Sv, Tu) \leq \beta d(Sv, Tu), \]

a contradiction to \( \beta < 1 \).

Therefore, \( Sv = Tu = Av \).

Thus, we have, \( Tu = Bu = Sv = Av \).

The weak compatibility of \( A \) and \( S \) implies that \( ASv = SAv = SSv = AAv \).

Now, we claim that \( Tu \) is the common fixed point of \( A, B, S \) and \( T \).

Suppose that, \( TTu \neq Tu \).

From (5.18), we have

\[ d(Tu, TTu) = d(Sv, TTu) \]

(5.33)

\[ \leq \alpha m(v, Tu) + \beta M(v, Tu), \]

with

\[ m(v, Tu) = d(BTu, TTu) = 0, \]

\[ M(v, Tu) = \max\{d(Av, BTu), d(Av, Sv), d(BTu, TTu)\} = d(Av, BTu) = d(Tu, TTu). \]

Thus, from (5.33), we have

\[ d(Tu, TTu) \leq \alpha 0 + \beta d(Tu, TTu), \]

that is,

\[ d(Tu, TTu) \leq \beta d(Tu, TTu), \]

a contradiction to \( \beta < 1 \).

Therefore, \( Tu = TTu = BTu \).

Hence \( Tu \) is the common fixed point of \( B \) and \( T \).

Similarly, we prove that \( Sv \) is the common fixed point of \( A \) and \( S \). Since \( Tu = Sv \), \( Tu \) is the common fixed point of \( A, B, S \) and \( T \). The proof is similar when \( AX \) is assumed to be a complete subspace of \( X \). The cases in which \( TX \) or \( SX \) is a complete subspace of \( X \) are similar to the cases in which \( AX \) or \( BX \), respectively is complete subspace of \( X \), since \( TX \subseteq AX \) and \( SX \subseteq BX \).

Now, we shall prove that the common fixed point is unique.

If possible, let \( p \) and \( q \) be two common fixed points of \( A, B, S \) and \( T \), such that, \( p \neq q \).

From (5.18), we have

\[ d(p, q) = d(Sp, Tq) \]

(5.34)

\[ \leq \alpha m(p, q) + \beta M(p, q), \]

with
and
\[ m(p, q) = d(Bq, Tq)^{1+\frac{\beta}{1+\beta}} = 0, \]

and
\[ M(p, q) = \max\{d(Ap, Bq), d(Ap, Sp), d(Bq, Tq)\} = d(p, q). \]

Thus, from (5.34), we have
\[ d(p, q) \leq \alpha 0 + \beta d(p, q), \]
that is,
\[ d(p, q) \leq \beta d(p, q), \]
a contradiction to \( \beta < 1. \)

Therefore, \( p = q. \)

Hence \( A, B, S \) and \( T \) have a unique common fixed point.

**Corollary 5.4.7.** Let \( B \) and \( S \) be two weakly compatible self maps of a complex valued metric space \((X, d)\) satisfying the following:

(i) \( SX \subseteq BX, \)

(ii) \( d(Sx, Sy) \leq \alpha m(x, y) + \beta M(x, y), \)

where
\[ m(x, y) = d(By, Sy)^{1+\frac{\beta}{1+\beta}} \]

and
\[ M(x, y) = \max\{d(Bx, By), d(Bx, Sx), d(By, Sy)\}, \]

for each \( x, y \) in \( X, \) and \( \alpha > 0, \beta > 0, \frac{1}{\alpha} + \frac{1}{\beta} < 1, \)

(iii) \( B \) and \( S \) satisfies the E.A. property.

If \( SX \) or \( BX \) is complete subspace of \( X, \) then \( B \) and \( S \) have a unique common fixed point.

**Proof.** By putting \( A = B \) and \( S = T \) in Theorem 5.4.6, we get the corollary 5.4.7.

**Theorem 5.5.1.** Let \( f \) and \( g \) be self mappings of a complex valued metric space \((X, d)\) satisfying (5.8), (5.21) and the following:

(5.35) \( f \) and \( g \) satisfy \((CLR_g)\) property.

Then \( f \) and \( g \) have a unique common fixed point.

**Proof.** Since \( f \) and \( g \) satisfy the \((CLR_g)\) property, there exists a sequence \( \{x_n\} \) in \( X \) such that

\[ \lim_{n \to \infty} R_{1+n}x_n = \lim_{n \to \infty} R_{1+n}x_n = gx, \text{ for some } x \text{ in } X. \]
From (5.8), we have
\[ d(fx, fx_n) \leq A d(gx, gx_n) + B + C + D + E. \]

Letting \( n \to \infty \), we have
\[ d(fx, gx) \leq A d(gx, gx) + B + C + D + E, \]
implies that, \( \bar{d}(f, g) \leq 0 \), that is, \( fx = gx \).

Now, let \( u = fx = gx \). Since \( f \) and \( g \) are weakly compatible mappings, therefore, \( fgx = gfx \), implies that, \( fu = fgx = gfx = gu \).

Now, we claim that \( gu = u \). Let, if possible, \( gu \neq u \).

From (5.8), we have
\[ d(u, gu) = A d(u, gu) + C, \]
that is,
\[ \bar{d}(u, gu) \leq A \bar{d}(u, gu) + C. \]

Since \( 1 + \bar{d}(u, gu) > \bar{d}(u, gu) \), we have
\[ \bar{d}(u, gu) \leq (A + C) \bar{d}(u, gu), \] implies that, \( A + C \geq 1 \), a contradiction.

Hence, \( gu = u = fu \).

Therefore, \( u \) is the common fixed point of \( f \) and \( g \).

For the uniqueness, let \( w \) be another common fixed point of \( f \) and \( g \) such that \( w \neq u \).

From (5.8), we have
\[ d(w, u) = A d(gw, gu) + B + C + D + E, \]
that is,
\[ \bar{d}(w, u) \leq A \bar{d}(w, u) + C. \]
1 + \beta(\beta, \beta)
Since \( 1 + \mathfrak{B}(\overline{a}, \overline{b}) > \mathfrak{B}(\overline{a}, \overline{b}) \), we have

\[
\mathfrak{B}(\overline{a}, \overline{b}) \leq (A + C) \mathfrak{B}(\overline{a}, \overline{b})
\]

implies that, \( A + C \geq 1 \), a contradiction.

Hence \( f \) and \( g \) have a unique common fixed point.

**Corollary 5.5.2.** Let \( f \) and \( g \) be self mappings of a complex valued metric space \((X, d)\) satisfying (5.21), (5.35) and the following:

\[
(5.37) \quad d(fx, fy) \leq A d(gx, gy) + C \frac{n}{1+\mathfrak{B}(\overline{a}, \overline{b})}, \text{ for all } x, y \text{ in } X,
\]

where \( A \) and \( C \) are non-negative constants with \( A + C < 1 \).

Then \( f \) and \( g \) have a unique common fixed point.

**Proof.** By putting \( B = D = E = 0 \) in Theorem 5.5.1, we get the Corollary 5.5.2.

**Corollary 5.5.3.** Let \( f \) and \( g \) be self mappings of a complex valued metric space \((X, d)\) satisfying (5.21), (5.35) and the following:

\[
(5.38) \quad d(fx, fy) \leq C \frac{n}{1+\mathfrak{B}(\overline{a}, \overline{b})}, \text{ for all } x, y \text{ in } X,
\]

where \( C \) is a non-negative constant with \( C < 1 \).

Then \( f \) and \( g \) have a unique common fixed point.

**Proof.** By putting \( A = 0 \) in Corollary 5.5.2, we get the Corollary 5.5.3.

**Corollary 5.5.4.** Let \( f \) and \( g \) be self mappings of a complex valued metric space \((X, d)\) satisfying (5.21), (5.35) and the following:

\[
(5.39) \quad d(fx, fy) \leq A d(gx, gy), \text{ for all } x, y \text{ in } X,
\]

where \( A \) is a non-negative constant with \( A < 1 \).

Then \( f \) and \( g \) have a unique common fixed point.

**Proof.** By putting \( C = 0 \) in Corollary 5.5.2, we get the Corollary 5.5.4.

**Example 5.5.5.** Let \( X = [0, 1] \) and define \( d : X \times X \to \mathbb{C} \) by \( d(x, y) = i \cdot \overline{x} - \overline{y} \), for all \( x, y \) in \( X \).

Then \((X, d)\) is a complex valued metric space.

Define the functions \( f, g : X \to X \) by \( fx = \overline{x} \) and \( gx = \overline{2} \).

Clearly \( fX = [0, \overline{1}] \subseteq [0, \overline{1}] = gX \).

Also, \( f \) and \( g \) are weakly compatible.

Consider the sequence \( \{x_n\} = \{\overline{1}, \overline{2}, \overline{3}, \ldots\} \), \( n \in \mathbb{N} \).

Since \( \lim_{n \to \infty} \overline{1} = \lim_{n \to \infty} \overline{x} = 0 \), so \( f \) and \( g \) satisfy (CLR) property.

Also, for \( A = \overline{0} < 1 \), we have

\[
d(fx, fy) \leq A d(gx, gy), \text{ for all } x, y \text{ in } X.
\]
Here $0$ is the unique common fixed point of $f$ and $g$.

Hence all the conditions of Corollary 5.5.4 are satisfied.

**Theorem 5.5.6.** Let $A$, $B$, $S$ and $T$ be self maps of a metric space $(X, d)$ satisfying (5.18), (5.28) and the following:

(5.40) $SX \subseteq BX$ and the pair $(A, S)$ satisfies (CLR$A$) property, or $TX \subseteq AX$ and the pair $(B, T)$ satisfies (CLR$B$) property.

Then $A$, $B$, $S$ and $T$ have a unique common fixed point.

**Proof.** Without loss of generality, assume that $SX \subseteq BX$ and the pair $(A, S)$ satisfies (CLR$A$) property, then there exists a sequence $\{x_n\}$ in $X$ such that

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = Ax,$$

for some $x$ in $X$.

Since $SX \subseteq BX$, there exists a sequence $\{y_n\}$ in $X$ such that $Sx_n = By_n$.

Hence $\lim_{n \to \infty} By_n = Ax$.

We shall show that $\lim_{n \to \infty} Ty_n = Ax$.

Let, if possible, $\lim_{n \to \infty} Ty_n = z \neq Ax$.

From (5.18), we have

$$d(Sx_n, Ty_n) \leq \alpha m(x_n, y_n) + \beta M(x_n, y_n).$$

Letting $n \to \infty$, we have

(5.41) $d(Ax, z) \leq \alpha \lim_{n \to \infty} m(x_n, y_n) + \beta \lim_{n \to \infty} M(x_n, y_n),$

with

$$\lim_{n \to \infty} m(x_n, y_n) = \lim_{n \to \infty} d(By_n, Ty_n)^{1+\frac{\alpha + \beta}{2}}$$

and

$$\lim_{n \to \infty} M(x_n, y_n) = \lim_{n \to \infty} \max \{ d(Ax_n, By_n), d(Ax_n, Sx_n), d(By_n, Ty_n) \}.$$ 

Thus, from (5.41), we have

$$d(Ax, z) \leq \alpha d Ax, z + \beta d Ax, z = \alpha + \beta \ d Ax, z,$$

that is,

$$d(Ax, z) \leq \alpha + \beta \ d(Ax, z),$$

a contradiction to $\alpha + \beta < 1$.

Therefore, $Ax = z$, that is, $\lim_{n \to \infty} Ty_n = Ax$.

Subsequently, we have

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = Ax = z.$$
Now, we shall show that $Sx = z$.
Let, if possible, $Sx \neq z$.
From (5.18), we have
\[ d(Sx, Ty_n) \leq \alpha m(x, y_n) + \beta M(x, y_n). \]
Letting $n \to \infty$, we have
\[ d(Sx, z) \leq \alpha \lim m(x, y_n) + \beta \lim M(x, y_n), \]
with
\[ \lim m(x, y_n) = \lim d(By_n, Ty_n) \to 0 \]
and
\[ \lim M(x, y_n) = \lim \max\{ d(Ax, By), d(Ax, Sx), d(By, Ty) \} \to 0. \]
Thus, from (5.42), we have
\[ d(z, Sx) = d(Sx, z). \]
\[ d(Sx, z) \leq \beta d(Sx, z), \text{ a contradiction to } \beta < 1. \]
Therefore, $Sx = z = Ax$.
Since, the pair $(A, S)$ is weakly compatible, it follows that $Az = Sz$.
Also, since $SX \subseteq BX$, there exists some $y$ in $X$ such that $Sx = By$, that is, $By = z$.
Now, we show that $Ty = z$.
Let, if possible, $Ty \neq z$.
From (5.18), we have
\[ d(Sx_n, Ty) \leq \alpha m(x_n, y) + \beta M(x_n, y). \]
Letting $n \to \infty$, we have
\[ d(Ax, z) \leq \alpha \lim m(x_n, y_n) + \beta \lim M(x_n, y_n), \]
with
\[ \lim m(x_n, y_n) = \lim d(By, Ty) \to 0 \]
and
\[ \lim M(x_n, y_n) = \lim \max\{ d(Ax_n, By), d(Ax_n, Sx_n), d(By, Ty) \} \to 0. \]
Thus, from (5.43), we have
\[ d(z, Ty) \leq \alpha d z, Ty + \beta d z, Ty = \alpha + \beta d z, Ty , \text{ that is,} \]
\[ d(z, Ty) \leq \alpha + \beta d(z, Ty), \text{ a contradiction to } \alpha + \beta < 1. \]
Thus, $z = Ty = By$.

Since the pair $(B, T)$ is weakly compatible, it follows that $Tz = Bz$.

Now, we claim that $Sz = Tz$.

Let, if possible, $Sz \neq Tz$.

From (5.18), we have

\[ d(Sz, Tz) \leq \alpha m(z, z) + \beta M(z, z), \]

with

\[ m(z, z) = d(Bz, Tz) \]

and

\[ M(z, z) = \max\{d(Az, Bz), d(Az, Sz), d(Bz, Tz)\} = d(Sz, Tz). \]

Thus, from (5.44), we have

\[ d(Sz, Tz) \leq \alpha 0 + \beta d(Sz, Tz), \]

that is,

\[ d(Sz, Tz) \leq \beta \ d(Sz, Tz), \]

a contradiction to $\beta < 1$.

Therefore, $Sz = Tz$, that is, $Az = Sz = Tz = Bz$.

Now, we shall show that $z = Tz$.

Let, if possible, $z \neq Tz$.

From (5.18), we have

\[ d(Sx, Tz) \leq \alpha m(x, z) + \beta M(x, z), \]

with

\[ m(x, z) = d(Bz, Tz) \]

and

\[ M(x, z) = \max\{d(Ax, Bz), d(Ax, Sx), d(Bz, Tz)\} = d(z, Tz). \]

Thus, from (5.45), we have

\[ d(z, Tz) \leq \alpha 0 + \beta d(z, Tz), \]

that is,

\[ d(z, Tz) \leq \beta \ d(z, Tz), \]

a contradiction to $\beta < 1$.

Therefore, $z = Tz = Bz = Az = Sz$.

Hence $z$ is the common fixed point of $A, B, S$ and $T$.

Now, we shall prove that the common fixed point is unique.

Let $u$ be another common fixed point of $A, B, S$ and $T$.

Let, if possible, $z \neq u$. 

\[d(u, z) = d(Su, Tz) \leq \alpha m(u, z) + \beta M(u, z),\]

\[m(u, z) = d(Bz, Tz)^{1+\frac{\beta}{1+\beta}} = 0,\]

and

\[M(u, z) = \max\{d(Au, Bz), d(Au, Su), d(Bz, Tz)\} = d(u, z).\]

Thus, from (5.46), we have

\[d(u, z) \leq \alpha 0 + \beta d(u, z), \text{ that is,} \]

\[d(u, z) \leq \beta d(u, z), \text{ a contradiction to } \beta < 1.\]

Therefore, \(u = z.\)

Hence \(A, B, S\) and \(T\) have a unique common fixed point.

**Corollary 5.5.7.** Let \(B\) and \(S\) be two weakly compatible self maps of a complex valued metric space \((X, d)\) satisfying (5.19), (5.20) and the following:

(5.47) \(B\) and \(S\) satisfies the \((\text{CLR}_B)\) property.

Then \(B\) and \(S\) have a unique common fixed point.

**Proof.** By putting \(A = B\) and \(S = T\) in Theorem 5.5.6, we get the corollary 5.5.7.